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SITUATION MODEL, TEXT BASE AND WHAT ELSE? FACTORS AFFECTING PROBLEM SOLVING

ABSTRACT. Our aim in this paper is to identify factors affecting problem solving strategies in the case of a multiplicative comparative situation, involving three unknown quantities whose sum is known, and in which comparison relations between two pairs of the quantities are given. We propose a model of the complexity of a word problem about such situation. The model has several variables, namely, 1) the ratio of the number of quantities that are being compared to the number of reference quantities (the 'reference ratio'); 2) the scheme of the situation, i.e. the type of relation between the two given elementary comparison relations; 3) the order of presenting the elementary comparison relations; 4) the words used to describe the multiplicative comparison relations (so many times 'more' Vs so many times 'less'). The choice of the values of these variables in the formulation of a problem is assumed to affect the solver's solution strategy. The most important construct of the model is what we have called the 'complexity level' of a solution, namely, the number of operations (theoretically) needed to go from the text of the problem to an algebraic representation of this solution. An analysis, in terms of the model, of word problems that can be formulated about this kind of situation makes it clear that, while there are several options in solving a given problem in algebraic terms, some of them are simpler (have a lower level of complexity), from the formal standpoint, than others. In fact, the most efficient choice for the independent variable is one, which, in the scheme of the situation, stands in the position of the 'connector' between the two elementary comparison relations. The model is used in analyzing solution strategies of a set of twelve problems about this type of multiplicative comparative situation by 104 teachers and 132 15 years old students. In particular, we identify their choices of the independent variable and discuss their solutions in terms of levels of complexity. We also look at their preferences for interpreting the texts of the problems in terms of the word 'more' rather than 'less'. One of our observations is that, even if they were not aware of it, the subjects in the study preferred solutions with a minimal route. There were, however, some exceptions in cases where subjects preferred to use the least value as the independent variable, as well as to translate the 'less' relations into the 'more' relations regardless of the high complexity of this solution.

KEY WORDS: comparison relation, complexity, multiplicative comparison, more – less, reference, scheme, text-base, two-step word problems

1. INTRODUCTION

Solving word problems in mathematics often means deducing new information (mostly quantitative) from the given data. The known information can be collected by the solver him or herself, or it can be provided by

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various means: verbally, graphically, or in any other form. We would like to begin this study one step earlier. To describe a real world situation the solver has to select those aspects of the situation that he or she wants to focus on, and decide which objects are to be explicitly mentioned. In the case of word problems, this initiates the formulation of the text of the problem.

For example, imagine a world consisting of 198 marbles distributed in the following manner: David has 22 marbles, Jirka – 44, and Peter – 132. What questions can be asked about this situation?

This situation can generate, among others, the following word problem:

A1. Peter, David and Jirka are playing marbles. They have 198 marbles altogether. Peter has 6 times more marbles than David, and Jirka has 2 times more marbles than David. How many marbles each one has?

For the sake of brevity, we shall use the letters P, D, and J to mean "the number of Peter's (resp. David's, Jirka's) marbles".

In this problem, the relation between P and D, the relation between J and D, as well as the total sum, are given. However, P and J are not compared at all. Moreover, the verbal description has already decided which value is the reference value and which values are to be compared with it (in this case D is the reference value). A choice has been made, too, regarding the nature of the multiplicative comparison. In both given relations, the lexical choice of the word 'more' rather than 'less' was made. Of course, one could choose to say that D is so many times less[1] than J or P and then obtain another formulation of the same situation. For example:

B1. Peter, David and Jirka are playing marbles. They have 198 marbles altogether. David has 6 times less marbles than Peter, and he has 2 times less marbles than Jirka. How many marbles each one has?

Our aim in this paper is to identify factors affecting problem solving strategies in the case of this type of multiplicative comparative situations, involving three unknown values whose sum is known, and in which comparison relations between two pairs of the values are given. We analyze the type of situations using a model of the cognitive processes through which the solver moves from the given description – the text of the problem – to the choice of its algebraic representation, focusing on the factors that affect this choice. Moving from the given text to an equation is a process, in the course of which several direct and indirect transformations can be involved. The course of this process depends both on the surface structure of the text and the underlying mathematical schemes. We devised a measure of complexity for this course, which can be used to measure the actual transition made by a solver from the given text to the produced formal language of algebra.

