

Enumerating Abelian Returns to Prefixes of Sturmian Words

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Return words

$\mathbf{u} = u_0 u_1 u_2 \cdots$ infinite word over an alphabet \mathcal{A} ($= \{0, 1\}$)

$w = u_i u_{i+1} \cdots u_{i+n-1}$ factor of \mathbf{u} of length $|w| = n$

$\mathcal{L}(\mathbf{u})$ language of \mathbf{u}

i occurrence of w in \mathbf{u}

$|w|_a$ number of occurrences of the letter a in w

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\mathbf{u} is recurrent if every factor w has infinitely many occurrences.

If $n_1 < \cdots < n_i < n_{i+1} < \cdots$ are occurrences of w in \mathbf{u} ,

then $v = u_{n_i}u_{n_i+1}\cdots u_{n_{i+1}-1}$ is a return word of w in u .

Abelian return words

w is abelian equivalent to w'

$w \sim_{\text{ab}} w'$, if $|w|_a = |w'|_a$ for every $a \in \mathcal{A}$.

Abelian return words

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$$w \sim_{\text{ab}} w', \text{ if } |w|_a = |w'|_a \text{ for every } a \in \mathcal{A}.$$

If $n_1 < \dots < n_i < n_{i+1} < \dots$ are occurrences of factors $\sim_{\text{ab}} w$, i.e.

$$u_{n_i} u_{n_i+1} \cdots u_{n_i+n-1} \sim_{\text{ab}} w,$$

$$u_j u_{j+1} \cdots u_{j+n-1} \not\sim_{\text{ab}} w \text{ for } j \neq n_i,$$

then $v = u_{n_i} u_{n_i+1} \cdots u_{n_{i+1}-1}$ is an **abelian return word** to w in \mathbf{u} .

Sturmian words – many equivalent definitions

- ▶ Aperiodic words with exactly $n + 1$ factors of each length n .
- ▶ Hedlund & Morse:

Balanced aperiodic infinite words:

$$||w|_a - |w'|_a| \leq 1$$

for every $w, w' \in \mathcal{L}(\mathbf{u})$, $|w| = |w'|$, and every $a \in \mathcal{A}$.

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- ▶ Vuillon:
Infinite words with exactly 2 return words for every $w \in \mathcal{L}(\mathbf{u})$.
- ▶ ...

Sturmian words and interval exchange

Irrational $\alpha \in (0, 1)$, $T : [0, 1) \rightarrow [0, 1)$

$$T(x) = \begin{cases} x + 1 - \alpha & \text{if } x \in [0, \alpha) =: J_0, \\ x - \alpha & \text{if } x \in [\alpha, 1) =: J_1. \end{cases}$$

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For any $\rho \in [0, 1)$ define $\mathbf{u}_{\alpha, \rho} = u_0 u_1 u_2 \cdots$ by

$$u_n = \begin{cases} 0 & \text{if } T^n(\rho) \in J_0, \\ 1 & \text{if } T^n(\rho) \in J_1. \end{cases}$$

Sturmian word with slope α and intercept ρ .

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Sturmian word with **slope** α and **intercept** ρ .

Language $\mathcal{L}(\mathbf{u}_{\alpha, \rho}) = \mathcal{L}(\alpha)$ depends only on α .

Abelian return words – Puzynina and Zamboni

Denote $\mathcal{AR}_{w,\mathbf{u}}$ the set of abelian return words of w in \mathbf{u}

Theorem (Puzynina & Zamboni) :

An aperiodic recurrent infinite word \mathbf{u} is sturmian if and only if

$$\#\mathcal{AR}_{w,\mathbf{u}} \in \{2, 3\} \text{ for any } w \in \mathcal{L}(\mathbf{u}).$$

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If $\#\mathcal{AR}_{w,\mathbf{u}} = 3$, then

$$\mathcal{AR}_{w,\mathbf{u}} = \{R_1, R_2, R_3\}, \text{ with } |R_1| + |R_2| = |R_3|.$$

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Experiment (Břinda) :

$R_3 = R_1R_2$ (for the Fibonacci word \mathbf{f} : $0 \mapsto 01, 1 \mapsto 0$)

\mathbf{f} : 01001010010010100101001001001001001...

