Multidimensional continued fractions

Jacobi-Perron algorithm for simultaneous rational approximation of d real numbers

Zuzana Masáková

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Outline

- Ordinary continued fractions
  - finiteness
  - approximation by convergents
  - uniqueness
  - periodicity and relation to units in quadratic fields

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- Other comments

Definition of continued fractions

Algorithm:

$$\alpha_0 = \alpha\,, \quad \alpha_n = a_n + \frac{1}{\alpha_{n+1}}\,, \text{ where } a_n = [\alpha_n]\,, \text{ for } n \in \mathbb{N}\,.$$

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Stops when $\alpha_n \in \mathbb{Z}$.

The sequence $(a_n)_{n=0}^{N}$, resp. $(a_n)_{n\in\mathbb{N}_0}$ denoted by

$$[a_0, a_1, \ldots, a_N]\,, \quad \text{resp.} \quad [a_0, a_1, \ldots, a_n, \ldots]$$

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is called the continued fraction of α.

Claim. $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$.

Finiteness of the algorithm

After n steps:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{\alpha_n}}}}$$

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Recall:

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Recall:

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}\,, \qquad a_n = [\alpha_n]\,.$$

Obviously $\alpha_n \in \mathbb{Z}$ for some $n \in \mathbb{N}_0$ implies $\alpha \in \mathbb{Q}$.

Conversely, if $\alpha = \frac{p}{q} \in \mathbb{Q}$, then Euclidean division leads

$$
\begin{array}{rclcrcl}
p & = & a_1 q + r_1 & \qquad & \frac{p}{q} & = & a_1 + \frac{r_1}{q} \\
q & = & a_2 r_1 + r_2 & & \frac{q}{r_1} & = & a_2 + \frac{r_2}{r_1} \\
 & \vdots & & & & \vdots & \\
r_{k-1} & = & a_{k+1} r_k + r_{k+1} & & \frac{r_{k-1}}{r_k} & = & a_{k+1} + \frac{r_{k+1}}{r_k}
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Claim. Algorithm stops if and only if $\alpha \in \mathbb{Q}$.

Approximation of irrational numbers

$\alpha \in \mathbb{R} \setminus \mathbb{Q} \quad \Longrightarrow \quad$ infinite continued fraction

($\alpha_n > 0$ for all $n \in \mathbb{N}$).

We approximate α_n in

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{\alpha_n}}}}$$

by $a_n = [\alpha_n]$.

What happens?

Convergents

Define the *n-th convergent* of α

$$\frac{p_n}{q_n} = \alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

Properties of convergents

Easy to show:

$$\begin{array}{llll} p_0 = a_0 & p_1 = 1 + a_0 a_1 & p_n = a_n p_{n-1} + p_{n-2} & n \geq 2\,, \\ q_0 = 1 & q_1 = a_1 & q_n = a_n q_{n-1} + q_{n-2} & n \geq 2\,. \end{array}$$

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$$\mathbb{A}_n := \begin{pmatrix} q_{n-1} & q_n \\ p_{n-1} & p_n \end{pmatrix} = \begin{pmatrix} q_{n-2} & q_{n-1} \\ p_{n-2} & p_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix},$$

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i.e. $\mathbb{A}_n = \mathbb{A}_{n-1} \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$ for $n \geq 0$, with $\mathbb{A}_{-1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

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i.e.

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n+1}}{q_n q_{n-1}}\,.$$

Claim. For $n \geq 0$, we have

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Proof by induction. For $n = 0$ we have $\mathbb{A}_{-1} \binom{1}{\alpha_0} = \binom{1}{\alpha}$. Let $n > 0$.

$$\mathbb{A}_n \begin{pmatrix} 1 \\ \alpha_{n+1} \end{pmatrix} = \mathbb{A}_{n-1} \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_{n+1} \end{pmatrix} = \mathbb{A}_{n-1} \begin{pmatrix} \alpha_{n+1} \\ a_n \alpha_{n+1} + 1 \end{pmatrix} =$$

$$= \alpha_{n+1} \mathbb{A}_{n-1} \begin{pmatrix} 1 \\ \alpha_n \end{pmatrix} = \alpha_1 \cdots \alpha_n \alpha_{n+1} \begin{pmatrix} 1 \\ \alpha \end{pmatrix}.$$