In the next section we discuss the theoretical background of our study. We continue by presenting the model, and particularly the construct of 'complexity level', which we explain using a set of twelve problems about the multiplicative comparative situation described above. We then present the results of an empirical study, in which the set of problems was administered to 104 teachers and 132 15 years old students. The solutions of the students and teachers are analyzed in terms of the proposed model. We conclude with an attempt to describe the complexity level of the subject's solutions by two basic cognitive factors: the underlying mathematical schemes , and the linguistic surface structure.

2. Theoretical background

The meaning of the term 'problem solving' is rather vague, a kind of umbrella under which different theoretical approaches co-exist. It is therefore necessary to specify explicitly at the very outset, which approach was employed in our work.

Mathematicians agree that problem solving occurs in cases where there is no clear algorithm to apply. Acknowledging that solving a genuine problem is not just a matter of following a given algorithm, first Polya (1945) and then Schoenfeld (1992) suggested general strategies for solving word problems, asking questions such as: What is the unknown? What are the data? What are the conditions? Do you know a related problem that was solved previously? Prepare a plan for the solution and, examine the solution obtained. Alongside the general strategies, Polya presented some mental operations typically useful for solving problems, such as: using analogies, decomposing the problem and recombining its elements in a different manner, generalization, induction, specialization and working backwards. In essence, these strategies have captured what is now known as the expert's knowledge, but whether it is teachable or not, still remains an open question. Our failure in teaching word problems is evidence that passing on the expert's knowledge to a novice is no simple matter.

Cognitive psychologists who tried to understand human cognitive performance via problem solving took a quite different approach. Most dramatic progress in this paradigm was achieved by approaching problem solving from two different angles.

The first was the linguistic approach, in which various constructs were proposed to account for understanding problem solving. Notable are the works of Kintsch (1986), who introduced notions such as 'text base' and 'situation model', and Nesher's notions of 'deep and surface structures in arithmetic' and 'semantic analysis' of arithmetic word problems (Nesher and Teubal, 1975; Nesher and Katriel, 1977; Kintsch and van Dijk, 1978; Nesher, Greeno and Riley, 1982; Nesher, 1998). Yet these works raised more questions than they answered.

The notion of 'text base' suggested by Kintsch deals with the propositional form of a problem, not with its surface structure phrasing. This approach leaves out the first interpretation of the given text. It may well be that the crux of the matter in solving a word problem lies in this first stage of interpretation, which is omitted in Kintsch's theoretical model. This author's concept of 'situation model' is also not satisfactory because it is too vague. What is meant by 'situation model'? Is it an untreated world before a verbal description, or, is it an imaginary world constructed by the reader through comprehending a given text? Our point here is that by the very fact of describing a situation verbally we already focus on some objects and relations (as mentioned in the introduction).

A second cognitive approach, not always separate from the first, emphasized, as its point of departure, the schematic analysis of arithmetic problems (Greeno, 1978; Fischbein, 1999; Hall, Kibler, Wenger and Truxaw, 1989). Those who adopted the schematic approach were influenced by notions such as 'frames', 'structures', 'analogies' emerging from the information processing research (Rumelhart, 1980; Thompson, 1985), or schemes within the constructivist approach (Reusser, 1992; Vergnaud, 1988). Their theoretical enterprise helped categorize word problems in arithmetic and algebra.

Several categorizations are now well established and employed in the educational research. For example, categorizations of additive problems into 'combine', 'change' and 'compare' (Carpenter, Moser and Romberg, 1982; De Corte and Verschaffel, 1981; De Corte, 1985; Nesher et al., 1982); categorization of multiplicative problems into 'ratio' or 'rate', 'mapping rule', 'Cartesian multiplication' or 'multiplicative comparison' (Greer, 1994; Vergnaud, 1983; Nesher, 1988); and the categorization into static and dynamic rate problems in algebra (Hall et al., 1989; Yerushalmy and Gilead, 1997). The gist of these categorizations lies in offering a parsimonious schematic approach that constricts the wealth of situations appearing as new on each occasion.

