Affine factor complexity of infinite words associated with simple Parry numbers

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Definition of u_β

Substitution φ over the alphabet $\mathcal{A} = \{0, 1, \ldots, m-1\}$, given by

$$\begin{aligned} \varphi(0) &= 0^{t_1}1 \\ \varphi(1) &= 0^{t_2}2 \\ &\vdots \\ \varphi(m-2) &= 0^{t_{m-1}}(m-1) \\ \varphi(m-1) &= 0^{t_m} \end{aligned} \tag{1}$$

substitution is primitive, has a unique fixed point $u_\beta := \lim_{n\to\infty} \varphi^n(0)$.

$$u_\beta = \underbrace{0^{t_1}1 \ \cdots \ 0^{t_1}1}_{t_1 \ \text{times}} \ 0^{t_2}2 \ \underbrace{0^{t_1}1 \ \cdots \ 0^{t_1}1}_{t_1 \ \text{times}} \ 0^{t_2}2 \ \cdots$$

Rényi expansion of unity in base β

$$d_\beta(1) = t_1 t_2 t_3 \cdots\,, \qquad t_i = \left\lfloor \beta T_\beta^{i-1}(1) \right\rfloor$$

t_i obtained using $\quad T_\beta : [0,1] \to [0,1), \quad T_\beta(x) := \beta x - \lfloor \beta x \rfloor = \{\beta x\}.$

Parry number: $d_\beta(1)$ eventually periodic; simple Parry number: $d_\beta(1) = t_1 \cdots t_m$.

β-expansion, irrational $\beta > 1$

For $x > 0$: $\qquad x = \sum_{i=-\infty}^{k} x_i \beta^i\,, \quad x_i \in \mathbb{N}, \qquad \langle x \rangle_\beta = x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots$

Digits x_i obtained by the greedy algorithm $\quad \Longrightarrow \quad x_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$

$\mathbb{Z}_\beta \ldots$ β-integers $\qquad \mathbb{Z}_\beta := \{\pm x \mid \langle x \rangle_\beta = x_k x_{k-1} \ldots x_0 \bullet,\ k \in \mathbb{N}\}\,,$

Thurston: Distances between consecutive β-integers form the set

$$\left\{ \sum_{i=1}^{\infty} \frac{t_{i+k}}{\beta^i} \;\middle|\; k \in \mathbb{N} \right\}, \quad \text{which is finite} \quad \Longleftrightarrow \quad \beta \text{ is Parry}$$

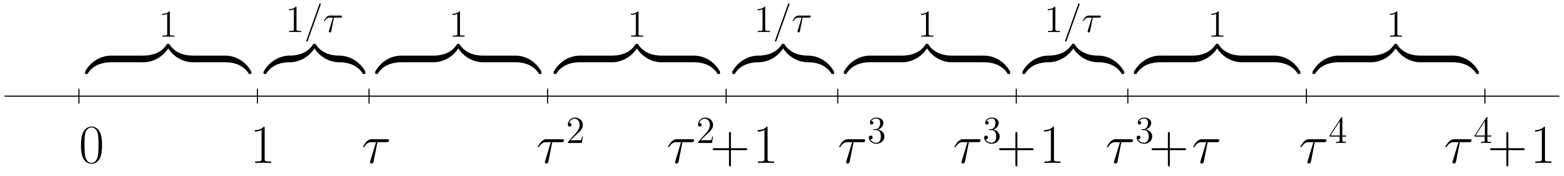
Example:

golden ratio $\tau = \frac{1}{2}(1+\sqrt{5}) = 1.618\cdots$ root of $x^2 - x - 1$.

Since $\lfloor \tau \rfloor = 1$, digits of τ-expansions are $0, 1$.

$$\mathbb{Z}_\tau = \left\{ \pm \sum_{i=0}^{k} x_i \tau^i \;\middle|\; x_i \in \{0,1\},\ x_i x_{i+1} = 0 \right\}$$

$$\mathbb{Z}_\tau = \pm\{0, 1, \tau, \tau^2, \tau^2+1, \tau^3, \tau^3+1, \tau^3+\tau, \tau^4, \tau^4+1, \dots\}$$



x	$\langle x \rangle_\tau$
0	0
1	1
τ	10
τ^2	100
τ^2+1	101
τ^3	1000
τ^3+1	1001
$\tau^3+\tau$	1010
τ^4	10000
τ^4+1	10001
$\vdots$	$\vdots$