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Consequences: For $n \geq 0$, we have

$$\begin{aligned} \alpha_1 \alpha_2 \cdots \alpha_n &= q_{n-2} + \alpha_n q_{n-1} , \\ \alpha &= \frac{p_{n-2} + \alpha_n p_{n-1}}{q_{n-2} + \alpha_n q_{n-1}} . \end{aligned}$$

Do convergents converge?

From the previous it follows that

$$\frac{p_n}{q_n} - \alpha = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n(\alpha_{n+1} q_n + q_{n-1})} = \frac{(-1)^{n+1}}{q_n(\alpha_{n+1} q_n + q_{n-1})}\,.$$

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Claim.

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \alpha, \qquad \text{even} \qquad \lim_{n\to\infty} (p_n - q_n\alpha) = 0$$

Uniqueness

For a given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there is a unique sequence $(a_n)_{n \in \mathbb{N}_0}$,

$$a_0 \in \mathbb{Z}\,, \qquad a_i \in \mathbb{N}\,, \quad \text{for } i \geq 1\,, \tag{1}$$

such that $\alpha = [a_0, a_1, a_2, \ldots]$.

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Not so for $\alpha \in \mathbb{Q}$: For example $\frac{9}{4} = 2 + \frac{1}{4} = 2 + \cfrac{1}{3 + \cfrac{1}{1}}$

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Conversely, every sequence $(a_n)_{n \in \mathbb{N}_0}$, satisfying (1) is a continued fraction of some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, namely $\alpha = \lim_{n \to \infty} \frac{p_n}{q_n}$.

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Claim. Let $(\frac{p_n}{q_n})$, $(\frac{\tilde{p}_n}{\tilde{q}_n})$ correspond to $(a_n)_{n \in \mathbb{N}_0}$, $(\tilde{a}_n)_{n \in \mathbb{N}_0}$ satisfying (1). Then

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} \frac{\tilde{p}_n}{\tilde{q}_n} \qquad \Longrightarrow \qquad a_n = \tilde{a}_n \quad \text{for all } \ n \in \mathbb{N}_0\,.$$

Periodic continued fractions

Let $\boldsymbol{\alpha_l = \alpha}$ for some $l \geq 1$. Equivalently, $(a_n)_{n \in \mathbb{N}_0}$ is purely periodic,

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From

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we have

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we have

$$\mathbb{A}_{l-1} \binom{1}{\alpha} = \alpha_1 \cdots \alpha_l \binom{1}{\alpha}\,.$$

Therefore $\mathbb{A}_{l-1} = \begin{pmatrix} q_{l-2} & q_{l-1} \\ p_{l-2} & p_{l-1} \end{pmatrix} \in \mathbb{N}^{2\times 2}$ has the eigenvector $\binom{1}{\alpha}$ to the (dominant) eigenvalue $\boldsymbol{\varrho = \alpha_1 \cdots \alpha_l}$.

ϱ is a Pisot unit

The characteristic polynomial

$$f(x) = \det(\mathbb{A}_{l-1} - x\mathbb{I}) = x^2 - x(p_{l-1} + q_{l-2}) + (-1)^l$$

is monic, in $\mathbb{Z}[x]$ and has absolute coefficient equal to

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$$\Longrightarrow \qquad \varrho \text{ is a quadratic Pisot unit}, \quad \alpha \in \mathbb{Q}(\varrho).$$

Note:

$$\varrho = [\overline{a_0, a_1, \ldots, a_{l-1}}] \cdot [\overline{a_1, \ldots, a_{l-1}, a_0}] \cdots [\overline{a_{l-1}, a_0, \ldots, a_{l-2}}] .$$

Theorem (Lagrange). The continued fraction of α is eventually periodic if and only if α is a quadratic number.