The above-mentioned categorizations represent only partly the degree of difficulty encountered by the solvers (De Corte, 1987; Kintsch et al., 1975; Kintsch, 1986; Nesher et al., 1982; Reusser, 1992). Some studies established the levels of difficulty of these problem categories. However, only a few gave a full account of the entire process, starting from reading a text given in natural language, and ending with the mathematical model that solves it. Kintsch, as already mentioned, started from a 'text base' and not from the surface structure (Kintsch, 1986). Nesher and Katriel offered a semantic analysis of additive problems and dealt with the relation between the surface formulation and the underlying propositional and logical structure of a text under mathematical constraints (Nesher and Katriel, 1977). Their studies did not combine with the schemes suggested by Greeno and Kintsch. Hershkovitz (Nesher and Hershkovitz, 1994), in her study of more complex problems, suggested a schematic structure for two-step problems, yet did not connect it to linguistic analysis.

The accumulated quantity of variables across the various problem-solving research paradigms does not yet form a satisfactory theory. It remains unclear what factors affecting the solution path were actually adopted by a given solver. The study described in the present paper is an effort to analyze one problem from multiple aspects: its linguistic surface structure; its underlying scheme; and the mathematical model selected by solvers in their attempts to solve it. We hope that the empirical evidence will shed light on additional cognitive aspects involved in problem solving.

3. The Situation Addressed in the Study and Possible Related Word Problems

The situation to which we refer has been described in the introduction and we recall it here.

Peter, David and Jirka have 198 marbles in total; of these, David has 22, Jirka has 44 and Peter has 132.

This kind of situation is traditionally used as a basis for several arithmetic or algebra problems. Novotna, for example, used it in three verbal forms and studied six graders' solutions. The forms used by Novotna[2] (Novotna, 1997; Schmidt and Bednarz, 1997; Bednarz, 2001; Kubinova et al., 1994) were:

A1. Peter, David and Jirka are playing marbles. They have 198 marbles altogether. Peter has 6 times more marbles than David, and Jirka has 2 times more marbles than David. How many marbles each one has?

A2. Peter, David and Jirka are playing marbles. They have 198 marbles altogether. Peter has 3 times more marbles than Jirka, and Jirka has 2 times more marbles than David. How many marbles each one has?

A3. Peter, David and Jirka are playing marbles. They have 198 marbles altogether. Peter has 6 times more marbles than David, and 3 times more than Jirka. How many marbles each one has?

TABLE I

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The above problems have different surface and underlying structures. All these problems are of the type known as 'multiplicative compare' problems. All use the word 'more' for describing, on the surface level, the underlying relation, which is a comparison relation.

Described in linguistic terms, this asymmetric comparison relation assigns the roles of the subject or of the predicate of the sentence to the values. In our case, the compared quantity serves as the subject in the sentence, and the reference is part of the predicate. Thus, a given situation can be described using different surface structures (See Appendix A for other possible descriptions of the same situation).

4. THE REFERENCE STRUCTURE OF A WORD PROBLEM

The problems differ in some important aspects. The reference structure in their texts can be described as follows. In problem A1, P and J are both compared to D. Thus two quantities are compared to a third and the 'reference ratio' is 2 [compared quantities] to 1 [reference quantity]. In problem A2, a first quantity is compared to a second, and the second is compared to a third. The reference ratio is 2 [compared quantities] to 2 [reference quantities]. In problem A3, one quantity is compared to the other two quantities, resulting in the reference ratio of 1 to 2. The reference

AUTHOR, PLEASE SUPPLY FIGURE 1

Figure 1. .

structure and the values of the reference ratio for all twelve problems used in our research can be found in Table I, columns 1 and 2.

5. The underlying schemes

The problems in our set differ also with respect to their underlying schemes (see column 4 in Table I). We use the term 'scheme' in the sense of Fischbein's definition of 'schema':

> The term 'schema' indicates a kind of condensed, simplified representation of a class of objects or events. (Fischbein, 1999: 36)

In earlier studies (Hershkovitz and Nesher, 1996; Nesher and Hershkovitz, 1994) we demonstrated the role of schemes in solving two-step word problems. According to the scheme approach, each problem in our study consists of a description of two comparison relations. For example, the two comparison relations given in problem A1 are: P is 6 times more than D and J is 6 times more than D. These relations can be represented in two separate schemes, as shown in Figure 1.

The two separate schemes are connected through David, creating a compound scheme represented in Figure 2.[3]

The compound scheme represents the two elementary relations mentioned in the text of the problem, as well as the way they are connected.

