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Superstatistics approach to path integral for a relativistic particle

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Superstatistics permits the calculation of the Feynman propagator of a relativistic particle in a novel way from a superstatistical average over nonrelativistic single-particle paths. We illustrate this for the Klein-Gordon particle in the Feshbach-Villars representation, and for the Dirac particle in the Schrödinger-Dirac representation. As a by-product we recover the worldline representation of Klein-Gordon and Dirac propagators, and discuss the role of the smearing distributions in fixing the reparametrization freedom. The emergent relativity picture that follows from our approach together with a novel representation of the Lorentz group for the Feshbach-Villars particle are also discussed.

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I. INTRODUCTION

There has been a recent upsurge of interest in the so-called superstatistics paradigm [1–11]. Superstatistics is a branch of statistical physics devoted originally to the study of nonequilibrium non-Gaussian systems. It is characterized by superpositions of different distribution functions which usually operate on vastly different time scales with a non-Gaussian distribution as an output. Such an approach has a long tradition. There are many examples of nonlinear or nonequilibrium systems that have been treated with such methods [12–14]. The recent revival of interest in this field has been caused by recognizing the ubiquitous character of such a statistical behavior in the nature. This has in turn led to a systematic classification of the various compound distributions [6,11]. Particularly important is the realization that there are only three major physically relevant universality classes of smearing distributions: χ^2 superstatistics, inverse χ^2 superstatistics, and lognormal superstatistics. These three classes arise as universal limit statistics in a majority of known superstatistical systems [6,11].

In an earlier paper [9], we have shed yet another light on the superstatistics paradigm by addressing the following question: Assume that a conditional probability distribution $P(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ describing a particle to move from the position $\boldsymbol{x}_a$ in a D-dimensional Euclidean space at time t_a to the position $\boldsymbol{x}_b$ at time t_b satisfies the Chapman-Kolmogorov equation for Markovian process, i.e.,

$$P(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) = \int d\boldsymbol{x} P(\boldsymbol{x}_b, t_b|\boldsymbol{x}, t)P(\boldsymbol{x}, t|\boldsymbol{x}_a, t_a), \quad t_b \geq t \geq t_a. \tag{1}$$

This equation implies that $P(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a)$ possesses a path-integral representation (see e.g. Ref. [15]). If we distinguish various distributions by a strength parameter v, then

$$P_v(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) = \int_{\boldsymbol{x}(t_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(t_b)=\boldsymbol{x}_b} \mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \times \exp\left\{\int_{t_a}^{t_b} d\tau[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - vH(\boldsymbol{p}, \boldsymbol{x})]\right\}. \tag{2}$$

Is it possible that also superpositions of such path integrals

$$\bar{P}(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) = \int_0^\infty dv\omega(v, t_{ba})P_v(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) \tag{3}$$

satisfy the Chapman-Kolmogorov equation (1)? The answer is affirmative, if the weight function $\omega(v, t)$ fulfills a certain simple functional equation. In Ref. [9] we have derived this equation as follows. We have first defined a rescaled weight function

$$w(v, t) \equiv \omega(v/t, t)/t, \tag{4}$$

and calculated its Laplace transform

$$\tilde{w}(p_v, t) \equiv \int_0^\infty dve^{-p_v v}w(v, t). \tag{5}$$

The condition that the superposition (3) satisfies (1) can then be recast as a simple factorization property

$$\tilde{w}(p_v, t_1 + t_2) = \tilde{w}(p_v, t_2)\tilde{w}(p_v, t_1). \tag{6}$$

Assuming continuity in t, the solution of (6) is unique and can be written as an exponential of some real "Hamiltonian" $H_v(p_v)$:

$$\tilde{w}(p_v, t) = \mathrm{e}^{-tH_v(p_v)}, \tag{7}$$

where $H_v(p_v)$ must increase monotonically for large p_v, and must satisfy the normalization condition $H_v(0) = 0$.

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The Laplace inverse of $\tilde{w}(p_v, t)$ yields the desired smearing function $\omega(v, t)$. This allows for a rather large variety of functions $\omega(v, t)$ to guarantee the Chapman-Kolmogorov equation (1). Once this is satisfied, the smeared distribution (3) possesses a path-integral representation on its own, associated with a new Hamiltonian $\bar{H}(\boldsymbol{p}, \boldsymbol{x})$.

In Ref. [9] we have exploited this relationship in the converse direction by observing that the path integral representing of the Euclidean version of the probability *amplitude* for a *relativistic scalar particle* to move from a position $\boldsymbol{x}_a$ at time t_a to position $\boldsymbol{x}_b$ at time t_b,

$$P(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}\boldsymbol{x} \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \times \exp\left\{\int_{t_a}^{t_b} d\tau [\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - c\sqrt{\boldsymbol{p}^2 + m^2c^2}]\right\}, \tag{8}$$

can be considered as a superposition of nonrelativistic free-particle path integrals, namely,

$$P(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) = \int_0^\infty dv\,\omega(v, t_{ba}) \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}\boldsymbol{x} \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \times \exp\left\{\int_{t_a}^{t_b} d\tau [\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - v(\boldsymbol{p}^2c^2 + m^2c^4)]\right\}. \tag{9}$$

Here $\boldsymbol{p}$ and $\boldsymbol{x}$ are vectors in D-dimensional Euclidean space. The weight function $\omega(v, t)$ is the Weibull distribution [14,16] of order 1 (also known as the scaled inverse χ^2 distribution [17]). The Weibull distribution of order a is defined by

$$\omega(v, a, t) = \frac{a\exp(-a^2t/4v)}{2\sqrt{\pi}\sqrt{v^3/t}}, \qquad a \in \mathbb{R}^+, \tag{10}$$

with $\omega(v, t) \equiv \omega(v, 1, t)$. From the superstatistics point of view, the relation (9) belongs to the inverse χ^2-superstatistics universality class.

In the literature, the representation (8) is often referred to as the Newton-Wigner propagator [18]. The name Klein-Gordon kernel used in Ref. [19] is misleading since the propagator of the Klein-Gordon field must include also *negative*-energy spectrum, reflecting the existence of the charge-conjugated solution—antiparticle.

It is the purpose of this paper to use the superstatistics relationship to find the Feynman propagator of the Klein-Gordon field in a novel way. Subsequently, the same method will be applied to the Dirac field. The superstatistics approach will allow us to circumvent the technically involved procedure of constrained quantization [15,20–22] that is inherent to any theory with reparametrization invariance. The result will be the worldline representations for the two relativistic propagators.

For a better understanding of the upcoming result, we begin by introducing, in Sec. II, the Feshbach-Villars representation of a Klein-Gordon particle. After this we present, in Sec. III, a derivation of the corresponding Feynman Green function in Euclidean spacetime from the superstatistics standpoint. It will be seen that one must invoke the Stückelberg-Feynman interpretation of antiparticles as particles of negative energy running in the reverse time direction in order to make sense of the Feshbach-Villars time evolution operator. On the mathematical side, the Stückelberg-Feynman interpretation is necessary to ensure that the Feshbach-Villars evolution operator forms a strongly continuous semigroup. As a by-product, we obtain the well-known worldline representation of the Klein-Gordon propagator [15,20]. The same approach produces also the propagator of the Dirac particle, due to a close analogy between Feshbach-Villars diagonalization, which brings the Hamiltonian into a form where the positive- and negative-energy parts are explicitly separated, and the Foldy-Wouthuysen transformation of Dirac's Hamiltonian. This is demonstrated in some detail in Sec. IV. In Sec. V, we discuss the role of smearing distributions in fixing the reparametrization freedom. In particular, we show that Weibull's distribution parameter a is closely related to the *einbein* characterizing the worldline.

In Sec. VI, we briefly comment on the relation of our superstatistical path integrals to the concept of "emergent relativity." Various remarks and generalizations are proposed in the concluding Sec. VII. For the reader's convenience, we relegate some technical issues concerning the Feshbach-Villars representation to two Appendices.

II. SPINLESS PARTICLE IN FESHBACH-VILLARS REPRESENTATION

We start with the observation of Feshbach and Villars [23] that the Klein-Gordon equation for a free spinless charged particle can be rewritten in a Schrödinger-like form as

$$\mathrm{i}\,\partial_t\Psi(\boldsymbol{x}, t) = H_{\mathrm{FV}}(\hat{\boldsymbol{p}})\Psi(\boldsymbol{x}, t), \tag{11}$$

where $\hat{\boldsymbol{p}} = -\mathrm{i}\partial/\partial\boldsymbol{x}$. The wave function $\Psi(\boldsymbol{x}, t)$ is a two-component object

$$\Psi(\boldsymbol{x}, t) = \begin{pmatrix} \phi(\boldsymbol{x}, t) \\ \chi(\boldsymbol{x}, t) \end{pmatrix}, \tag{12}$$

and the Hamiltonian operator a 2×2 matrix

$$H_{\mathrm{FV}}(\hat{\boldsymbol{p}}) = (\sigma_3 + \mathrm{i}\sigma_2)\frac{\hat{\boldsymbol{p}}^2}{2m} + \sigma_3 mc^2 \equiv \hat{H}_{\mathrm{FV}}. \tag{13}$$

To see the equivalence with the Klein-Gordon equation, we rewrite (11) for the to components as

$$\mathrm{i}\,\partial_t(\phi + \chi) = mc^2(\phi - \chi), \tag{14}$$

$$\mathrm{i}\,\partial_t(\phi - \chi) = \frac{\hat{\boldsymbol{p}}^2}{m}(\phi + \chi) + mc^2(\phi + \chi), \tag{15}$$

from which we obtain

$$(\Box + m^2c^2)(\phi + \chi) = 0 \quad \text{and} \quad (\Box + m^2c^2)(\phi - \chi) = 0, \tag{16}$$

showing that both ϕ and χ obey a Klein-Gordon equation of mass m.

The physical role of the components ϕ and χ can be understood by introducing the electromagnetic potential $\boldsymbol{A}(\boldsymbol{x}, t)$ via the minimal substitution $\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{x}, t)/c$, and noting that the charge-conjugated wave function has the form [23]

$$\Psi_c(\boldsymbol{x}, t) = \sigma_1 \Psi^*(\boldsymbol{x}, t) = \begin{pmatrix} \chi^*(\boldsymbol{x}, t) \\ \phi^*(\boldsymbol{x}, t) \end{pmatrix}. \tag{17}$$

Thus, the two-component form of the wave function reflects the presence of particles and antiparticles of opposite charge.