Proof: for $\Rightarrow$ see above; $\Leftarrow$ is little technical using Dirichlet's princip

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- Continued fraction of α is finite $\iff$ $\alpha \in \mathbb{Q}$.

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  $\lim_{n\to\infty}(p_n - \alpha q_n) = 0$.
- Uniqueness.
- Continued fraction of α is periodic $\iff$ α is quadratic.
- In that case the eigenvalue ϱ given by the period is a quadratic Pisot unit.

Jacobi-Perron Algorithm for multidimensional continued fractions

Finding $\gcd(x^{(0)}, x^{(1)}, \ldots, x^{(d)})$

$$\begin{cases} x_0^{(1)} &= a_0^{(1)} x_0^{(0)} + x_1^{(0)} \\ x_0^{(2)} &= a_0^{(2)} x_0^{(0)} + x_1^{(1)} \\ & \vdots \\ x_0^{(d)} &= a_0^{(d)} x_0^{(0)} + x_1^{(d-1)} \\ x_0^{(0)} &= x_1^{(d)} \end{cases}$$

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Input: $(\alpha^{(1)}, \ldots, \alpha^{(d)}) \in \mathbb{R}^d$, $d \geq 1$.

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If $\alpha_n^{(1)} \in \mathbb{Z}$ for some n, one cannot define $\alpha_{n+1}^{(d)}$

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A d-dimensional JPA can have $0, 1, \ldots, d$ interruptions.

Theorem (Perron). If JPA of $(\alpha^{(1)}, \ldots, \alpha^{(d)})$ admits m interruptions, then there are at least m independent relations between $1, \alpha^{(1)}, \ldots, \alpha^{(d)}$ with integer coefficients.

Consequences:

- If JPA of $(\alpha^{(1)}, \ldots, \alpha^{(d)})$ admits *d* interruptions, then $\alpha^{(1)}, \ldots, \alpha^{(d)} \in \mathbb{Q}$.

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Theorem (Dubois, Farhane, Paysant-LeRoux).

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- The JPA of $(\alpha^{(1)}, \alpha^{(2)})$ admits no interruption if and only if $1, \alpha^{(1)}, \alpha^{(2)}$ are linearly independent over $\mathbb{Q}$.
- For every $d \geq 3$ there exist $\alpha^{(1)}, \ldots, \alpha^{(d)}$ such that their JPA does not admit interruptions, but $1, \alpha^{(1)}, \ldots, \alpha^{(d)}$ are linearly dependent over $\mathbb{Q}$.

Lexicographic condition on coefficients of JPA

From the algorithm obviously

$$a_0^{(i)} \in \mathbb{Z}\,, \quad \text{and} \quad a_n^{(i)} \in \mathbb{N} \quad \text{for all } n \in \mathbb{N}\,, \quad 1 \le i \le d\,.$$

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Claim. For all $n \in \mathbb{N}$ and $i = 1, \dots, d-1$ we have

$$\left(a_n^{(d)}, a_{n+1}^{(d-1)}, \dots, a_{n+i}^{(d-i)}\right) \succeq \left(a_n^{(i)}, a_{n+1}^{(i-1)}, \dots, a_{n+i-1}^{(1)}, 1\right).$$

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Simultaneous rational approximation

Define $(q_n, p_n^{(1)}, \ldots, p_n^{(d)})_{n=-d-1}^{\infty}$ which gives rational approximations

$$\left(\frac{p_n^{(1)}}{q_n}, \ldots, \frac{p_n^{(d)}}{q_n}\right)_{n\in\mathbb{N}_0} \quad \text{to} \quad (\alpha^{(1)}, \ldots, \alpha^{(d)})$$

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For $n \geq -1$ denote

$$\mathbb{A}_n := \begin{pmatrix} q_{n-d} & q_{n-d+1} & \cdots & q_n \\ p_{n-d}^{(1)} & p_{n-d+1}^{(1)} & \cdots & p_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-d}^{(d)} & p_{n-d+1}^{(d)} & \cdots & p_n^{(d)} \end{pmatrix}$$

Properties of convergents

Set $\mathbb{A}_{-1} = \mathbb{I}_{d+1}$ and

$$\mathbb{A}_n = \mathbb{A}_{n-1} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_n^{(1)} \\ 0 & 1 & \dots & 0 & a_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_n^{(d)} \end{pmatrix} \quad \text{for} \quad n \geq 0.$$

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Proof by induction.