AUTHOR, PLEASE SUPPLY FIGURE 2

Figure 2. .

AUTHOR, PLEASE SUPPLY FIGURE 3

Figure 3. .

AUTHOR, PLEASE SUPPLY FIGURE 4

Figure 4. .

In Problem A2 the compound scheme is built around J (see Figure 3).

In a third possible compound scheme, describing problem A3, the connection is built around P (see Figure 4).

In principle (Nesher and Hershkovitz, 1994), there are only three ways of combining such two relationships, called 'hierarchical' (denoted by H in Table I, column 4), as in Problem A2, 'shared parts' (S-P), as in Problem A1, and, 'shared whole' (S-W), as in Problem A3.

Each of the three compound schemes (H, S-P, S-W) appears wherever the same relations exist in the problem (see Table I, column 4). Thus, the problems indexed by 1 (i.e. A1, B1, C1 and E1) have been constructed on the principle of the S-P scheme; the elementary given relations are connected through D (Figure 2). The problems indexed by 2 used the H scheme; the elementary relations are connected through J (Figure 3). Finally, the S-W scheme appears in problems indexed by 3; here the elementary comparison relations are connected by P (Figure 4).

Focusing on the schemes helped us understand what we came to call the 'complexity level' of a solution. We develop this notion in the next section. Here, let us only mention the main underlying observation, which is that choosing the connection point between the two elementary schemes as the independent variable of an equation (for a given problem) allows one to avoid using the compound function. Choosing any other argument as the independent variable requires additional elaboration, thus leading to a higher level of complexity. Thus, the schemes capture the underlying (deep) structure of the described relationships.

6. BETWEEN THE TEXT AND ITS FORMAL REPRESENTATION

The text is presented to the students in natural language, whose syntactic constraints have to be respected: the compared quantity appears as the subject of the sentence, and the reference is part of the predicate. However, the student's task is to write an equation in algebraic notation, which will lead to the mathematical solution. In mathematical language, the 'compared quantities' and the 'reference quantities' assume the form of, resp., the 'dependent' and the 'independent' variables.

In analyzing the text from the expert's point of view, it should be kept in mind that an actual solver reading the text need not make a direct translation from the subject and predicate to the corresponding dependent and independent variables in the formal representation. The cognitive variables that affect the choice of the 'independent variable' (or the 'x' of the equation) by the solvers were the target of our empirical study, which will be discussed later in this paper. At this point, we propose an expert theoretical model for the given problem texts. We do not claim that it is necessary to make the solver aware of this kind of analysis. But, from the research point of view, it is useful to trace the solvers' intuitive choices of the independent variable in their formal written work to disclose their tacit knowledge. We shall weigh the solvers' choices as they appear in the empirical study against our present theoretical analysis.

There is no one-to-one mapping between the given text and its solution. As any expert would note, there are several options for expressing the conditions of a problem by an equation.

We claim that the choice of the independent variable is based not only on the subject-predicate distinction but also on additional considerations, such as: the priority of 'more' over 'less', and the simplicity of the mathematical expression. For example, choosing the smallest quantity (i.e. D, in our case) yields a simple equation with whole numbers, while choosing the largest quantity (P) requires using fractions.

In the sequel, we use a shorthand notation for the surface structure of the natural language text. We denote the two comparisons given in each text by the functions f(x) and g(x), f(x) representing the first relation mentioned and g(x) – the second mentioned relation, retaining their order in the text. P, J, and D serve as the arguments of these functions, (see Table I, column 3). Using this notation, the conditions of Problem A1 can be written as follows: P = f(D), J = g(D), D + f(D) + g(D) = 198. We shall use this notation when describing solvers' solutions in our empirical study.

7. Possible solutions: An analysis

As noted before, despite the syntactic structure of the text, the text is neutral with respect to the equation that can be written by the solver. There are real options for selecting an independent variable and only this then dictates the form of the equation. In order to better understand the alternative options, let us examine Problem A2 as a detailed example. Here we present three options that exhaust all the theoretical possibilities from an expert point of view.

A2. Peter, David and Jirka are playing marbles. They have 198 marbles altogether. Peter has 3 times more marbles than Jirka, and Jirka has 2 times more marbles than David. How many marbles each one has?

The first comparison relation mentioned in the text states: "*Peter has 3 times more marbles than Jirka";* in short P=f(J). The second relation is: "*Jirka has 2 times more marbles than David";* in short: J=g(D).