If we define the conjugate Hamiltonian operator as

$$\bar{\hat{H}}_{\rm FV} \equiv \sigma_3 \hat{H}^\dagger_{\rm FV} \sigma_3, \tag{18}$$

then we see from (13) that $\hat{H}_{\rm FV}$ is conjugate to itself, i.e., *Hermitian* under the scalar product in D spatial dimensions:

$$(\Psi, \Psi') \equiv \int d\boldsymbol{x} \Psi^\dagger(\boldsymbol{x}, t) \sigma_3 \Psi'(\boldsymbol{x}, t), \qquad d\boldsymbol{x} \equiv d^D x. \tag{19}$$

The Hamiltonian $H_{\rm FV}(\boldsymbol{p})$ can be diagonalized via the similarity transformation

$$H_{\rm FV}(\boldsymbol{p}) = U_{\boldsymbol{p}} \begin{pmatrix} H_{\boldsymbol{p}} & \\ 0 & -H_{\boldsymbol{p}} \end{pmatrix} U_{\boldsymbol{p}}^{-1} = U_{\boldsymbol{p}} \sigma_3 U_{\boldsymbol{p}}^{-1} H_{\boldsymbol{p}}, \tag{20}$$

with

$$H_{\boldsymbol{p}} \equiv c\sqrt{\boldsymbol{p}^2 + m^2c^2}, \tag{21}$$

and $U_{\boldsymbol{p}}$ denoting the nonunitary Hermitian matrix

$$\begin{aligned} U_{\boldsymbol{p}} &= \frac{(mc^2 + H_{\boldsymbol{p}})\sigma_0 + (mc^2 - H_{\boldsymbol{p}})\sigma_1}{2\sqrt{mc^2 H_{\boldsymbol{p}}}} \\ &= \frac{(1+\gamma_{\boldsymbol{v}})\sigma_0 + (1-\gamma_{\boldsymbol{v}})\sigma_1}{2\sqrt{\gamma_{\boldsymbol{v}}}} = \exp\left(-\tfrac{1}{2}\sigma_1 \ln\gamma_{\boldsymbol{v}}\right) \\ &= \exp\left[\frac{1}{2}\sigma_1 \operatorname{arcosh}\left(\frac{1}{2}(\gamma_{\boldsymbol{v}} + 1/\gamma_{\boldsymbol{v}})\right)\right]. \end{aligned} \tag{22}$$

Here σ_0 is Pauli's two-dimensional unit matrix, and $\gamma_{\boldsymbol{v}}$ the usual Lorentz factor of relativistic motion $\gamma_{\boldsymbol{v}} \equiv (1 - \boldsymbol{v}^2/c^2)^{-1/2} = H_{\boldsymbol{p}}/mc^2$, where $\boldsymbol{v} = c^2\boldsymbol{p}/H_{\boldsymbol{p}}$ is the velocity of the particle. Note that the similarity transformation $U_{\boldsymbol{p}}$ converts the non-Hermitian Hamiltonian matrix $H_{\rm FV}$ into the Hermitian Hamiltonian matrix $\sigma_3 H_{\boldsymbol{p}}$. In this form the positive- and negative-energy solutions are decoupled. We also observe that if Ψ is a positive-energy eigenstate of the operator $\hat{H}_{\rm FV}$, then the associated charge-conjugated wave function Ψ^c corresponds to a negative-energy eigenstate of $\hat{H}_{\rm FV}$, and vice versa. This is because a positive-energy solution of momentum $\boldsymbol{p}$ can be written as

$$\Psi_{\boldsymbol{p}}(\boldsymbol{x}) = u(p)e^{ipx} \equiv U_{\boldsymbol{p}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ipx}, \tag{23}$$

while the charge-conjugated solution reads

$$\begin{aligned} \Psi^c_{\boldsymbol{p}}(\boldsymbol{x}) &= \sigma_1 \Psi^*_{\boldsymbol{p}}(\boldsymbol{x}) = \sigma_1 U_{\boldsymbol{p}} \sigma_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ipx} = v(p) e^{-ipx} \\ &\equiv U_{\boldsymbol{p}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ipx}, \end{aligned} \tag{24}$$

showing that it corresponds to the negative energies. The two-component objects

$$\xi\left(+\frac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi\left(-\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{25}$$

play the role of "pseudospinors" in a charge space.

From Eqs. (19), (23), and (24) we see that the wave functions can be normalized according to

$$(\Psi, \Psi) = \pm 1, \tag{26}$$

where the plus/minus sign corresponds to particle/antiparticle. Here we have used the relation $U_{\boldsymbol{p}} \sigma_3 U_{\boldsymbol{p}} = \sigma_3$.

Equation (23) suggests that $U_{\boldsymbol{p}}$ may be viewed as a boost transformation that brings a "pseudospinor" of a spinless particle at rest, $\xi(\frac{1}{2})$, to the pseudospinor $u_{\boldsymbol{p}}$ of a particle with velocity $\boldsymbol{v}$. As usual, the Lorentz boosts are, in contrast to rotations, nonunitary, as all finite-dimensional representations of noncompact group transformations should be. However, as in the case of Dirac spinors, they are pseudounitary with respect to the conjugation operation (18), namely, $U_{\boldsymbol{p}}^{-1} = \sigma_3 U_{\boldsymbol{p}}^\dagger \sigma_3 \equiv \bar{U}_{\boldsymbol{p}}$. More details will be provided in Appendices A and B.

III. FEYNMAN GREEN FUNCTION

Let us calculate the Feynman Green function $\mathcal{G}(x, y)$ associated with the Schrödinger-like equation (11). It is defined by

$$(\mathrm{i}\partial_t - \hat{H}_{\rm FV})\mathcal{G}(\boldsymbol{x}, t; \boldsymbol{x}', t') = \mathrm{i}\delta^{(D)}(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t'). \tag{27}$$

The solution has the Fourier decomposition

$$\begin{aligned} \mathcal{G}(\boldsymbol{x}, t; \boldsymbol{x}', t') &= \mathrm{i} \int_{\mathbb{R}^{D+1}} \frac{dp_0}{2\pi} \frac{d\boldsymbol{p}}{(2\pi)^D} \mathrm{e}^{-ip(x-x')} \\ &\quad \times \left[p_0 c - (\sigma_3 + \mathrm{i}\sigma_2)\frac{\boldsymbol{p}^2}{2m} - \sigma_3 mc^2\right]^{-1} \\ &= \mathrm{i} \int_{\mathbb{R}^{D+1}} \frac{d^{D+1}p}{(2\pi)^{D+1}} \frac{\mathrm{e}^{-ip(x-x')}}{p^2c^2 - m^2c^4 + \mathrm{i}\epsilon} \\ &\quad \times \left[p_0 c + (\sigma_3 + \mathrm{i}\sigma_2)\frac{\boldsymbol{p}^2}{2m} + \sigma_3 mc^2\right]. \end{aligned} \tag{28}$$

The Feynman boundary conditions are ensured by the usual $\mathrm{i}\epsilon$ prescription with infinitesimal $\epsilon > 0$.

Equivalently, we can perform a Wick rotation, which makes the denominator regular, and the imaginary-time Green function $\mathcal{G}(\boldsymbol{x}, -it; \boldsymbol{x}', -it') \equiv P(\boldsymbol{x}, t|\boldsymbol{x}', t')$ satisfies the Fokker-Planck-like equation:

$$(\partial_t + \hat{H}_{\mathrm{FV}})P(\boldsymbol{x}, t|\boldsymbol{x}, t') = \delta(t - t')\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'). \qquad (29)$$

The solution is obtained from the local matrix element of the time evolution operator $e^{-t\hat{H}_{\mathrm{FV}}}$:

$$P(\boldsymbol{x}, t|\boldsymbol{x}', t') = \langle \boldsymbol{x}|\mathrm{e}^{-(t-t')\hat{H}_{\mathrm{FV}}}|\boldsymbol{x}'\rangle. \qquad (30)$$

Recalling the matrix relation (20), this is equal to

$$P(\boldsymbol{x}, t|\boldsymbol{x}', t') = \langle \boldsymbol{x}|U_{\hat{\boldsymbol{p}}}\mathrm{e}^{-(t-t')\sigma_3 H_{\hat{\boldsymbol{p}}}}U_{\hat{\boldsymbol{p}}}^{-1}|\boldsymbol{x}'\rangle. \qquad (31)$$

Since $U_{\hat{\boldsymbol{p}}}$ and $H_{\hat{\boldsymbol{p}}}$ are diagonal in the momentum basis $|\boldsymbol{p}\rangle$, we use the completeness relation

$$\int_{\mathbb{R}^D} \frac{d\boldsymbol{p}}{(2\pi)^D}|\boldsymbol{p}\rangle\langle\boldsymbol{p}| = \mathbb{1} \qquad (32)$$

to rewrite (31) as

$$P(\boldsymbol{x}, t|\boldsymbol{x}', t') = \int_{\mathbb{R}^D} \frac{d\boldsymbol{p}}{(2\pi)^D}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x})'}U_{\boldsymbol{p}}\langle\boldsymbol{p}|\mathrm{e}^{-(t-t')\sigma_3 H_{\boldsymbol{p}}}|\boldsymbol{p}\rangle U_{\boldsymbol{p}}^{-1}. \qquad (33)$$

Alternatively we may rewrite (31) as

$$P(\boldsymbol{x}, t|\boldsymbol{x}', t') = U_{\hat{\boldsymbol{p}}}\langle \boldsymbol{x}|\mathrm{e}^{-(t-t')\sigma_3 H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle(\overleftarrow{U}_{\hat{\boldsymbol{p}}'})^{-1}, \qquad (34)$$

where the arrow on top of the operator indicates the direction in which the momentum operator $\hat{\boldsymbol{p}}$ acts.

Let us now express the amplitude in (34) as a path integral. This is not straightforward, since the formal expression

$$\langle \boldsymbol{x}''|\mathrm{e}^{-(t-t')\sigma_3 H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle = \int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''} \mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D}\mathrm{e}^{\int_{t'}^{t} d\tau[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - c\sigma_3\sqrt{\boldsymbol{p}^2+m^2c^2}]} \qquad (35)$$

diverges for the lower components of the imaginary-time evolution operator

$$\mathrm{e}^{-t\sigma_3 H_{\boldsymbol{p}}} = \begin{pmatrix} \mathrm{e}^{-tH_{\boldsymbol{p}}} & 0 \\ 0 & \mathrm{e}^{tH_{\boldsymbol{p}}} \end{pmatrix}. \qquad (36)$$

This difficulty can be circumvented by using different superpositions of Gaussian path integrals, written as in Eq. (9), both with the same positive Hamiltonian, but with different weight functions $\omega(v, t)$ for upper and lower components of $\mathrm{e}^{-t\sigma_3 H_{\boldsymbol{p}}}$. In particular, we write

$$\langle \boldsymbol{x}''|\mathrm{e}^{-t\sigma_3 H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle = \int_0^\infty dv\,\omega(v, t)\int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''} \mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \times \mathrm{e}^{\int_0^t d\tau[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - v(\boldsymbol{p}^2c^2+m^2c^4)]}, \qquad (37)$$

with a matrix weight function

$$\omega(v, t) = \frac{1}{2\sqrt{\pi}\sqrt{v^3/|t|}}\begin{pmatrix} \theta(t)\mathrm{e}^{-t/4v} & 0 \\ 0 & \theta(-t)\mathrm{e}^{t/4v} \end{pmatrix}. \qquad (38)$$

Here we have invoked the Feynman-Stuckelberg causality condition [24–26] that negative-energy solutions propagate backward in time.