Abelian return words – Rigo et al.

Denote $APR_{\mathbf{u}} = \bigcup \{AR_{w,\mathbf{u}} : w \text{ is a prefix of } \mathbf{u}\}$.

Theorem (Rigo, Salimov & Vandomme) :

$$APR_{\mathbf{u}_{\alpha,\rho}} < +\infty \iff \rho \neq 0.$$

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Theorem (Rigo, Salimov & Vandomme) :

$$APR_{\mathbf{u}_{\alpha,\rho}} < +\infty \iff \rho \neq 0.$$

Fibonacci word \mathbf{f} : $APR_{\mathbf{f}} = \{0, 1, 01, 10, 001\}$.

Our results

- ▶ If $\#\mathcal{AR}_{w,\mathbf{u}} = 3$, then $\mathcal{AR}_{w,\mathbf{u}} = \{R_1, R_2, R_1R_2\}$.
- ▶ $\#\mathcal{APR}_{\mathbf{u}}$ dependingly on ρ and α .
- ▶ Algorithm for listing elements of $\mathcal{APR}_{\mathbf{u}}$.
- ▶ $\mathcal{APR}_{\mathbf{c}_\alpha}$ where $\mathbf{c}_\alpha = \mathbf{u}_{\alpha,1-\alpha}$

Factors of sturmian words

$\alpha, T^{-1}(\alpha), \dots, T^{-n+1}(\alpha)$ defines a partition of $[0, 1)$,

$$[0, 1) = \bigcup \{J_w : w \in \mathcal{L}_n(\alpha)\}.$$

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For $w \in \mathcal{L}_n(\alpha)$, we have $|w|_1 \in \{\lceil n\alpha \rceil, \lceil n\alpha \rceil - 1\}$.

w is **heavy** if $|w|_1 = \lceil n\alpha \rceil$,

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Lemma: w is light $\iff J_w \subset [0, T^{-n+1}(\alpha))$.

First return map to $I \subset [0, 1)$

Return time to I by $r : I \rightarrow \{1, 2, 3, \dots\}$

$$r(x) = \min\{n \in \mathbb{N}, n \geq 1 : T^n(x) \in I\}.$$

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First return map $T_I : I \rightarrow I$,

$$T_I(x) = T^{r(x)}(x).$$

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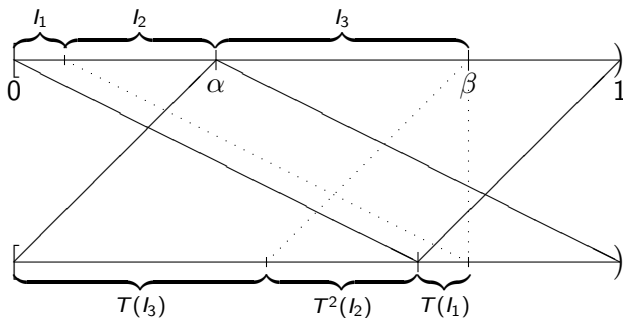
$$T_I(x) = T^{r(x)}(x).$$

I -itinerary of x under T

$$R(x) = \text{the prefix of length } r(x) \text{ of } \mathbf{u}_{\alpha, x}.$$

Example

Let $I = [0, \beta)$, where $\max\{\alpha, 1 - \alpha\} \leq \beta \leq 1$.



$x \in l_1$	$r(x) = 1$	$T_I(x) = x + 1 - \alpha$	$R(x) = 0$
$x \in l_2$	$r(x) = 2$	$T_I(x) = x + 1 - 2\alpha$	$R(x) = 01$
$x \in l_3$	$r(x) = 1$	$T_I(x) = x - \alpha$	$R(x) = 1$

Abelian return words as I -itineraries

Lemma:

Let $w \in \mathcal{L}_n(\alpha)$. Put

$$I = \begin{cases} [0, T^{-n+1}(\alpha)) & \text{if } w \text{ is light,} \\ [T^{-n+1}(\alpha), 1) & \text{otherwise.} \end{cases}$$

Then

v is an abelian return to w



v is an I -itinerary of some $x \in I$ under T .