Words, factors, complexity

$\mathcal{A} = \{0, 1, \ldots, m-1\}$	alphabet
$u = (u_i)_{i\in\mathbb{N}}$, $u_i \in \mathcal{A}$	infinite word over $\mathcal{A}$
$w = u_i u_{i+1} \cdots u_{i+n-1}$	factor of u of length n
$\mathcal{L}_n(u)$	the set of factors of u of length n
$\mathcal{L}(u) = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(u)$	the language of u

Factor complexity of u is the function $\mathcal{C} : \mathbb{N} \to \mathbb{N}$, given by $\quad \mathcal{C}(n) := \#\mathcal{L}_n(u)$.

For $w \in \mathcal{L}(u)$

$$\mathrm{Rext}(w) = \{a \in \mathcal{A} \mid wa \in \mathcal{L}(u)\}\,.$$

$$\Delta\mathcal{C}(n) = \mathcal{C}(n+1) - \mathcal{C}(n) = \sum_{w\in\mathcal{L}_n(u)} \big(\#\mathrm{Rext}(w) - 1\big)\,.$$

w is a right special factor if $\#\mathrm{Rext}(w) \geq 2$.

Similarly w is LSF if $\#\mathrm{Lext}(w) \geq 2$.

Known results on complexity of u_β

- $\mathcal{C}(n)$ is a sublinear function (φ is primitive).

- Tijdeman: lower bound $\mathcal{C}(n) \geq (d-1)n+1$, where d is the dimension of the vector space generated by the densities of letters over $\mathbb{Q}$.

  Often $d = m$, but: minimal Pisot number, root of $x^3 = x + 1$

  $$d_\beta(1) = 10001, \qquad \text{i.e.} \quad d = 3,\ m = 5.$$

- Frougny, M., Pelantová: exact values of $\mathcal{C}(n)$ under the conditions

  $$t_1 = t_2 = \cdots = t_{m-1} \quad \text{or} \quad t_1 > \max\{t_2, \ldots, t_{m-1}\}\,, \qquad (\star)$$

  in particular, $\mathcal{C}(n)$ satisfies

  $$(m-1)n + 1 \leq \mathcal{C}(n) \leq mn, \qquad \text{for all} \ \ n \geq 1\,.$$

- If $(\star)$ and moreover $t_m = 1$, then $\mathcal{C}(n)$ is affine.

Main result

Theorem. Let $d_\beta(1) = t_1 \cdots t_m$. Then u_β has affine factor complexity $\iff$

1) $t_m = 1$

and

2) no non-empty word can be both a proper prefix and a proper suffix of $t_1 \cdots t_{m-1}$,

or $t_1 \cdots t_{m-1} = v^k$ for $k \in \mathbb{N}$, $k \geq 2$.

N.B.: Condition $(\star)$ is a special case of 2).

Description of LSF:

- Every prefix of u_β is a LSF.
- Every LSF of length $\leq t_1$ is a prefix of u_β.
- u_β has affine complexity $\iff$ every LSF is a prefix of u_β.

Comments:

Affine factor complexity : $\mathcal{C}(n) = (m-1)n + 1$ for all $n \in \mathbb{N}$.

$d_\beta(1) = v^k 1$, $|v| = r$ $\Longrightarrow$ for every $n \in \mathbb{N}$:
one LSF of length n with #Lext $= m$,
r RSF of length n with #Rext $= k+1$;

$\Longrightarrow$ for every $w \in \mathcal{L}(u_\beta)$ exactly m return words.

$r = 1$: corresponds to $t_1 = t_2 = \cdots t_{m-1}$, i.e. $d_\beta(1) = t^{m-1}1$

one LSF and one RSF of each length (Arnoux-Rauzy words)

m palindromes of every odd length, one palindrome of every even length

$r > 1$: $\mathcal{L}(u_\beta)$ is not closed under reversal, i.e. finitely many palindromes in u_β.

Open problems

- Factor complexity of u_β for non-simple Parry β.

- Explicit formula for the factor complexity in the non-affine case.

- Is it possible that the factor complexity is ultimately affine, $\mathcal{C}(n) = an + b$ for $n \geq n_0$?

- Other combinatorial properties of u_β (e.g. balance function - known only for quadratic simple Parry β).