Note that the amplitude (37) has the time-ordered form

$$\begin{aligned}\langle \boldsymbol{x}''|\mathrm{e}^{-t\sigma_3 H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle &= \theta(t)\frac{1+\sigma_3}{2}\langle \boldsymbol{x}''|\mathrm{e}^{-tH_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle \\ &\quad + \theta(-t)\frac{1-\sigma_3}{2}\langle \boldsymbol{x}''|\mathrm{e}^{tH_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle \\ &= \frac{1+\mathrm{sgn}(t)\sigma_3}{2}\langle \boldsymbol{x}''|\mathrm{e}^{-|t|H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle \\ &= \frac{1}{2}\left(1 - \frac{H_{\hat{\boldsymbol{p}}''}\sigma_3}{\partial_t}\right)\langle \boldsymbol{x}''|\mathrm{e}^{-|t|H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle. \end{aligned} \qquad (39)$$

Representation (37) can be given a familiar relativistic form by functionally integrating out $\mathcal{D}\boldsymbol{p}$ and changing variables from v to $\nu \equiv 1/v$. This gives ($t > 0$)

$$\langle \boldsymbol{x}''|\mathrm{e}^{-tH_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle = \int_0^\infty d\nu\frac{\mathrm{e}^{-m^2c^4t/\nu}\mathrm{e}^{-t\nu/4}}{2\sqrt{\pi}\sqrt{\nu/t}} \times \int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''} \mathcal{D}\boldsymbol{x}\mathrm{e}^{-\int_0^t d\tau\nu\dot{\boldsymbol{x}}^2/4c^2}. \qquad (40)$$

By setting $\tau = \bar{\lambda}\nu/2mc^2$, the right-hand side becomes

$$\langle \boldsymbol{x}''|\mathrm{e}^{-tH_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle = \int_0^\infty d\lambda\frac{\mathrm{e}^{-\lambda mc^2/2}\mathrm{e}^{-t\nu/4}\sqrt{2mc^2}t}{2\sqrt{\pi}\sqrt{\lambda^3}} \times \int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(\lambda)=\boldsymbol{x}''} \mathcal{D}\boldsymbol{x}\mathrm{e}^{-\int_0^\lambda d\bar{\lambda}m[\boldsymbol{x}'(\bar{\lambda})]^2/2}. \qquad (41)$$

We can now use a trivial Gaussian path integral for an auxiliary zeroth component $x_0(\lambda)$ associated with the path $\boldsymbol{x}(\lambda)$:

$$\sqrt{\frac{2\pi\lambda}{m}}\int_{x_0(0)=0}^{x_0(\lambda)=ct} \mathcal{D}x_0\mathrm{e}^{-\int_0^\lambda d\lambda m[x_0'(\bar{\lambda})]^2/2} = \mathrm{e}^{-mc^2t^2/2\lambda} = \mathrm{e}^{-t\nu/4}, \qquad (42)$$

whose time derivative is

$$-\partial_t\left[\sqrt{\frac{2\pi\lambda}{m}}\int_{x_0(0)=0}^{x_0(\lambda)=ct} \mathcal{D}x_0\mathrm{e}^{-\int_0^\lambda d\bar{\lambda}m[x_0'(\bar{\lambda})]^2/2}\right] = \frac{v}{4}\mathrm{e}^{-tv/4}. \qquad (43)$$

With (43) we can finally express (39) in the form

$$\begin{aligned}\langle \boldsymbol{x}''|\mathrm{e}^{-t\sigma_3 H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle &= (\sigma_3 H_{\boldsymbol{p}} - \partial_t) \\ &\quad \times \left[\int_0^\infty d\lambda\mathrm{e}^{-\lambda mc^2/2}\int_{x(0)=x'}^{x(\lambda)=x} \mathcal{D}x\mathrm{e}^{-\int_0^\lambda d\bar{\lambda}m[x_\mu'(\bar{\lambda})]^2/2}\right],\end{aligned} \qquad (44)$$

where $x_\mu = (ct, \boldsymbol{x})$, $x_\mu^2 = c^2t^2 + \boldsymbol{x}^2$, and $x_0'' = ct$. The path integral on the right-hand side describes a free nonrelativistic particle in $D + 1$ dimensions.

The expression in brackets has a Fourier representation (see, e.g. Chapter 19 in Ref. [15])

$$\int_0^\infty d\lambda \mathrm{e}^{-\lambda mc^2/2} \int_{x(0)=x'}^{x(\lambda)=x''} \mathcal{D}x \mathrm{e}^{-\int_0^\lambda d\bar{\lambda}(m/2)(dx^\mu/d\bar{\lambda})^2} = \int_{\mathbb{R}^{D+1}} \frac{d^{D+1}p}{(2\pi)^{D+1}} \frac{\mathrm{e}^{-\mathrm{i}p(x''-x')}}{p^2 + m^2c^2}. \tag{45}$$

Inserting this into formula (44), and the result further into (34), we find

$$\begin{aligned} P(\boldsymbol{x}, t|\boldsymbol{x}', t') &= \int_{\mathbb{R}^{D+1}} \frac{d^{D+1}p}{(2\pi)^{D+1}} \mathrm{e}^{-\mathrm{i}p(x-x')} U_{\boldsymbol{p}}(\mathrm{i}p_0 c + \sigma_3 H_{\boldsymbol{p}}) \\ &\quad \times U_{\boldsymbol{p}}^{-1} \frac{1}{p^2 + m^2c^2} \\ &= \int_{\mathbb{R}^{D+1}} \frac{d^{D+1}p}{(2\pi)^{D+1}} \mathrm{e}^{-\mathrm{i}p(x-x')} \\ &\quad \times \left[\mathrm{i}p_0 c + (\sigma_3 + \mathrm{i}\sigma_2) \frac{\boldsymbol{p}^2}{2m} + \sigma_3 mc^2 \right] \\ &\quad \times \frac{1}{p^2 + m^2c^2}. \end{aligned} \tag{46}$$

It is straightforward to verify that (46) satisfies the differential equation (29). One could, of course, arrive at (45) directly by comparing (33) and (44) with (28).

Note that by reading Eq. (45) from right to left we obtain the well-known path-integral representation of the Klein-Gordon propagator $\Delta(x - x')$ [15,27], also known as the Feynman-Fock worldline representation. Normally this is derived with the help of a spurious dynamical variable—einbein, that makes the path integral manifestly reparametrization invariant. Such a gauge freedom is then treated with the usual methods of constrained quantization [15,20,21]. Our use of Weibull's distribution brought us automatically to what is sometimes called Polyakov gauge [20]—i.e., the gauge where the einbein variable is fixed to be the velocity of light (for details see Ref. [15]). In Sec. V we shall see how $\omega(v, t)$ must be modified to account for a general gauge. By going back from Euclidean times to real times, we can now recover the Green function associated with the initial real-time Schrödinger equation (11).

IV. DIRAC PARTICLE AND FOLDY-WOUTHUYSEN TRANSFORMATION

It is instructive to use the same procedure for calculating the Green function of the Dirac particle. Here the role of Feshbach-Villars diagonalization matrix $U_{\boldsymbol{p}}$ in Eq. (22) is played by the 4×4 Foldy-Wouthuysen transformation in spinor space [28],

$$V_{\boldsymbol{p}} = \mathrm{e}^{-\mathrm{i}S_{\boldsymbol{p}}}, \qquad S_{\boldsymbol{p}} = -\mathrm{i}\boldsymbol{\gamma} \cdot \frac{\boldsymbol{p}}{|\boldsymbol{p}|} \frac{\theta_{\boldsymbol{p}}}{2}, \tag{47}$$

where $\boldsymbol{\gamma}$ denotes the three-vector of the Hermitian spatial Dirac matrices and

$$\begin{aligned} \cos\theta_{\boldsymbol{p}} &= \frac{mc}{\sqrt{\boldsymbol{p}^2 + m^2c^2}} = \frac{1}{\gamma_{\boldsymbol{v}}}, \\ \sin\theta_{\boldsymbol{p}} &= \frac{|\boldsymbol{p}|}{\sqrt{\boldsymbol{p}^2 + m^2c^2}} = \frac{|\boldsymbol{v}|}{c}. \end{aligned} \tag{48}$$

The matrix $V_{\boldsymbol{p}}$ brings the Hermitian Dirac Hamiltonian

$$H_{\mathrm{D}}(\boldsymbol{p}) = c\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{p} + \gamma^0 mc^2 = \begin{pmatrix} mc & \boldsymbol{p} \cdot \boldsymbol{\sigma} \\ \boldsymbol{p} \cdot \boldsymbol{\sigma} & -mc \end{pmatrix} c, \tag{49}$$

to the diagonal form,

$$\begin{aligned} H_{\mathrm{diag}}(\boldsymbol{p}) &= \begin{pmatrix} c\sqrt{\boldsymbol{p}^2 + m^2c^2}\sigma_0 & 0 \\ 0 & -c\sqrt{\boldsymbol{p}^2 + m^2c^2}\sigma_0 \end{pmatrix} \\ &= \gamma_0 H_{\boldsymbol{p}}, \end{aligned} \tag{50}$$

with the similarity transformation,

$$H_{\mathrm{D}}(\boldsymbol{p}) = V_{\boldsymbol{p}} H_{\mathrm{diag}}(\boldsymbol{p}) V_{\boldsymbol{p}}^{-1}. \tag{51}$$

Here γ_0 is the Dirac matrix

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} = \sigma_3 \otimes \sigma_0, \tag{52}$$

composed of Pauli's two-dimensional unit matrices σ_0 ($\otimes$ denotes a tensor product).

Note that the matrix $V_{\boldsymbol{p}}$ is now unitary, in contrast to the spinless case where it was nonunitary but Hermitian, since there the Hamiltonian was non-Hermitian.