Abelian returns to prefixes

Theorem:

Let $\alpha, \rho \in [0, 1)$, α irrational. Then satisfies

$$\mathcal{APR}_{\mathbf{u}_{\alpha, \rho}} = \mathcal{R}_{\rho}^{\alpha} \cup \mathcal{R}'_{\rho}{}^{\alpha},$$

where

$$\mathcal{R}_{\rho}^{\alpha} = \bigcup_{\rho \leq \beta < 1} \{R : R = R(x) \text{ is the } [0, \beta)\text{-itinerary for an } x \in [0, \beta)\},$$

$$\mathcal{R}'_{\rho}{}^{\alpha} = \bigcup_{0 < \gamma \leq \rho} \{R : R = R(x) \text{ is the } [\gamma, 1)\text{-itinerary for an } x \in [\gamma, 1)\}.$$

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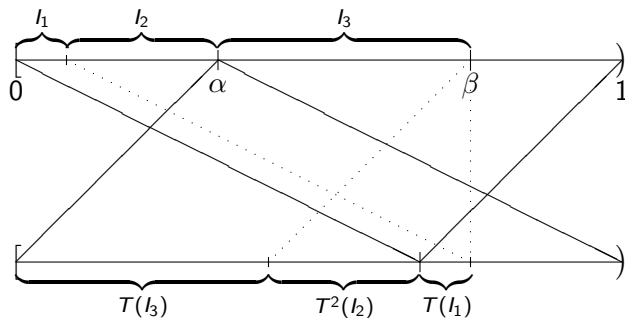
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Note: $\mathcal{R}_{\rho}^{\prime\alpha} = E(\mathcal{R}_{1-\rho}^{1-\alpha})$ where $E : 0 \leftrightarrow 1$.

Return time vs. itineraries – example 1

Let $I = [0, \beta)$, where $\max\{\alpha, 1 - \alpha\} \leq \beta \leq 1$.



$x \in l_1$	$r(x) = 1$	$T_I(x) = x + 1 - \alpha$	$R(x) = 0$
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$x \in l_3$	$r(x) = 1$	$T_I(x) = x - \alpha$	$R(x) = 1$

Return time vs. itineraries – example 2

Let $\alpha = \frac{1}{\tau}$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$,

$$T(x) = \begin{cases} x + 1 - \frac{1}{\tau} & \text{if } x \in [0, \frac{1}{\tau}), \\ x - \frac{1}{\tau} & \text{if } x \in [\frac{1}{\tau}, 1). \end{cases}$$

Let $I = [\frac{1}{\tau^3}, \frac{1}{\tau} + \frac{1}{\tau^4})$.

$x \in I_1$	$r(x) = 1$	$T_I(x) = x + \frac{1}{\tau^2}$	$R(x) = 0$
$x \in I_2$	$r(x) = 3$	$T_I(x) = x + \frac{1}{\tau^4}$	$R(x) = 010$
$x \in I_3$	$r(x) = 2$	$T_I(x) = x - \frac{1}{\tau^3}$	$R(x) = 01$
$x \in I_4$	$r(x) = 2$	$T_I(x) = x - \frac{1}{\tau^3}$	$R(x) = 10$

where

$$I_1 = [\frac{1}{\tau^3}, \frac{1}{\tau^2}), I_2 = [\frac{1}{\tau^2}, \frac{1}{\tau^2} + \frac{1}{\tau^5}), I_3 = [\frac{1}{\tau^2} + \frac{1}{\tau^5}, \frac{1}{\tau}), I_4 = [\frac{1}{\tau}, \frac{1}{\tau} + \frac{1}{\tau^4}).$$

First return map for Sturmian words

Interval exchange:

Partition $J_0 \cup J_1 \cup \dots \cup J_{k-1} = [0, 1)$,
 $t_0, t_1, \dots, t_{k-1} \in \mathbb{R}$ such that

$S(x) = x + t_j$ for $x \in J_j$, is a bijection $S : [0, 1) \rightarrow [0, 1)$.

Theorem:

First return map T_I to $I \subset [0, 1)$ induced by a 2iet
 $T : [0, 1) \rightarrow [0, 1)$ is always an exchange of two or three intervals.