Properties of convergents

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$$\begin{pmatrix} 1 \\ \alpha^{(1)} \\ \vdots \\ \alpha^{(d)} \end{pmatrix} = \frac{1}{\alpha_1^{(d)} \cdots \alpha_n^{(d)}} \mathbb{A}_{n-1} \begin{pmatrix} 1 \\ \alpha_n^{(1)} \\ \vdots \\ \alpha_n^{(d)} \end{pmatrix} \qquad (\star)$$

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$$\alpha_1^{(d)} \cdots \alpha_n^{(d)} = q_{n-d-1} + q_{n-d}\alpha_n^{(1)} + \cdots + q_{n-1}\alpha_n^{(d)},$$

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for $1 \leq i \leq d$.

Proof:

$$\mathbb{A}_n \begin{pmatrix} 1 \\ \alpha_{n+1}^{(1)} \\ \alpha_{n+1}^{(2)} \\ \vdots \\ \alpha_{n+1}^{(d)} \end{pmatrix} = \mathbb{A}_{n-1} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_n^{(1)} \\ 0 & 1 & \dots & 0 & a_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_n^{(d)} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_{n+1}^{(1)} \\ \alpha_{n+1}^{(2)} \\ \vdots \\ \alpha_{n+1}^{(d)} \end{pmatrix} =$$

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$$= \mathbb{A}_{n-1} \begin{pmatrix} \alpha_{n+1}^{(d)} \\ 1 + a_n^{(1)} \alpha_{n+1}^{(d)} \\ \alpha_{n+1}^{(1)} + a_n^{(2)} \alpha_{n+1}^{(d)} \\ \vdots \\ \alpha_{n+1}^{(d-1)} + a_n^{(d)} \alpha_{n+1}^{(d)} \end{pmatrix} = \alpha_{n+1}^{(d)} \mathbb{A}_{n-1} \begin{pmatrix} 1 \\ \alpha_n^{(1)} \\ \alpha_n^{(2)} \\ \vdots \\ \alpha_n^{(d)} \end{pmatrix} =$$

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Convergence of convergents

Claim. For every JPA we have

$$\lim_{n\to\infty}\left|\frac{p_n^{(i)}}{q_n}-\alpha^{(i)}\right|=0\,.$$

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For any 2-dimensional JPA $\left(\left|p_n^{(1)} - \alpha^{(1)} q_n\right|, \left|p_n^{(2)} - \alpha^{(2)} q_n\right|\right)$ are bounded.

Will say more for periodic JPA.

Uniqueness

Theorem. The sequence $\left(a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(d)}\right)_{n \in \mathbb{N}_0}$ of non-negative integer n-tuples is a JPA development if and only if it satisfies the lexicographic condition.

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Theorem. Let $\left(a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(d)}\right)_{n\in\mathbb{N}_0}$, $\left(\tilde{a}_n^{(1)}, \tilde{a}_n^{(2)}, \ldots, \tilde{a}_n^{(d)}\right)_{n\in\mathbb{N}_0}$ satisfy the lexicographic condition, and let $\left(\frac{p_n^{(1)}}{q_n}, \ldots, \frac{p_n^{(d)}}{q_n}\right)$, $\left(\frac{\tilde{p}_n^{(1)}}{\tilde{q}_n}, \ldots, \frac{\tilde{p}_n^{(d)}}{\tilde{q}_n}\right)$ be the corresponding convergents. If

$$\lim_{n\to\infty} \frac{p_n^{(i)}}{q_n} = \lim_{n\to\infty} \frac{\tilde{p}_n^{(i)}}{\tilde{q}_n}, \qquad \text{for all} \quad i = 1, \ldots, d,$$

then

$$a_n^{(i)} = \tilde{a}_n^{(i)}, \qquad \text{for all } i = 1, \ldots, d, \quad n \in \mathbb{N}_0 .$$

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WLOG: $(\alpha_l^{(1)}, \ldots, \alpha_l^{(d)}) = (\alpha_0^{(1)}, \ldots, \alpha_0^{(d)})$ for some $l \in \mathbb{N}$.