In the Dirac case, we may calculate the probability $P(\boldsymbol{x}, t|\boldsymbol{x}', t')$ as a 4×4 matrix following the same strategy as in Sec. III. In particular, we write $\langle \boldsymbol{x}|\mathrm{e}^{-(t-t')\gamma_0 H_{\boldsymbol{p}}}|\boldsymbol{x}'\rangle$ by analogy with (37) as a superposition of nonrelativistic single-particle path integrals,

$$\begin{aligned} \langle \boldsymbol{x}|\mathrm{e}^{-t\gamma_0 H_{\hat{\boldsymbol{p}}}}|\boldsymbol{x}'\rangle &= \int_0^\infty dv \omega(v, t) \int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}} \mathcal{D}\boldsymbol{x} \frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \\ &\quad \times \mathrm{e}^{\int_{t'}^{t} d\tau[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - v(\boldsymbol{p}'c^2 + m^2c^4)]}, \end{aligned} \tag{53}$$

with a matrix of Weibull distributions:

$$\omega(v, t) = \frac{1}{2\sqrt{\pi}\sqrt{v^3/|t|}} \begin{pmatrix} \theta(t)\mathrm{e}^{-t/4v}\sigma_0 & 0 \\ 0 & \theta(-t)\mathrm{e}^{t/4v}\sigma_0 \end{pmatrix}. \tag{54}$$

Applying the Foldy-Wouthuysen transformation (51) we obtain for the matrix elements $\langle \boldsymbol{x}|\mathrm{e}^{-t\gamma_0 H_{\mathrm{D}}(\hat{\boldsymbol{p}})}|\boldsymbol{x}'\rangle$ the path integral

$$\begin{aligned}\langle \boldsymbol{x}|\mathrm{e}^{-t\gamma_0 H_{\mathrm{D}}(\hat{\boldsymbol{p}})}|\boldsymbol{x}'\rangle &= \int_{\mathbb{R}^{D+1}} \frac{d^{D+1}p}{(2\pi)^{D+1}} \mathrm{e}^{-\mathrm{i}p(x-x')} \\ &\quad \times V_{\boldsymbol{p}}(\mathrm{i}cp_0 + H_{\boldsymbol{p}}\gamma_0)V_{\boldsymbol{p}}^{-1} \frac{1}{p^2+m^2c^2} \\ &= \frac{\gamma_{\mathrm{E}}^0}{c}\int_{\mathbb{R}^{D+1}} \frac{d^{D+1}p}{(2\pi)^{D+1}} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{\mathrm{i}\not{p}_{\mathrm{E}} + mc}{p^2+m^2c^2}.\end{aligned} \tag{55}$$

To obtain the last line we have used the simple matrix identity $V_{\boldsymbol{p}}(\mathrm{i}cp_0 + H_{\boldsymbol{p}}\gamma_0)V_{\boldsymbol{p}}^{-1} = \mathrm{i}cp_0 + c\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p} + \gamma^0 mc^2$, and introduced the Euclidean gamma matrices $\gamma_{\mathrm{E}}^0 \equiv \gamma^0$ and $\boldsymbol{\gamma}_{\mathrm{E}} \equiv \mathrm{i}\boldsymbol{\gamma}$ fulfilling the Clifford algebra $\{\gamma_{\mathrm{E}}^\mu, \gamma_{\mathrm{E}}^\nu\} = 2\delta^{\mu\nu}$. Similarly $\not{p}_{\mathrm{E}} \equiv \gamma_{\mathrm{E}}^0 p^0 + \boldsymbol{\gamma}_{\mathrm{E}}\cdot\boldsymbol{p}$.

The Euclidean amplitude (55) satisfies the Fokker-Planck-like equation analogous to (29):

$$(\partial_t + H_{\mathrm{D}})\langle \boldsymbol{x}|\mathrm{e}^{-t\gamma_0 H_{\mathrm{D}}(\hat{\boldsymbol{p}})}|\boldsymbol{x}'\rangle = \delta(t-t')\delta^{(D)}(\boldsymbol{x}-\boldsymbol{x}'). \tag{56}$$

By multiplying this with γ_{E}^0, and defining $\tilde{P}(\boldsymbol{x},t|\boldsymbol{x}',t') \equiv \langle \boldsymbol{x}|\mathrm{e}^{-t\gamma_0 H_{\mathrm{D}}(\hat{\boldsymbol{p}})}|\boldsymbol{x}'\rangle c\gamma_{\mathrm{E}}^0$, we obtain the covariant expression

$$\begin{aligned}(\gamma_{\mathrm{E}}^0\partial_{ct} + c^{-1}\gamma_{\mathrm{E}}^0 H_{\mathrm{D}})\tilde{P}(\boldsymbol{x},t|\boldsymbol{x}',t') &= (\mathrm{i}\not{p}_{\mathrm{E}} + mc)\tilde{P}(\boldsymbol{x},t|\boldsymbol{x}',t') \\ &= \delta^{(D+1)}(x-x'),\end{aligned} \tag{57}$$

showing that $\tilde{P}(\boldsymbol{x},t|\boldsymbol{x}',t')$ is the Green function for the Euclidean Dirac equation.

Let us finally emphasize the well-known fact that using only the positive energies in relativistic path integrals such as Eq. (8) leads immediately to pathologies, such as non-conservation of probability, lack of Zitterbewegung [29], loss of relativistic invariance [18], and, of course, the inability to find the correct Klein-Gordon propagator. These difficulties do not arise when the *full matrix structure* of the Weibull distribution is taken into account. Such a matrix structure takes complete care of both particles and antiparticles and it highlights the key role of the Feynman-Stuckelberg boundary condition.

V. REPARAMETRIZATION FREEDOM

Let us now turn to the case of a general reparametrization invariance. In this connection it is instructive to observe various degrees of freedom in the representation (9) of the conditional probability (8). First we note that by substituting $v \mapsto v/a^2$ we obtain the identity

$$\begin{aligned}&\int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''} \mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D}\exp\left\{\int_0^t d\tau\left[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - c\sqrt{\boldsymbol{p}^2+m^2c^2}\right]\right\} \\ &\quad = \int_0^\infty dv\,\omega(v,a,t)\int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''}\mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \\ &\quad \times \exp\left\{\int_0^t d\tau\left[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \frac{v}{a^2}(\boldsymbol{p}^2c^2+m^2c^4)\right]\right\},\end{aligned} \tag{58}$$

where $\omega(v,a,t)$ is the Weibull distribution of order a in Eq. (10). The right-hand side can be further integrated functionally over $\boldsymbol{p}$ to become

$$\begin{aligned}&\int_0^\infty dv\,\omega(v,a,t)\int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}''}\mathcal{D}\boldsymbol{x} \\ &\quad\times\exp\left\{-\int_0^t d\tau\left[\frac{a^2}{4c^2v}\dot{\boldsymbol{x}}^2 + \frac{c^2v}{a^2}m^2c^2\right]\right\} \\ &= \int_0^\infty dL\frac{ct\mathrm{e}^{-c^2t^2/2L}}{\sqrt{2\pi L^3}}\int_{\boldsymbol{x}(0)=\boldsymbol{x}''}^{\boldsymbol{x}(t)=\boldsymbol{x}''}\mathcal{D}\boldsymbol{x} \\ &\quad\times\exp\left\{-\int_0^\lambda d\bar{\lambda}\left[\frac{1}{2e}[\boldsymbol{x}'(\bar{\lambda})]^2 + \frac{e}{2}m^2c^2\right]\right\}.\end{aligned} \tag{59}$$

Here we have defined a new variable $\bar{\lambda} \equiv \tau 2c^2v/a^2e$, so that the length of a particle orbit is $L \equiv \int_0^\lambda d\bar{\lambda}e$. In this expression, e may be viewed as a constant "einbein," i.e., a square root of the intrinsic metric along the worldline. As in Sec. III we can rewrite (59) in a relativistic form by utilizing an auxiliary Gaussian path integral for x_0 similar to (43), as

$$\begin{aligned}&\partial_t\int_{x_0(0)=0}^{x_0(\lambda)=ct}\mathcal{D}x_0\exp\left\{-\int_0^\lambda d\bar{\lambda}\frac{1}{2e}[x_0'(\bar{\lambda})]^2\right\} \\ &= -c^2t\sqrt{\frac{1}{2\pi L^3}}\mathrm{e}^{-c^2t^2/2L}.\end{aligned} \tag{60}$$

With this we can rewrite the right-hand side of (58) as

$$\begin{aligned}&-\frac{\partial_t}{c}\int_0^\infty dL\int_{x(0)=x'}^{x(\lambda)=x''}\mathcal{D}x \\ &\quad\times\exp\left\{-\int_0^\lambda d\bar{\lambda}\left[\frac{1}{2e}[x_\mu'(\bar{\lambda})]^2 + \frac{e}{2}m^2c^2\right]\right\}.\end{aligned} \tag{61}$$

Analogous steps to those in Sec. III allow us to find for the Klein-Gordon propagator (45) the worldline representation,

$$\begin{aligned}&\int_0^\infty dL\int_{x(0)=x'}^{x(\lambda)=x''}\mathcal{D}x \\ &\quad\times\exp\left\{-\int_0^\lambda d\bar{\lambda}\left[\frac{1}{2e}[x_\mu'(\bar{\lambda})]^2 + \frac{e}{2}m^2c^2\right]\right\} \\ &\quad= \int_{\mathbb{R}^{D+1}}\frac{d^{D+1}p}{(2\pi)^{D+1}}\frac{\mathrm{e}^{-\mathrm{i}p(x''-x')}}{p^2+m^2c^2}.\end{aligned} \tag{62}$$

So the different choices of the parameter a of the Weibull smearing distribution correspond to different constant einbeins e in the worldline representations of the Klein-Gordon propagator.

The freedom of choice of e in (62) can be generalized further to a full gauge freedom, i.e., a freedom to change the worldline parametrization. However, this cannot be done straightforwardly just by assuming that a depends on $\bar{\lambda}$. This is because the Hamiltonian we wish to smear out would become explicitly "time dependent" [see Eq. (58)], and for such cases our superstatistics argument [9] is not valid. It is, however, not difficult to tackle the problem indirectly. To see this we use a simple identity for a functional δ function [30]: let $e(\bar{\lambda})$ be a dynamical variable and let $F_{\bar{\lambda}}(e) = 0$ be a system of equations that for each $\bar{\lambda}$

provides a constant solution e_s. Let, in addition, $F_{\bar\lambda} = F_{\bar\lambda}(e)$ be a one-to-one map in some neighborhood of $F_{\bar\lambda} = 0$ which can be inverted to $e(\bar\lambda) \equiv e_{\bar\lambda} = e_{\bar\lambda}(F)$. Any functional $G[e_s]$ can be then written as

$$\begin{aligned} G[e_s] &= \int\Big[\prod_{\bar\lambda} dF_{\bar\lambda}\delta(F_{\bar\lambda})\Big]G[e(F)] \\ &= \int\Big[\prod_{\bar\lambda} de_{\bar\lambda}\delta(F_{\bar\lambda}(e))\Big]\mathcal{J}(e)G[e] \\ &\equiv \int \mathcal{D}e\,\delta[F(e)]\mathcal{J}(e)G[e], \end{aligned} \tag{63}$$

with the functional Jacobian

$$\mathcal{J}(e) = \det F_{\bar\lambda\bar\lambda'}, \qquad F_{\bar\lambda\bar\lambda'} = \frac{\partial F_{\bar\lambda}}{\partial e(\bar\lambda')}. \tag{64}$$