Option 1

In selecting the equations for solving this problem one could choose D for the independent variable and write: D = X, J = 2X, P = 3(2X) = 6X. The last equation involves an intermediate calculation to describe P in terms of D. This relation is not explicitly mentioned in the text. In this case one has to rely on Jirka's relation to David, thus employing a compound function. The final equation is X + 2X + 6X = 198.
Thus, in general terms: If D is the independent variable, the equation type is

D + g(D) + f(g(D)) = N

Option 2

For the same problem, one could select J as the independent variable, thus J = X, P = 3X and, therefore, D = 1/2 X. Here, D = G^{-1}(J). We have denoted the second function by the symbol G^{-1} because it is not the direct function described in the text. The text states that "*Jirka has 2 times more marbles than David",* and what was written formally was a translation of "*David has two times less marbles than Jirka"*. The equation in this case is X + 3X + 1/2 X = 198, and the general form of the choice of J as an independent variable in the equation is: $J + G^{-1}(J) + f(J) = N$. In this case the equation includes fractions, which some solvers consider as more complicated.

Option 3

If Peter is selected to be the independent variable, the equation is X + 1/3 X + 1/6 X = 198. In general terms,

$$P + f^{-1}(P) + G^{-1}(f^{-1}(P)) = N.$$

Of course, in the empirical study, each solver chose only one of the options, and the difference between the given text and the option selected by the solver was used in analyzing the data.

8. Complexity levels

We now introduce an intermediate variable named 'the complexity level of the solution', which describes the number of assumed mental transformations that have to be made in the text for writing a certain equation with a certain independent variable. Presenting a mathematical expression in terms of an equation with *one variable* means having the situation described in terms of one reference quantity with all the other quantities dependent on it. Not all the problems in our set are presented in this format. In such cases, in order to write an equation with one variable one should go through one or more mental transformations.

There are many ways of theorizing about the complexity of the solution of a word problem. We chose to construe complexity in terms of the number of mental transformations that one has to execute in moving from the written text to a representation using formal notation. We created a theoretical model of complexity, against which we then compared solutions obtained in the empirical study.

Before presenting our way of assigning a level to the complexity of a solution of a problem, we admit that this is arbitrary and we consider it as a first approximation of the idea that one solution is more complex than another.

We suggest calculating the level of complexity of a solution as follows. Each direct function (one that does not require a mental transformation) counts as a single complexity notch since it is one direct step away from the given text to its algebraic representation. For example, if it is stated in the text, as in problem A1: "*Peter has 6 times more marbles than David*", and it is written formally $P = 6 \times D$, then this is a direct translation which

TABLE II

AUTHOR, PLEASE SUPPLY TABLE II

therefore obtains the score of 1. But if, for example, the same text is written as D = 1/6 P, it becomes an inverse relation (since the subject of the sentence was Peter and not David). Thus it consists of two mental steps and obtains the score of 2. Dealing with a compound function (f(g(x))), as in the detailed example of the first option of problem A2, where no relation between P and D is mentioned in the text, yet the solver makes such connection in the formal writing, thus arriving at the relation between P and D through the previously mentioned relation between P and J. This is assigned the score of 2, as well. The level of complexity of an equation is defined as the sum of all the scores. For example, in the problem A2, selecting D as the independent variable results in a complexity level of 4; selecting J – a complexity level of 3; and selecting P – a complexity level of 8 (See Appendix B for possible solutions and their respective complexity levels for all problems in the set). This again comprises the entire domain of theoretically possible strategies for solving Problem A2. Each solver could select one of these.

Table II presents all possible levels of complexity of the various solutions for the problems used in our study.

The mental transformations that we have taken into account in our definition of level of complexity of a solution are more than just a technical matter. They touch upon some cognitive processes already dealt with in the research literature. We have defined complexity with the aid of the inverse and compound functions. The *inverse* function grew out of changes in the lexical terms of the text, thus, tacitly moving from 'less' in the text to operating as if it was 'more' (changing the reference and the compared) and vise versa. The *compound* function grew out of shifting from an explicit reference mentioned in the text to an implied reference, which was not mentioned (thus, operating differently on the underlying scheme).