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From $(\star)$ we have

$$\mathbb{A}_{l-1} \begin{pmatrix} 1 \\ \alpha^{(1)} \\ \vdots \\ \alpha^{(d)} \end{pmatrix} = \alpha_1^{(d)} \cdots \alpha_l^{(d)} \begin{pmatrix} 1 \\ \alpha^{(1)} \\ \vdots \\ \alpha^{(d)} \end{pmatrix}$$

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i.e. $(\alpha^{(1)}, \dots, \alpha^{(d)})^T$ is an eigenvector of $\mathbb{A}_{l-1}$ corresponding to the dominant eigenvalue

$$\boldsymbol{\varrho = \alpha_1^{(d)} \cdots \alpha_l^{(d)}} = q_{l-d-1} + q_{l-d}\alpha^{(1)} + \cdots + q_{l-1}\alpha^{(d)} \,.$$

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We have $\varrho \in \mathbb{Q}(\alpha^{(1)}, \ldots, \alpha^{(d)})$, but also $\alpha^{(i)} \in \mathbb{Q}(\varrho)$, whence

$$\mathbb{Q}(\boldsymbol{\alpha^{(1)}, \ldots, \alpha^{(d)}}) = \mathbb{Q}(\boldsymbol{\varrho}) .$$

The characteristic polynomial $f(x) = \det(\mathbb{A}_{l-1} - xI)$ is not necessarily irreducible.

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Note:

- ϱ is an algebraic integer (unit) of degree at most $d + 1$,
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Theorem.
The characteristic polynomial of JPA of $(\alpha^{(1)}, \dots, \alpha^{(d)})$ is irreducible if and only if $(1, \alpha^{(1)}, \dots, \alpha^{(d)})$ are linearly independent over $\mathbb{Q}$.

Case $d = 2$.

Claim. If JPA of $(\alpha^{(1)}, \alpha^{(2)})$ is purely periodic with period l, then $\varrho = \alpha_1^{(2)} \cdots \alpha_l^{(2)}$ is a cubic Pisot unit.

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Proof. The characteristic polynomial

$$f(x) = \begin{vmatrix} q_{l-3} - x & q_{l-2} & q_{l-1} \\ p_{l-3}^{(1)} & p_{l-2}^{(1)} - x & p_{l-1}^{(1)} \\ p_{l-3}^{(2)} & p_{l-2}^{(2)} & p_{l-1}^{(2)} - x \end{vmatrix} =$$

$$= -x^3 \; + \; x^2(q_{l-3} + p_{l-2}^{(1)} + p_{l-1}^{(2)}) \; + \; x(\cdots) \; + \; 1$$

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$$= -x^3 + x^2(q_{l-3} + p_{l-2}^{(1)} + p_{l-1}^{(2)}) + x(\cdots) + 1$$

has roots $\varrho, \varrho', \varrho''$, satisfying

$$\varrho = \alpha_1^{(2)} \cdots \alpha_l^{(2)} > 1 \quad \text{and} \quad \varrho\varrho'\varrho'' = 1 \qquad \Longrightarrow \qquad |\varrho'\varrho''| < 1 .$$

Since $$\varrho = q_{l-3} + q_{l-2}\alpha^{(1)} + q_{l-1}\alpha^{(2)}\,,$$

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The same done for period *nl* leads to

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Therefore

$$|\varrho'| < 1 \quad \text{and} \quad |\varrho''| < 1\,.$$

QED.