By setting

$$G[e_s] = \exp\Big\{-\int_0^\lambda d\bar\lambda\Big[\frac{1}{2e_s}[x'_\mu(\bar\lambda)]^2 + \frac{e_s}{2}mc^2\Big]\Big\}, \tag{65}$$

we can rewrite (62) in the form

$$\begin{aligned} &\int_0^\infty dL \int \mathcal{D}e\,\delta[F(e)]\mathcal{J}(e)\int_{x(0)=x'}^{x(\lambda)=x''}\mathcal{D}x \\ &\quad\times \exp\Big\{-\int_0^\lambda d\bar\lambda\Big[\frac{1}{2e(\bar\lambda)}[x'_\mu(\bar\lambda)]^2 + \frac{e(\bar\lambda)}{2}m^2c^2\Big]\Big\} \\ &= \int_{\mathbb{R}^{D+1}} \frac{d^{D+1}p}{(2\pi)^{D+1}}\frac{\mathrm{e}^{-\mathrm{i}p(x''-x')}}{p^2+m^2c^2}. \end{aligned} \tag{66}$$

The action in (66) is clearly *reparametrization invariant* under $\bar\lambda \mapsto \bar\lambda' = f(\bar\lambda)$, if we transform

$$\begin{aligned} x^\mu(\bar\lambda) &\mapsto \tilde x^\mu(\bar\lambda) = x^\mu(f^{-1}(\bar\lambda)), \\ e(\bar\lambda) &\mapsto e'(\bar\lambda) = \frac{df^{-1}(\bar\lambda)}{d\bar\lambda}e(f^{-1}(\bar\lambda)). \end{aligned} \tag{67}$$

Here $f(\bar\lambda)$ is an arbitrary monotonically increasing function of $\bar\lambda$. A general gauge fixing, say $\tilde F(e) = 0$, can be now implemented by performing the change of the einbein variable $e \mapsto e'$ via $e = F^{-1}\circ\tilde F(e')$. As a consequence of the rules of functional differentiation, we have

$$\begin{aligned} \mathcal{D}e\,\delta[F(e)]\mathcal{J}(e) &= \mathcal{D}e'\det\Big[\frac{\partial(F^{-1}\circ\tilde F)}{\partial e'}\Big]\delta[\tilde F(e')] \\ &\quad\times\det\Big[\frac{\partial(F\circ F^{-1}\circ\tilde F)}{\partial(F^{-1}\circ\tilde F)}\Big] \\ &= \mathcal{D}e'\delta[\tilde F(e')]\tilde{\mathcal{J}}(e'), \end{aligned} \tag{68}$$

where $\delta[\tilde F(e')] = \prod_{\bar\lambda'}\delta(\tilde F_{\bar\lambda'}(e'))$, and the functional Jacobian $\tilde{\mathcal{J}}$ has the form

$$\tilde{\mathcal{J}}(e') = \det\tilde F_{\bar\lambda\bar\lambda'}, \qquad \tilde F_{\bar\lambda\bar\lambda'} = \frac{\partial\tilde F_{\bar\lambda}}{\partial e'(\bar\lambda')}. \tag{69}$$

Note also that due to einbein identity, $d\bar\lambda e(\bar\lambda) = d\bar\lambda' e'(\bar\lambda')$ (see, e.g., Refs. [15,31]), the action in (66) changes to

$$\int_0^\lambda d\bar\lambda\Big[\frac{1}{2e'(\bar\lambda)}[\tilde x'_\mu(\bar\lambda)]^2 + \frac{e'(\bar\lambda)}{2}mc^2\Big]. \tag{70}$$

We can now relabel e' back to e and $\tilde x^\mu$ back to x^μ, and write the left-hand side of (66) in the gauge-fixed form

$$\begin{aligned} &\int_0^\infty dL \int \mathcal{D}e\,\delta[\tilde F(e)]\tilde{\mathcal{J}}(e)\int_{x(0)=x'}^{x(\lambda)=x''}\mathcal{D}x \\ &\quad\times \exp\Big\{-\int_0^\lambda d\bar\lambda\Big[\frac{1}{2e(\bar\lambda)}[x'_\mu(\bar\lambda)]^2 + \frac{e(\bar\lambda)}{2}m^2c^2\Big]\Big\}, \end{aligned} \tag{71}$$

with a particle orbit length

$$L = \int_0^\lambda d\bar\lambda e(\bar\lambda). \tag{72}$$

The reader may notice that the gauge $e \equiv$ const is recovered by setting $\tilde F_{\bar\lambda}(e) = e(\bar\lambda) - e$.

Let us finally observe that (71) can be rewritten as

$$\begin{aligned} &\int_0^\infty dL \int \mathcal{D}e\,\delta[\tilde F(e)]\tilde{\mathcal{J}}(e)\frac{\exp[-(x_0''-x_0')/2L]}{\sqrt{2\pi}L} \\ &\quad\times\int_{x(0)=x'}^{x(\lambda)=x''}\mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \\ &\quad\times\exp\Big\{\int_0^\lambda d\bar\lambda\Big[\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} - \frac{e(\bar\lambda)}{2}(\boldsymbol{p}^2+m^2c^2)\Big]\Big\}, \end{aligned} \tag{73}$$

which indicates that the smearing-distribution functional corresponding to the einbein $e(\bar\lambda)$ reads

$$\varrho[e; x_0'', x_0'] = \delta[\tilde F(e)]\tilde{\mathcal{J}}(e)\frac{\exp[-(x_0''-x_0')/2L]}{\sqrt{2\pi}L}. \tag{74}$$

In deriving (73) we have used the fact that

$$\begin{aligned} &\int_{x_0(0)=x_0'}^{x_0(\lambda)=x_0''}\mathcal{D}\Big(\frac{x_0}{\sqrt{e}}\Big)\exp\Big\{-\int_0^\lambda d\bar\lambda\frac{1}{2e(\bar\lambda)}[x_0'(\bar\lambda)]^2\Big\} \\ &\quad= \frac{\exp[-(x_0''-x_0')^2/2L]}{\sqrt{2\pi L}}, \end{aligned} \tag{75}$$

with (see, e.g., Refs. [15,20])

$$\mathcal{D}\Big(\frac{x_0}{\sqrt{e}}\Big) \equiv \sqrt{\frac{1}{2\pi\Delta\bar\lambda_0 e_{\bar\lambda_0}}}\prod_{\bar\lambda_i}\frac{dx_0(\bar\lambda_i)}{\sqrt{2\pi\Delta\bar\lambda_i e_{\bar\lambda_i}}}, \tag{76}$$

and the identity [19]

$$\begin{aligned} &\int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(\lambda)=\boldsymbol{x}''}\mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D}\exp\Big\{\int_0^\lambda d\bar\lambda\Big[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \frac{e(\bar\lambda)}{2}\boldsymbol{p}^2\Big]\Big\} \\ &\quad= \int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(\lambda)=\boldsymbol{x}''}\mathcal{D}\Big(\frac{\boldsymbol{x}}{e^{D/2}}\Big)\exp\Big\{-\int_0^\lambda d\bar\lambda\frac{1}{2e(\bar\lambda)}[\boldsymbol{x}'(\bar\lambda)]^2\Big\}. \end{aligned} \tag{77}$$

In the above definition the interval $0 \le \bar\lambda \le \lambda$ is split into not necessarily equal slices $\Delta\bar\lambda_i$ in order to preserve the integration measure under the reparametrization transformations (67). In particular, while $\Delta\bar\lambda_i$ are nonequal slices, $\Delta\bar\lambda_i e_{\bar\lambda_i}$ is constant for all i because $e^2(\bar\lambda)$ is the one-dimensional version of the metric "tensor" along the path.

VI. CONNECTION WITH EMERGENT RELATIVITY

The identity (58) can be interpreted in yet another interesting way. To this end, we rewrite Eq. (58) as

$$\begin{aligned}
&\int_{x(0)=x'}^{x(t)=x''} \mathcal{D}x \frac{\mathcal{D}p}{(2\pi)^D} \exp\left\{\int_0^t d\tau [\mathrm{i}p\cdot\dot{x} - c\sqrt{p^2+m^2c^2}]\right\}\\
&\quad = \int_0^\infty d\tilde{m}\sqrt{\frac{c^2 t}{2\pi\tilde{m}}}\,\mathrm{e}^{-tc^2(\tilde{m}-m)^2/2\tilde{m}} \int_{x(0)=x'}^{x(t)=x''} \mathcal{D}x \frac{\mathcal{D}p}{(2\pi)^D}\\
&\quad\times \exp\left\{\int_0^t d\tau\left[\mathrm{i}p\cdot\dot{x} - \frac{p^2}{2\tilde{m}} - mc^2\right]\right\}\\
&= \int_0^\infty d\tilde{m} f_{1/2}(\tilde{m}, tc^2, tc^2m^2) \int_{x(0)=x'}^{x(t)=x''} \mathcal{D}x \frac{\mathcal{D}p}{(2\pi)^D}\\
&\quad\times \exp\left\{\int_0^t d\tau\left[\mathrm{i}p\cdot\dot{x} - \frac{p^2}{2\tilde{m}} - mc^2\right]\right\},
\end{aligned}\tag{78}$$

where

$$f_p(z,a,b) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} z^{p-1}\mathrm{e}^{-(az+b/z)/2} \tag{79}$$

(with K_p = modified Bessel function of the second kind) is the *generalized inverse Gaussian* distribution [32] (known also as *Sichel*'s distribution). From the form of the Hamiltonian in (78) we see that the mass $\tilde{m}$ plays the role of the ordinary Newtonian mass which takes on continuous values distributed according to distribution $f_{1/2}(\tilde{m}, tc^2, tc^2m^2)$ with the expectation value $\langle\tilde{m}\rangle = m + 1/tc^2$. Relation (78) can then be given the following heuristic interpretation: Single-particle relativistic theory might be viewed as a single-particle *nonrelativistic* theory whose Newtonian mass $\tilde{m}$ (which is not invariant under Lorentz transformations) is a fluctuating parameter whose average approaches the true relativistic mass m in the large t limit. In view of the results of Ref. [9], we can more formally state that a stochastic process described by the Kramers-Moyal equation with the relativistic Hamiltonian $c\sqrt{p^2+m^2c^2}$ is equivalent to a doubly stochastic process in which the fast-time dynamics of a free nonrelativistic particle (Brownian motion) is coupled with the long-time dynamics describing fluctuations of the particle's Newtonian mass. On a more speculative vein, one can fit the above observation into the currently much debated ''emergent (special) relativity.'' The emergent relativity tries to view a special theory of relativity as a theory that statistically emerges from a deeper (essentially nonrelativistic) level of dynamics. It dates back to works of Bohm [33,34] in the early 1950s, but it received a real boost with the advancement of quantum-gravity approaches. In recent years it has appeared under various disguises in quantum-gravity models based on spacetime foam pictures [35], in loop quantum-gravity models [36,37], in noncommutative geometry models [38–41], or in black-hole physics [42].

At a strictly phenomenological level, one can understand fluctuations of the Newtonian mass as originating from the idea that the medium in which propagation occurs (''spacetime'') involves some sort of ''granularity'' (usually considered in quantum-gravity models). On the basis of experience with condensed-matter systems, one can expect that granularity of the medium might lead to corrections in the local dispersion relation and hence to modifications in local effective mass of a particle.