The comparison relation, like many other relations, is asymmetric. In most cases we have the choice of how to linguistically describe the same situation. Commonly used in natural language is the description of the same underlying situation as either "*David has more marbles than Peter*", or "*Peter has less marbles than David*". Our notion of "complexity' variable is affected by this kind of considerations. Moreover we also need to take into account the privileged status of the term 'more', as is well known from the literature. Several studies suggest that the word 'more' is comprehended easier than the word 'less' (Donaldson and Balfour, 1968; Nesher and Teubal, 1975; Riley and Greeno, 1988). In the present study the word 'more' is applied in three problems (A1, A2, A3). In three problems (B1, B2, B3) we have employed the word 'less' and in the rest we used both in each problem (C1-E3). (See Appendix A).

9. The Empirical Study

The research question was: What variables affect the choice of the independent variable (the X of the equation)?

In the given problems each unknown quantity, P or J or D can serve as the independent variable. Each choice is regarded as a strategy. Our aim is to understand the choice of strategies and reveal the variables that explain these choices. Is there a privileged and preferred candidate to be selected as the independent variable? One would naturally expect that the independent variable would be chosen so as to lead to a minimal level of complexity in the solution. But there are other variables at play, which may explain why the lowest level of complexity is not selected. Sometimes the solver violates the lowest complexity level rule if it is easier for him to deal with 'more' and whole numbers rather than with 'less' and fractions. Thus, our hypotheses are:

1. The solvers of each problem choose a strategy that leads to the minimal level of complexity.

2. Deviation from the above claim can be explained by the preference of 'more than' over 'less than' in interpreting the text (i.e. making an inverse transformation).

9.1. *The experiment*

We first ran a pilot study with 104 teachers who solved 12 problems (A1 to E3). We relate to it briefly in our analysis of the results. A replication of the same set of problems was given to 167 teachers in an in-service workshop in Israel and the Czech Republic. All teachers were experienced in teaching mathematics in primary schools. The problems were also given to 132 15-year old students in Israel, who already studied equations with one variable. The problems were presented on separate sheets, containing only four problems, each from a different class of problems, with regard to the use of 'more', 'less' or both. Each teacher or student solved only one sheet of problems. The sheets were distributed at random. About 30 teachers and 20 students solved each problem. It took less than 40 minutes for each to complete the task.

9.2. *Findings*

We name the selection of the independent variable (the X) of the given problem among the unknown quantities D, P, J, as 'the solver's strategy'. As noted in our analysis, each quantity was a possible option. Though it may be viewed as a technical choice, we believe that this choice reveals the complexity of the solution and the mental transformation chosen for execution.

Table III presents the percentage of strategies employed for each problem, treating teachers and students separately.

As can be seen from Table III, each problem elicited a different distribution of strategies. Thus, there is no direct translation of the text. Each text produces different strategies for different solvers, meaning that, rather than a direct translation of the text into an equation, we have here a kind of elaboration.

The selection of D as the independent variable was the best-preferred strategy (except for problems: C2, C3, and E2). D was most likely to become the X of the equation for both teachers and students. One should recall that in the described situation D represents the smallest quantity, thus, choosing D as the independent variable means thinking in terms of the relation 'more than' and working with whole numbers, choosing J or P as the independent variable means working with rational numbers. The table emphasizes that working with simple equations and whole numbers

TABLE III

AUTHOR, PLEASE SUPPLY TABLE III

is preferred in most problems. We observe that in problems C2 and C3 the priority is different. This is explained by the fact that choosing D as the independent variable in these cases leads to a much more complex solution (complexity level of 6 and 8, respectively).

Comparison between teachers and students shows that they react similarly to the cognitive tasks they face. Thus, most problems elicit comparable strategies in teachers and students. The minor exceptions (such as in problem E2) are discussed below. A distinct difference between teachers and students was found in the analysis of the 'other' strategies column (which is not the target of our study). The 'other' column consists of different behaviors of teachers and students. Most teachers solved all the problems, but those who employed numerical strategies were included in 'other'. The students included in the 'other' column mostly did not use numerical strategies but gave up and did not solve the problems at all.

Next we tried to observe the relationship between 'strategy' and 'complexity level'. We are aware that the 'complexity level' is a theoretical construct while the 'strategy' is an empirical fact. We assume, however, that such presentation will help us understand the mental transformations that need to be executed in solving a problem in a given strategy.