Quality of convergence for periodic JPA

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Theorem (Perron). Let $\varrho_0 =, \varrho_1, \ldots, \varrho_d$ be roots of the characteristic polynomial,

$$\varrho_0 > |\varrho_1| = \max\{|\varrho_i| \mid i = 1, \ldots, d\} .$$

Then $\exists\, C, \quad \forall\, \varepsilon > 0, \quad \forall\, n \in \mathbb{N}_0, \quad \forall\, k = 0, 2, \ldots, l-1$

$$|p_{nl+k}^{(i)} - q_{nl+k}\alpha^{(i)}| < C|\varrho_1|^n(1+\varepsilon)^n \qquad (i = 1, \ldots, d) .$$

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$$|p_{nl+k}^{(i)} - q_{nl+k}\alpha^{(i)}| < C|\varrho_1|^n(1+\varepsilon)^n \qquad (i = 1, \ldots, d).$$

On the other hand, for at least one pair (i, k) there exists C' such that

$$|p_{nl+k}^{(i)} - q_{nl+k}\alpha^{(i)}| > C'|\varrho_1|^n$$

is verified for infinitely many n.

Strong convergence and Pisot numbers

Corollary. For a periodic JPA, $\lim_{\to\infty}(p_n^{(i)} - q_n\alpha^{(i)}) = 0$ for all i if and only if the characteristic polynomial is irreducible with a Pisot number as its root.

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Note: For $d \geq 3$ there exist non strongly convergent periodic JPA.

Note:

It is not known whether or not a *d*-dimensional JPA of algebraic irrationals $\alpha^{(1)}, \ldots, \alpha^{(d)} \in \mathbb{R}$, $d \geq 2$, in a number field K of degree $d + 1$ always becomes periodic.

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It is not known whether or not a d-dimensional JPA of algebraic irrationals $\alpha^{(1)}, \ldots, \alpha^{(d)} \in \mathbb{R}$, $d \geq 2$, in a number field K of degree $d + 1$ always becomes periodic.

But in any number field of degree $d + 1$ there exists a d-tuple with periodic JPA.

$\longrightarrow$ can be used for calculation of units in numbers fields

Purely periodic JPA with period length 1

Let $(a^{(1)}, \ldots, a^{(d)}) \in \mathbb{N}^d$ such that for $i = 0, \ldots, d-1$,

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HA!

Comparison of JPA for $d = 1$ and $d \geq 2$

Finiteness

$d = 1$ Algorithm is finite $\iff \alpha \in \mathbb{Q}$.

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no interruption	$\Leftarrow$	$1, \alpha^{(1)}, \dots, \alpha^{(d)}$ LN over $\mathbb{Q}$
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Conditions on coefficients

$d = 1$ $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$ for $i \geq 1$

$d \geq 2$ lexicographic

Comparison of JPA for $d = 1$ and $d \geq 2$

Uniqueness

$d = 1$	yes	$d \geq 2$	yes

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Matrix of convergents

$d = 1$ $\det \mathbb{A}_n = (-1)^{n+1}$ $d \geq 2$ $\det \mathbb{A}_n = (-1)^{(n+1)d}$

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Convergence

$d = 1$ always strong
$d \geq 2$ always weak, sometimes strong?

Comparison of JPA for $d = 1$ and $d \geq 2$

Periodic expansion

$d = 1$

Characteristic polynomial has an algebraic unit ϱ as a root.

$\mathbb{Q}(\varrho) = \mathbb{Q}(\alpha)$ of degree $d + 1 = 2$.

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$d \geq 2$

Characteristic polynomial has an algebraic unit ϱ as a root.

$\mathbb{Q}(\varrho) = \mathbb{Q}(\alpha^{(1)}, \ldots, \alpha^{(d)})$ of degree $\leq d + 1$.

Of degree $d + 1 \iff 1, \alpha^{(1)}, \ldots, \alpha^{(d)}$ LN over $\mathbb{Q}$.

Then strong convergent $\iff \varrho$ is Pisot.