Suppose, now, that on the fast-time level a nonrelativistic particle propagates through grains with a different local $\tilde{m}$ in each grain (e.g., crystalline grains with a different local lattice structures or lattice spacings). Assume that the probability of the distribution of $\tilde{m}$ in various grains is $f_{1/2}(\tilde{m}, tc^2, tc^2m^2)$. Because the fast-time scale motion is Brownian, the local probability density matrix (PDM) conditioned on some fixed $\tilde{m}$ in a given grain is Gaussian:

$$\hat{\rho}(p, t|\tilde{m}) \propto \mathrm{e}^{-t\hat{p}^2/2\tilde{m}}. \tag{80}$$

The joint PDM is then $\hat{\rho}(p, t; \tilde{m}) = f_{1/2}(\tilde{m}, tc^2, tc^2m^2)\hat{\rho}(p, t|\tilde{m})$ and the marginal PDM describing the mass-averaged (i.e. long-time) behavior of the particle is

$$\hat{\rho}(p, t) = \int_0^\infty d\tilde{m} f_{1/2}(\tilde{m}, tc^2, tc^2m^2)\hat{\rho}(p, t|\tilde{m}). \tag{81}$$

Local matrix element in the position basis, i.e. $\langle x|\hat{\rho}(p, t-t')|x'\rangle$, then corresponds to transition probability $P(x, t|x', t')$ which has the Newton-Wigner path-integral representation (8).

It should be noted that these conclusions extend also to less trivial situations. One may, for instance, consider the Klein-Gordon or Dirac particle coupled to an external electromagnetic field $A_\mu(x, t)$ and to a scalar potential $V(x, t)$. In such a case Dirac's Hamiltonian is

$$H_{\mathrm{D}}^{A,V} = c\gamma_0\boldsymbol{\gamma}\cdot(p - eA/c) + \gamma_0(mc^2 + V) + eA_0. \tag{82}$$

and the Feshbach-Villars Hamiltonian reads

$$H_{\mathrm{FV}}^{A,V} = (\sigma_3 + \mathrm{i}\sigma_2)\frac{1}{2m}(p - eA/c)^2 + \sigma_3(mc^2 + V) + eA_0. \tag{83}$$

For the purpose of illustrating our point we will focus on an electron in a magnetostatic field $B = \mathrm{rot}A$ with $V = 0$. In this case, there exists a 4×4 Foldy-Wouthuysen-like transformation [43,44] that brings Dirac's Hamiltonian (82) to a quasidiagonal form:

$$H_{\mathrm{D}}^{A,V}(p, x) = V_{p,x} H_{\mathrm{diag}}^{A,V}(p, x) V_{p,x}^{-1}, \tag{84}$$

where

$$H^{A,V}_{\text{diag}}(\boldsymbol{p},\boldsymbol{x}) = \gamma_0\sqrt{c^2(\boldsymbol{p}-e\boldsymbol{A}/c)^2 + m^2c^4 - e\hbar c\boldsymbol{B}\cdot\boldsymbol{\Sigma}},$$
$$\times\ \Sigma_i = \frac{i}{4}\epsilon_{ijk}[\gamma_j,\gamma_k], \quad (85)$$

and

$$V_{\boldsymbol{p},\boldsymbol{x}} = \mathrm{e}^{-\mathrm{i}S_{p,x}},\ S_{\boldsymbol{p},\boldsymbol{x}} = -\frac{1}{2}\arctan\left(\frac{\mathrm{i}\boldsymbol{\gamma}\cdot(c\boldsymbol{p}-e\boldsymbol{A})}{mc^2}\right). \quad (86)$$

For $\boldsymbol{A}=0$ this reduces to the Foldy-Wouthuysen transformation (47) as one can easily check by comparing respective Taylor series. Our analysis will further simplify when the magnetic field is also spatially constant. In this case the vector potential can be taken as $A_x = -By$ and $A_y = A_z = 0$ (the z axis is chosen to be in the $\boldsymbol{B}$ direction, $B_z \equiv B$) and then

$$\begin{aligned} &H^{A,V}_{\text{diag}}(\boldsymbol{p},\boldsymbol{x}) \\ &= \gamma_0\sqrt{c^2(p_x+eBy/c)^2 + c^2p_y^2 + c^2p_z^2 + m^2c^4 - e\hbar cB\Sigma_z} \\ &= \sigma_3\otimes\sqrt{c^2\boldsymbol{p}^2 + e^2B^2y^2 - ecB(\hbar\sigma_3 - 2yp_x) + m^2c^4}. \end{aligned} \quad (87)$$

Let us observe that the latter is already in a diagonal form. Following the procedure from Secs. III and IV, the key object to be evaluated is the imaginary-time propagator,

$$\begin{aligned} &\langle\boldsymbol{x}|\mathrm{e}^{-(t-t')H^{A,V}_{\text{diag}}(\boldsymbol{p},\boldsymbol{x})}|\boldsymbol{x}'\rangle \\ &= \int_0^\infty dv\,\omega(v,t)\otimes\int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}}\mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \\ &\quad\times \mathrm{e}^{\int_{t'}^{t}d\tau\{\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - v[\boldsymbol{p}^2c^2 + e^2B^2y^2 - ecB(\hbar\sigma_3 - 2yp_x) + m^2c^4]\}} \\ &= \int_0^\infty d\tilde{m}f_{1/2}(\tilde{m},tc^2,tc^2m^2)\otimes\int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}}\mathcal{D}\boldsymbol{x}\frac{\mathcal{D}\boldsymbol{p}}{(2\pi)^D} \\ &\quad\times\exp\left\{\int_{t'}^{t}d\tau[\mathrm{i}\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - H_{\text{SP}} - mc^2]\right\}, \end{aligned} \quad (88)$$

where H_{SP} corresponds to the nonrelativistic Hamiltonian for a particle in a constant uniform magnetic field (Schrödinger-Pauli Hamiltonian)

$$H_{\text{SP}} = \frac{1}{2\tilde{m}}\left[\left(p_x + \frac{e}{c}By\right)^2 + p_y^2 + p_z^2\right] - \mu_{\text{B}}B\sigma_3, \quad (89)$$

with $\mu_{\text{B}} = e\hbar/2\tilde{m}c$ representing the Bohr magneton.

Note, in particular, that the smearing distributions ω and $f_{1/2}$ stay the same as in the free-particle case [cf. Eq. (38) and (78)]. Diagonalization analogous to (84) and (85) can be performed also for charged spin-0 particles, such as, e.g., $\pi^{\pm}$ mesons [43].

Two points hinder this program to be carried further in a full generality: first, general x dependence of A_μ and V leads to a notorious ordering problem. Second, and most importantly, transformation that would bring the Hamiltonian into a form where the positive and negative parts are explicitly separated is no longer possible for a general interaction. This last point makes it impossible to carry over straightforwardly our reasonings from Secs. III and IV.

VII. CONCLUDING REMARKS

In this paper we have extended an earlier paper [9] on superstatistics to a calculation of the worldline representations of Feynman propagators for spin-0 an spin-$\frac{1}{2}$ particles via a superstatistical average of nonrelativistic single-particle paths. For conceptual reasons we have found it useful to describe the spin-0 particle by the less-known Feshbach-Villars rather than the usual Klein-Gordon equation. The Feshbach-Villars representation casts the Klein-Gordon equation into two equations, both of which are first order in time. Because of this first-order nature we could use the Feynman-Kac formula to set up the path-integral representation for the corresponding Feynman's propagator. The two-component nature of the wave function, in addition, allowed one to treat the positive- and negative-energy solutions on equal footing and easily accommodated the Feynman-Stuckelberg boundary condition. This considerably facilitated our calculations. Although we have discussed only spin-0 and spin-$\frac{1}{2}$ particles, the method could have been also employed to discuss the Proca equation for spin-1 particle. This is because for Proca's Hamiltonian one can find an analogous diagonalization as in spin-0 and spin-$\frac{1}{2}$ cases [43].

From the superstatistics viewpoint, the relations (9) and (58), as well as their matrix generalizations (37) and (53), belong to the inverse χ^2-superstatistics universality class [6,11]. This is a rather interesting result, in particular when we realize that Weibull's smearing distribution was unambiguously forced upon us by requiring that the smeared Gaussian Markovian process (i.e., nonequilibrium Markovian processes) should be identical with the Newton-Wigner Markovian process [9], i.e., a Markovian process with the square-root Hamiltonian $H_{\boldsymbol{p}}$. In fact, in Ref. [9] we have shown that mere requirement that a smeared Markovian process should be again a Markovian process naturally resulted in both inverse χ^2- and χ^2-superstatistics universality classes. In this view it is plausible to conjecture that superstatistics equivalence classes are closely related to smearing distributions in nonequilibrium Markovian systems.

In passing we remark that our approach is instructive in yet another respect, namely, that our smearing distribution $\omega(v,a,t)$ is inevitably time dependent. Though superstatistics does not prohibit *per se* time-dependent smearing distributions, common practice is to assume (at least in first approximation) that the fluctuation parameter (inverse temperature, volatility, energy dissipation rate, etc.) as well as its moments do not have explicit time dependence. This is not the case here since $\langle v^\alpha\rangle \propto t^\alpha$ for $\alpha < 1/2$. In our considerations we have two well-separated time scales: a short time scale of order Δt which corresponds to the size of the time mesh, and a macroscopic time t ($t \gg \Delta t$) over which ω changes significantly—t is proportional to a statistical dispersion (scale parameter) of ω (see, e.g., [16]). An explicit use of the macroscopic time t in ω is mandatory and we should not ignore it if we want to obtain correct relativistic propagators. Note also that because $\langle v^\alpha \rangle$ diverges for $\alpha > 1/2$, one cannot apply any form of truncated cumulant expansion (often used in perturbative superstatistics) to obtain, e.g., a nonrelativistic limit. The path-integral identity (58) is fully nonperturbative in v.

In the end we wish to add few more comments concerning worldline path integrals. Worldline representations of field-theoretic propagators, as considered here, are an aspect of the so-called "worldline quantization" of particle physics. In this approach the process of second quantization is reversed. Second quantization, or field quantization, was introduced to represent a grand-canonical ensemble of quantum particles by a single quantum field. Since each quantum particle possesses a fluctuating worldline, quantized field theory is the most efficient way of studying grand-canonical ensembles of fluctuating lines. These can be, for instance, worldlines of elementary particles as emphasized by Feynman [27] and Schwinger [45], or lines of a completely different physical nature, such as polymers, vortices, or defect lines. In the latter case it is possible to study the phase transitions caused by the proliferation of such vortices or defect lines with the help of a single quantum field. The associated quantum field theory is known as *disorder field theory* [46]. In the first case, the phase transitions in polymer ensembles become tractable by the efficient methods of quantum field theory [15]. At a phase transition, an order or a disorder field can acquire a nonzero expectation value. This phenomenon is very hard to describe in a first-quantized worldline approach [47].