First return map for Sturmian words

If $\alpha = [0, a_1, a_2, \dots]$, denote for $k \geq 0$, $1 \leq s \leq a_{k+1}$,

$$\delta_{k,s} := |(s-1)(p_k - \alpha q_k) + p_{k-1} - \alpha q_{k-1}|,$$

where $\frac{p_k}{q_k}$ are convergents of α .

Theorem:

Let $I = [c, c + \delta)$, where $0 \leq c < c + \delta \leq 1$. Then T_I is

- ▶ a 2iet, if $\delta = \delta_{k,s}$ for some $k \geq 0$, $1 \leq s \leq a_{k+1}$;
- ▶ a 3iet with permutation (321), otherwise.

Three $[0, \beta)$ -itineraries

Theorem

Let $I = [0, \beta) \subset [0, 1)$. There exist words R and R' , $R \prec_{\text{lex}} R'$, such that the I -itinerary $R(x)$ of every $x \in I$ under T satisfies

$$R(x) \in \{R, R', RR'\}.$$

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Corollary:

For every factor w of a sturmian word \mathbf{u} there exist factors w_1, w_2 such that $\mathcal{AR}_{w, \mathbf{u}} \in \{w_1, w_2, w_1 w_2\}$.

Main result

Recall

$$\mathcal{R}_\rho^\alpha = \bigcup_{\rho \leq \beta < 1} \{R : R = R(x) \text{ is the } [0, \beta)\text{-itinerary for an } x \in [0, \beta)\}$$

Corollary:

Let $k, s \in \mathbb{N}$ be minimal such that $\rho \geq \delta_{k,s}$. Then

$$\#\mathcal{R}_\rho^\alpha = 1 + a_1 + a_2 + \cdots + a_k + s.$$

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Corollary:

Let $k, s \in \mathbb{N}$ be minimal such that $\rho \geq \delta_{k,s}$. Then

$$\#\mathcal{R}_\rho^\alpha = 1 + a_1 + a_2 + \cdots + a_k + s.$$

Note: $\#\mathcal{R}_0^\alpha = +\infty$ and hence also $\#\mathcal{APR}_{\mathbf{u}_\alpha, 0} = +\infty$.

Main result

Recall $APR_{\mathbf{u}_{\alpha,\rho}} = \mathcal{R}_{\rho}^{\alpha} \cup E(\mathcal{R}_{1-\rho}^{1-\alpha})$.

Theorem:

Let $\alpha, \rho \in (0, 1)$, $\alpha = [0, a_1, a_2, \dots]$ irrational, $a_1 \geq 2$, $\mathbf{u} = \mathbf{u}_{\alpha,\rho}$.

(i) Let $\rho \in (\alpha, 1 - \alpha)$. Then

$$\#APR_{\mathbf{u}} \in \{a_1 + 3, a_1 + 4\}.$$

(ii) Let $\rho \notin (\alpha, 1 - \alpha)$. Then

$$\#APR_{\mathbf{u}} = 2 + a_1 + \dots + a_k + s,$$

where $k \geq 0$ and $1 \leq s \leq a_{k+1}$ are minimal such that $\min\{\rho, 1 - \rho\} \geq \delta_{k,s}$.

APR_u for characteristic Sturmian words

Characteristic Sturmian words $\mathbf{c}_\alpha = \mathbf{u}_{\alpha, 1-\alpha}$.

Theorem:

Let $\alpha = [0, a_1, a_2, \dots]$ be an irrational in $(0, 1)$. Then

$$APR_{\mathbf{c}_\alpha} = \begin{cases} \{1, 10, 0, 01, 001, \dots, 0^{a_2+1}1\} & \text{if } \alpha > \frac{1}{2}, \\ \{0, 01, 1, 10, 110, \dots, 1^{a_1}0\} & \text{otherwise.} \end{cases}$$

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Applying $\alpha = \frac{1}{\tau} = [0, 1, 1, \dots]$, we obtain

$$APR_{\mathbf{f}} = \{0, 1, 01, 10, 001\}.$$

Děkuji za pozornost.