TABLE IV

AUTHOR, PLEASE SUPPLY TABLE IV

Table IV presents the distribution of selected strategies in all problems by complexity level for teachers and students (In each cell the result for teachers is given first in the brackets, followed by the result for students).

The fact that the distribution of strategies within each problem varies and is similar in most problems for both teachers and students means that we are dealing with a distinct cognitive demand in each problem, which might arise, from our general linguistic training and cognitive capabilities. Thus, despite the difference between teachers and students, we can consider the two samples to be a replication of the same tasks.[4]

Before delving into the details of our findings, let us look at Table V, which clearly marks the order of preference for each strategy within each problem, with 1 standing for the most preferred strategy for each specific problem, and 3 for the least preferred. As to the exact percentage of solvers that selected each strategy the reader is referred to Table IV.

From Tables IV and V we learn that the factors we analyzed are related to cognitive processing that one probably runs through when solving this type of word problems. We assume that while, in the course of solving the problems, the solver may be unaware of such factors, they do affect the choice of his or her strategy (the choice of X). The fact that there are preferable strategies for given problems suggests that there are cognitive factors that interfere in the process. One concerns the ease of processing

TABLE V

AUTHOR, PLEASE SUPPLY TABLE V

the given text. Solvers probably choose, perhaps not consciously, the direct translation that obviates the need for linguistic transformations. The second major factor is the solvers' preference for expressions containing 'more'. Choosing an expression with the term 'more' means that the reference quantity is smaller, which leads to an equation with whole numbers (See Table V). This is especially interesting in cases where the complexity level is quite high (8) as in problems B2 and C3.

There are, however, a few cases that go against this finding (B3, for example). In such cases, other surface structure variables such as 'the order of the information' and an easy transformation from 'less' to 'more', dictate the selection of the strategy.

Also in Table V we see that when the complexity level is very low (2) or very high (8) teachers and students use similar solutions, but at a medium complexity level, the choice of solution may, at times, differ between the teachers and the students.

In addition to the written work, we also discussed the issue of preferred strategies with the teachers. After solving the problems we asked the teachers to reflect on the reasons for their choice of strategy. First, when we asked whether they see any difference between the problems, their answers were: "The problems speak of different relations"; "The set of relations among the figures in the stories are different". Thus, the teachers were aware that there is a difference between the problems and that there is a transition from one set of relations (scheme in our theoretical discussion) to another within the same situation.

Asked, "What strategies did you use?" the teachers responded in two ways: (a) the order of the information, "I chose the X according to the order of the information"; and (b) identifying the smallest set, "I chose the smallest quantity as the X". This emphasizes the high priority of D as the independent variable.

Finally we asked "What makes a question easy or difficult?" Several teachers replied: "Finding the smallest quantity to compare the others to", "The identification of the given sets of marbles", meaning their role in the relation.

To summarize, the teachers viewed the problems as being distinct, with varying degrees of difficulty. The surface structure order of the text affects their choice of strategy, and they prefer the smallest quantity to be the independent variable to which other quantities will be compared. As mentioned before, teachers and students mostly behaved in a similar manner. However, in cases of discrepancy (see C3), students demonstrated clearly that they preferred a direct translation of the text with a low complexity level, even if they had to write an equation with fractions. Elementary school teachers preferred to arrive at an equation with whole numbers and probably were more at ease in making the linguistic transformations from 'less' to 'more', arriving at a complexity level of 6.

10. Discussion

Our study is an effort to find factors that affect the way solvers construct their equations to solve a simple multiplicative comparison problem. We based our work on cognitive theories related to solving word problems such as that of Kintsch, which distinguishes between the text base and the situation model (Kintsch and van Dijk, 1978; Kintsch, 1986), and work emphasizing the schemes approach (Greeno, 1978; Nesher, Greeno et al., 1982; Shalin, 1985; Vergnaud, 1988; Hall, Kibler et al., 1989; Reusser, 1992; Vergnaud, 1998; Fischbein, 1999).