In many recent works, this development has been turned around. The motivation for this comes from the inability to develop a second-quantized field theory for strings, whose "worldlines" are fluctuating surfaces (world sheets). In string theory, calculations have so far remained restricted to the first-quantized formulation [48]. In order to gain more insight, people [20,49] have returned to well-understood quantum field-theoretic problems of point particles and reconsidered them in the first-quantized formulation in which fluctuating worldlines play the essential role. This, so-called "string-inspired" approach has led to a great number of publications initiated by Bern and Kosower [49,50]. They shed an alternative light on calculations within quantum electrodynamics (QED) [51] and quantum chromodynamics (QCD) [49,52], on calculations of anomalies [53], and of index densities in the Atiyah-Singer theorem [54]. Besides, worldline quantization forms an integral part of the so-called operator regularization scheme of McKeon *et al.* [55].

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APPENDIX A

Here we briefly discuss the Lorentz properties of the two-component wave function,

$$\Psi(\boldsymbol{x}, t) = \begin{pmatrix} \phi(\boldsymbol{x}, t) \\ \chi(\boldsymbol{x}, t) \end{pmatrix}. \tag{A1}$$

We first observe that the components ϕ and χ can be represented as

$$\phi = \frac{1}{\sqrt{2}}\left(\psi - \frac{1}{\mathrm{i}mc^2}\frac{\partial \psi}{\partial t}\right), \qquad \chi = \frac{1}{\sqrt{2}}\left(\psi + \frac{1}{\mathrm{i}mc^2}\frac{\partial \psi}{\partial t}\right), \tag{A2}$$

where ψ is a Klein-Gordon field fulfilling

$$(\Box + m^2c^2)\psi = 0. \tag{A3}$$

Combining (A2) with (A3), one can easily check that Ψ satisfies the Schrödinger equation (11) with the Hamiltonian $H_{\rm FV}$ given by (13). Using the fact that ψ is a Lorenzian scalar, one can deduce the transformation properties of Ψ under the Lorentz group as follows: Under finite Lorentz transformation Λ the field Ψ should transform as

$$\Psi(x) \xrightarrow{\Lambda} \Psi'(x) = S(\Lambda)\Psi(\Lambda^{-1}x), \tag{A4}$$

where $S(\Lambda)$ represents an operator of intrinsic field transformations. Equation (A4) implies an infinitesimal Lorentz transformation

$$\Psi(x) \xrightarrow{\Lambda} \Psi(x) + \delta_\Lambda \Psi(x). \tag{A5}$$

Here

$$\begin{aligned} &\delta_\Lambda \Psi(x) = \Psi'(x) - \Psi(x) \quad \text{and} \\ &\delta_\Lambda x^\mu = x'^\mu - x^\mu = \omega^\mu_\nu x^\nu = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}(S^{\mu\nu})x, \end{aligned} \tag{A6}$$

where the antisymmetric matrix $\omega_{\mu\nu} = -\omega_{\nu\mu}$ collects both rotation angles and rapidities, i.e., $\omega_{ij} = \epsilon_{ijk}\varphi^k$ and $\omega_{0i} = \zeta^i = p^i/mc$, respectively. $(S^{\mu\nu})_\alpha{}^\beta = \mathrm{i}(\eta_{\mu\alpha}\eta^\beta{}_\nu - \eta_{\nu\alpha}\eta^\beta{}_\mu)$ represent generators of the Lorentz group for vectors.

We may now employ the fact that $\delta_\Lambda\psi = -\frac{i}{2}\omega_{\mu\nu}\hat{L}^{\mu\nu}\psi$, where $\hat{L}_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)$ represent the generators of the Lorentz group for scalar fields, and write

$$\delta_\Lambda \Psi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{i}{2}\omega_{\mu\nu}\hat{L}^{\mu\nu}\psi - \frac{1}{\mathrm{i}mc}\delta_\Lambda \partial_0 \psi \\ -\frac{i}{2}\omega_{\mu\nu}\hat{L}^{\mu\nu}\psi + \frac{1}{\mathrm{i}mc}\delta_\Lambda \partial_0 \psi \end{pmatrix}. \tag{A7}$$

If we now utilize the property

$$\delta_\Lambda \partial_0 \psi = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}\hat{L}^{\mu\nu}\partial_0\psi - \frac{\mathrm{i}}{2}(\omega_{\mu\nu}S^{\mu\nu})_0{}^\alpha\partial_\alpha\psi, \quad \text{(A8)}$$

we can cast (A7) into a form

$$\delta_\Lambda\Psi(x) = -\frac{i}{2}\omega_{\mu\nu}\left[\hat{L}^{\mu\nu} + \frac{1}{2mc}(S^{\mu\nu})_0{}^\alpha\hat{p}_\alpha(\sigma_3 + \mathrm{i}\sigma_2)\right]\Psi(x), \quad \text{(A9)}$$

which identifies the generators of the Lorentz transformations on the two-component wave Feshbach-Villars wave function $\Psi(x)$ as

$$\hat{M}^{\mu\nu} = \hat{L}^{\mu\nu} + \frac{1}{2mc}(S^{\mu\nu})_0{}^\alpha\hat{p}_\alpha(\sigma_3 + \mathrm{i}\sigma_2). \quad \text{(A10)}$$

In particular, if Λ describes rotations, then $\omega_{\mu\nu}$ has only spatial indices and $(S^{\mu\nu})_0{}^\alpha \mapsto (S^{ij})_0{}^\alpha = 0$. This implies that $\hat{M}_{ij} = \hat{L}_{ij} = \mathrm{i}(x_i\partial_j - x_j\partial_i)$, which are standard generators of rotation for scalar fields. If Λ corresponds to boost transformations, then $(S^{\mu\nu})_0{}^\alpha \mapsto (S^{0j})_0{}^i \neq 0$, and the boost generators read

$$\hat{K}_i \equiv \hat{M}^{0i} = \hat{L}^{0i} + \frac{1}{2mc}S^{0i}\hat{\boldsymbol{p}}(\sigma_3 + \mathrm{i}\sigma_2). \quad \text{(A11)}$$

Here the product $S^{0i}\hat{\boldsymbol{p}}$ is defined by the contraction $(S^{0i})_0^k\hat{p}_k$.

Let us now show that the generators $\hat{M}^{\mu\nu}$ close the $SO(3, 1)$ algebra. As for generators $\hat{M}_{ij}$, these clearly constitute the rotational subalgebra $SO(3)$, i.e.,

$$[\hat{J}_i, \hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{J}_k \quad \text{with } \hat{J}_i = \tfrac{1}{2}\epsilon_{ijk}\hat{M}^{jk}. \quad \text{(A12)}$$

The generators $\hat{K}_i \equiv \hat{M}^{0i}$ yield commutators

$$\begin{aligned}
[\hat{K}_i, \hat{K}_j] &= \left[\hat{L}^{0i} + \frac{1}{2mc}S^{0i}\hat{\boldsymbol{p}}(\sigma_3 + \mathrm{i}\sigma_2), \hat{L}^{0j} + \frac{1}{2mc}S^{0j}\hat{\boldsymbol{p}}(\sigma_3 + \mathrm{i}\sigma_2)\right] \\
&= [\hat{L}^{0i}, \hat{L}^{0j}] + \frac{1}{2mc}(S^{0i})^0{}_k[\hat{p}^k, \hat{L}^{0j}](\sigma_3 + \mathrm{i}\sigma_2) + \frac{1}{2mc}(S^{0j})^0{}_k[\hat{L}^{0i}, \hat{p}^k](\sigma_3 + \mathrm{i}\sigma_2) \\
&= [\hat{L}^{0i}, \hat{L}^{0j}] + \frac{i}{2mc}(S^{0i})^0{}_k\hat{p}^0\delta^{kj}(\sigma_3 + \mathrm{i}\sigma_2) - \frac{i}{2mc}(S^{0j})^0{}_k\hat{p}^0\delta^{ik}(\sigma_3 + \mathrm{i}\sigma_2) \\
&= [\hat{L}^{0i}, \hat{L}^{0j}] = -\mathrm{i}\epsilon_{ijk}\hat{J}_k, \quad \text{(A13)}
\end{aligned}$$

which is a familiar boost commutator. In the derivation we have used the fact that $(\sigma_3 + \mathrm{i}\sigma_2)^2 = 0$ and that $\hat{L}^{0i}$ are boost generators for scalar fields. Finally, the mixed commutators read

$$\begin{aligned}
[\hat{J}_i, \hat{K}_j] &= \left[\hat{J}_i, \hat{L}^{0j} + \frac{1}{2mc}S^{0j}\hat{\boldsymbol{p}}(\sigma_3 + \mathrm{i}\sigma_2)\right] \\
&= [\hat{J}_i, \hat{L}^{0j}] + \frac{1}{2mc}(S^{0j})^{0k}[\hat{J}_i, \hat{p}_k](\sigma_3 + \mathrm{i}\sigma_2) \\
&= \mathrm{i}\epsilon_{ijk}\hat{L}^{0k} - \frac{\mathrm{i}}{2mc}(S^{0j})^{0k}\epsilon_{ikl}\hat{p}^l(\sigma_3 + i\sigma_2) \\
&= \mathrm{i}\epsilon_{ijk}\left(\hat{L}^{0k} + \frac{1}{2mc}S^{0k}\hat{\boldsymbol{p}}(\sigma_3 + \mathrm{i}\sigma_2)\right) = \mathrm{i}\epsilon_{ijk}\hat{K}_k. \quad \text{(A14)}
\end{aligned}$$

Here we have utilized the identity $(S^{0j})^{0k}\epsilon_{ikl}\hat{p}^l = \mathrm{i}\eta^{jk}\epsilon_{ikl}\hat{p}^l = -\mathrm{i}\epsilon_{ijl}\hat{p}^l = -(S^{0k})^0_l\epsilon_{ijk}\hat{p}^l$. As a result we see the commutators (A12)–(A14) close the Lorentz algebra $SO(3, 1)$.

APPENDIX B

In this Appendix we show that $U_{\boldsymbol{p}}$ may be viewed as a boost transformation that brings a wave function $\Psi(\boldsymbol{x}, t)$ of a spinless particle at rest to the velocity $\boldsymbol{v}$. To this end we seek the positive- and negative-energy plane wave solutions of the Schrödinger-like equation (11) in the form

$$\Psi^{(+)}(\boldsymbol{x}, t) = u(p)\mathrm{e}^{-\mathrm{i}px}, \qquad \Psi^{(-)}(\boldsymbol{x}, t) = v(p)\mathrm{e}^{\mathrm{i}px}, \quad \text{(B1)}$$

with

$$\begin{aligned}
(cp_0 - H_{\boldsymbol{p}}U_{\boldsymbol{p}}\sigma_3U_{\boldsymbol{p}}^{-1})u(p) &= 0, \\
(cp_0 + H_{\boldsymbol{p}}U_{\boldsymbol{p}}\sigma_3U_{\boldsymbol{p}}^{-1})v(p) &= 0,
\end{aligned} \quad \text{(B2)}$$

where $p_0 = \sqrt{\boldsymbol{p}^2 + m^2c^2}$. For the rest momentum $p_R \equiv (mc, \boldsymbol{0})$, these equations simplify and the respective amplitudes $u(p_R)$ and $v(p_R)$ satisfy

$$(\sigma_0 - \sigma_3)u(p_R) = 0, \qquad (\sigma_0 + \sigma_3)v(p_R) = 0. \quad \text{(B3)}$$

The solutions are

$$u(p_R) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad v(p_R) = \begin{pmatrix}0\\1\end{pmatrix}. \quad \text{(B4)}$$

They are normalized to ensure the unit normalization (26),

$$\begin{gathered}
(u, u) = u^\dagger\sigma_3u = 1, \qquad (v, v) = v^\dagger\sigma_3v = -1, \\
(u, v) = u^\dagger\sigma_3v = 0,
\end{gathered} \quad \text{(B5)}$$

making positive- and negative-energy states orthogonal to each other. Using the identity

$$\begin{aligned}
&(cp_0 \pm H_{\boldsymbol{p}}U_{\boldsymbol{p}}\sigma_3U_{\boldsymbol{p}}^{-1})(cp_0 \mp H_{\boldsymbol{p}}U_{\boldsymbol{p}}\sigma_3U_{\boldsymbol{p}}^{-1}) \\
&\quad = c^2p_0^2 - H_{\boldsymbol{p}}^2 = 0,
\end{aligned} \quad \text{(B6)}$$

we can write the amplitudes $u(p)$ and $v(p)$ at arbitrary momentum [cf. Eqs. (B2)] as

$$u(p) = N_p(cp_0 + H_{\boldsymbol{p}}U_{\boldsymbol{p}}\sigma_3 U_{\boldsymbol{p}}^{-1})\begin{pmatrix}1\\0\end{pmatrix} = N_p\begin{pmatrix}cp_0 + \boldsymbol{p}^2/2m + mc^2\\ -\boldsymbol{p}^2/2m\end{pmatrix}, \tag{B7}$$

and

$$v(p) = N_p(cp_0 - H_{\boldsymbol{p}}U_{\boldsymbol{p}}\sigma_3 U_{\boldsymbol{p}}^{-1})\begin{pmatrix}0\\1\end{pmatrix} = N_p\begin{pmatrix}-\boldsymbol{p}^2/2m\\ cp_0 + \boldsymbol{p}^2/2m + mc^2\end{pmatrix}, \tag{B8}$$

with some normalization constant N_p. The normalization conditions (B5) require

$$N_p = \sqrt{\frac{mc}{p_0}}\frac{1}{cp_0 + mc^2}, \tag{B9}$$

so that Eqs. (B7) and (B8) become

$$u(p) = \frac{1}{2\sqrt{mcp_0}}\begin{pmatrix}mc + p_0\\ mc - p_0\end{pmatrix}, \quad v(p) = \frac{1}{2\sqrt{mcp_0}}\begin{pmatrix}mc - p_0\\ mc + p_0\end{pmatrix}. \tag{B10}$$

Equations (B10) define boost transformations

$$u(p) = U_{\boldsymbol{p}}\begin{pmatrix}1\\0\end{pmatrix}, \qquad v(p) = U_{\boldsymbol{p}}\begin{pmatrix}0\\1\end{pmatrix}. \tag{B11}$$

Here we have denoted the boost matrix as $U_{\boldsymbol{p}}$ because it appears here in the form

$$U_{\boldsymbol{p}} = \frac{p_0 + mc}{2\sqrt{mcp_0}}\sigma_0 - \frac{p_0 - mc}{2\sqrt{mcp_0}}\sigma_1, \tag{B12}$$

which is identical to the diagonalization matrix $U_{\boldsymbol{p}}$ as defined by Eq. (22). If we introduce a parameter

$$\alpha_{\boldsymbol{v}} \equiv \ln\sqrt{\frac{p_0}{mc}} = \frac{1}{2}\ln\gamma_{\boldsymbol{v}}, \tag{B13}$$

which satisfies

$$\cosh\alpha_{\boldsymbol{v}} = \frac{p_0 + mc}{2\sqrt{mcp_0}} \quad \text{and} \quad \sinh\alpha_{\boldsymbol{v}} = \frac{p_0 - mc}{2\sqrt{mcp_0}}, \tag{B14}$$

then we can write the boost matrix (B12) as an exponential [cf. also Eq. (22)]

$$U_{\boldsymbol{p}} = \exp(-\alpha_{\boldsymbol{v}}\sigma_1). \tag{B15}$$

Connection of $U_{\boldsymbol{p}}$ with boost generators (A11) can be established when we rewrite (A9) for boost transformation in the form

$$\Psi'(x') = \left(1 - \frac{\mathrm{i}}{2mc}\zeta^i(S^{0i})\hat{\boldsymbol{p}}(\sigma_3 + \mathrm{i}\sigma_2)\right)\Psi(x). \tag{B16}$$

For positive-energy plane wave solutions this can be written as

$$u(p') = \left(1 - \frac{\mathrm{i}}{2mc}\zeta^i(S^{0i})\boldsymbol{p}(\sigma_3 + \mathrm{i}\sigma_2)\right)u(p) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 + p^0/mc + \zeta^i p^i/mc\\ 1 - p^0/mc - \zeta^i p^i/mc\end{pmatrix}\tilde{\psi}^+(p). \tag{B17}$$

Here, $\tilde{\psi}^+(p)$ stands for an amplitude of the positive-energy plane wave solution of the Klein-Gordon equation. Term $p^0 + \zeta^i p^i$ can be recognized as a first-order term in the Lorentz boost transformation,

$$\Lambda(\boldsymbol{\zeta})^0{}_\mu p^\mu = p^0\cosh\zeta + (\hat{\boldsymbol{\zeta}}\cdot\boldsymbol{p})\sinh\zeta = p'^0. \tag{B18}$$

Here $\hat{\boldsymbol{\zeta}} = \boldsymbol{u}/|\boldsymbol{u}|$ denotes the unit vector in the direction of the boost velocity $\boldsymbol{u}$ ($\boldsymbol{u} \oplus \boldsymbol{v} = \boldsymbol{v}'$). If we further employ the identities,

$$\cosh\zeta = \gamma_{\boldsymbol{u}} \quad \text{and} \quad \hat{\boldsymbol{\zeta}}\sinh\zeta = \gamma_{\boldsymbol{u}}\frac{\boldsymbol{u}}{c}, \tag{B19}$$

we may cast (B17) into form

$$u(p') = \frac{1}{\sqrt{2}}\begin{pmatrix}1 + \gamma_{\boldsymbol{u}}\gamma_{\boldsymbol{v}}(1 + \boldsymbol{u}\cdot\boldsymbol{v}/c^2)\\ 1 - \gamma_{\boldsymbol{u}}\gamma_{\boldsymbol{v}}(1 + \boldsymbol{u}\cdot\boldsymbol{v}/c^2)\end{pmatrix}\tilde{\psi}^+(p) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 + \gamma_{\boldsymbol{u}\oplus\boldsymbol{v}}\\ 1 - \gamma_{\boldsymbol{u}\oplus\boldsymbol{v}}\end{pmatrix}\tilde{\psi}^+(p), \tag{B20}$$

which clearly shows that the original amplitude was boosted from the velocity $\boldsymbol{v}$ to the amplitude with the velocity $\boldsymbol{v}' = \boldsymbol{u}\oplus\boldsymbol{v}$.

In the particular case when the initial momentum is p_r, the relation (B20) acquires the form

$$u(p) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 + \gamma_{\boldsymbol{u}}\\ 1 - \gamma_{\boldsymbol{u}}\end{pmatrix}\tilde{\psi}^+(p_r). \tag{B21}$$

By utilizing the normalization condition (B5) we have that $\tilde{\psi}^+(p_r) = 1/\sqrt{2\gamma_{\boldsymbol{u}}}$, which finally gives

$$u(p) = \frac{1}{2}\begin{pmatrix}1/\sqrt{\gamma_{\boldsymbol{u}}} + \sqrt{\gamma_{\boldsymbol{u}}}\\ 1/\sqrt{\gamma_{\boldsymbol{u}}} - \sqrt{\gamma_{\boldsymbol{u}}}\end{pmatrix} = U_{\boldsymbol{u}}u(p_r). \tag{B22}$$

Analogous analysis applies for negative-energy plane waves in which case we obtain

$$v(p) = \frac{1}{2}\begin{pmatrix}1/\sqrt{\gamma_{\boldsymbol{u}}} - \sqrt{\gamma_{\boldsymbol{u}}}\\ 1/\sqrt{\gamma_{\boldsymbol{u}}} + \sqrt{\gamma_{\boldsymbol{u}}}\end{pmatrix} = U_{\boldsymbol{u}}v(p_r). \tag{B23}$$

Results (B22) and (B23) establish the promised identification between $U_{\boldsymbol{u}}$ and boost transformations from the rest frame velocity $\boldsymbol{v} = 0$ to the velocity $\boldsymbol{u}$.

Let us finally comment on a nonrelativistic limit of the Feshbach-Villars wave function. To this end we approximate $H_{\boldsymbol{p}} = c\sqrt{\boldsymbol{p}^2 + m^2c^2} \approx mc^2 + \boldsymbol{p}^2/2m$. With this the positive- and negative-energy solution (B1) becomes

$$\begin{aligned}\Psi^{(+)}(\boldsymbol{x},t) &\overset{c\to\infty}{\approx} \begin{pmatrix} 1 \\ -\boldsymbol{v}^2/4c^2 \end{pmatrix} \exp[\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{x} - H_{\boldsymbol{p}}t)] \\ &\equiv \Phi^{(+)}(\boldsymbol{x},t)e^{-\mathrm{i}mc^2t}, \\ \Psi^{(-)}(\boldsymbol{x},t) &\overset{c\to\infty}{\approx} \begin{pmatrix} -\boldsymbol{v}^2/4c^2 \\ 1 \end{pmatrix} \exp[\mathrm{i}(H_{\boldsymbol{p}}t - \boldsymbol{p}\cdot\boldsymbol{x})] \\ &\equiv \Phi^{(-)}(\boldsymbol{x},t)e^{\mathrm{i}mc^2t}. \end{aligned} \tag{B24}$$

In particular, we see that, for plane particle waves, the upper components are much larger than the lower components. The opposite holds for antiparticle waves. This is analogous to the situation for Dirac wave function. For particle waves, Eqs. (14) and (15) reduce to

$$\mathrm{i}\,\partial_t\Phi^{(+)} = -\frac{\nabla^2}{2m}\sigma_3\Phi^{(+)}, \qquad \mathrm{i}\partial_t\Phi^{(-)} = -\frac{\nabla^2}{2m}\sigma_3\Phi^{(-)}. \tag{B25}$$

By neglecting the small component in $\Phi^{(+)}$, this implies the Schrödinger equation for the large component in $\Phi^{(+)}$ with $\hat{H} = -\nabla^2/2m$. An analogous situation holds for $\Phi^{(-)}$.

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