In Kintsch's terms, a "situation model" is "a mental representation of the situation described by the text". We now believe that this notion is ambiguous. The same situation can be described by different texts as exemplified by the set of our 12 problems. By saying "a different text, we mean different not merely on the surface structure level, but also in its propositional structure with its coherent macrostructure" (see Kintsch 1986, p.89). Column 2 of Table I presents the equivalent of a propositional structure in terms of the considered relations. A visual representation of the coherence of the macrostructure can be found in Table I, Column 1, where the propositions are combined in the three possible compound schemes. The schemes are general and can be employed in many word problem formulations. The schemes describe all situation components in three place relations, as a comparison situation actually requires (Hershkovitz, Nesher et al., 1990; Nesher and Hershkovitz, 1994; Hershkovitz and Nesher, 1996).

We note that the 'situation model' concept could have at least three different interpretations:

1. The world as it is (in our case David, Jirka, and Peter each have a certain quantity of marbles).
2. A relation that exists between the members of that world (a comparison relation within each pair of the above-mentioned quantities).
3. The world is described propositionally. The 'compared' and 'reference' quantities are defined.

Kintsch (1986) named the third interpretation as the 'text base' while the second seems to be the 'situation model'. We think that the first interpretation is missing in Kintsch's analysis. The first level, however, was proven empirically to be necessary. Observing the strategies employed by teachers and students, we saw that although the text implied certain relationships, the solver added other relations that were not mentioned in the text. The ability to add such relations comes from the comprehension of the entire world situation, namely, the first interpretation above. For example, in Problem A3 there is a description of the relation between the possessions of Peter and David, and Peter and Jirka (A3). Yet, the solvers who chose 'D' as their independent variable (X) did not hesitate to solve the problem by bringing in the relation between the possessions of David and Jirka (not mentioned in the text at all). Moreover, this was even the preferable strategy used by 62% of the teachers.

Other findings in our study show that it is unjustified, in analyzing problem solving, to neglect surface structure variables. Starting the analysis from the 'text base', which is frequently done by cognitive researchers, means missing some of the influential variables. Of the surface structure variables we would like to mention three:

(a) Order of presentation of the comparison relations (Column 3, Table I); this variable was found in our study to be negligible. This supports the claim that the solver attends to the entire text (Kintsch et al., 1975; Nesher and Katriel, 1977; Reusser, 1992).

(b) The syntactic structure of the sentences, whether a simple conjunction, a compound sentence, or anaphora, affected the level of complexity. This is connected to the fact that being the 'compared' or the 'reference' quantity dictates who will appear as the subject of the sentence on the surface of the word problem text, and who will be part of the predicate. The subject-predicate relation for each problem is described in Table I, Column 1 (graphically) and Column 3 (in a functional notation). From a syntactic point of view, problems A2, B2, C1, C3, E1, E3 are compound sentences in which the predicate of the first sentence becomes the subject of the second sentence. Problems A1, B3, C2 are conjunctions that share the predicates. Problems A3, B1, E2 are making anaphoric use of the same subject that plays a role in both comparisons. These various syntactic structures denote the 'reference ratio' and have affected the level of complexity of the solution. They impact the number of transformations needed for the selection of the independent variable.

(c) The lexical items 'more' and 'less' (Column 5, Table I) were also influential. The tacit alternation between them was the tool that solvers used in selecting their own strategy.

Our study has some limitations, of course. We introduced a manifold theoretical construct, namely, the 'complexity level': and we analyzed the components affecting it (the schemes, the reference ratio, and the lexical items 'more' or 'less'). Future research is needed to find out more about the relative weight of the components affecting the complexity level. It would also be interesting to find out whether there are individual profiles of solvers who prefer certain strategies over others. As this study engaged in only one world situation, the generalizations of our conclusions need to be also studied within a variety of other situations.

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NOTES

1. It should be noted that there are languages, such as English, that do not use the term 'less' as symmetrical to 'more' in multiplicative cases. The common English expression relevant to multiplicative comparison is 'five times as many as'. However, in many other languages (Hebrew and Czech and Arabic included) the multiplicative comparison, similar to the additive comparison, employs the words 'more' and 'less' in the following manner: 'five times more' and 'five times less' in a most natural fashion.

2. This paper is based on the work Novotna started with N. Descamps-Bednarz and B. Dufour-Janvier at CIRADE and UQAM in Montreal.
3. Information shared by all problems, i.e. that the total of marbles is 198, is not mentioned in our analysis.
4. We have noticed another replication for the 12 problems (A1-E3) in our pilot study (Cerme 2, Marianske Lazne, February 2001). The results for each problem in the pilot study are very similar to those of the main study.

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