Arithmetics on beta-expansions*

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Abstract

In this paper we consider representation of numbers in an irrational basis $\beta > 1$. We study the arithmetic operations on β-expansions and provide bounds on the number of fractional digits arising in addition and multiplication, $L_\oplus(\beta)$ and $L_\odot(\beta)$, respectively. We determine these bounds for irrational numbers β which are algebraic with at least one conjugate in modulus smaller than 1. In the case of a Pisot number β we derive the relation between β-integers and cut-and-project sequences and then use the properties of cut-and-project sequences to estimate $L_\oplus(\beta)$ and $L_\odot(\beta)$. We generalize the results known for quadratic Pisot units to other quadratic Pisot numbers.

1 Beta-expansions

Let β be a real number strictly greater than 1. A real number $x \geq 0$ can be represented using a sequence $(x_i)_{k\geq i>-\infty}$, $x_i \in \mathbb{Z}$, $0 \leq x_i < \beta$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_a\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for certain $k \in \mathbb{Z}$. It is denoted by

$$(x)_\beta = x_k x_{k-1} \dots x_1 x_0 \bullet x_{-1} x_{-2} \dots$$

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A particular representation is the β-expansion of x, see [7]. The digits x_i of the β-expansion are computed by the 'greedy' algorithm: Let $[y]$ denote the largest integer smaller or equal to y. Find $k \in \mathbb{Z}$, for which $\beta^k \leq x < \beta^{k+1}$. Put $x_k = [x/\beta^k]$ and $r_k = x/\beta^k \bmod 1$. For $i \in \mathbb{Z}$, $i < k$ put $x_i = [\beta r_{i+1}]$ and $r_i = \beta r_{i+1} \bmod 1$. If $k < 0$, i.e. $0 < x < 1$ we put $x_0, x_1, \ldots x_{k+1} = 0$ and write $(x)_\beta = 0 \bullet 00\ldots 0 x_k x_{k-1} \ldots$. If an expansion ends in infinitely many zeros, it is said to be finite and the ending zeros are omitted.

We denote by $\mathrm{Fin}(\beta)$ the set of all x for which $|x|$ has a finite β-expansion. The β-expansion of every $x \in \mathrm{Fin}(\beta)$ has therefore the form

$$(x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots x_{-\ell}\,,$$

where $x_k x_{k-1} \ldots x_1 x_0 \bullet$ is the β-integer part and $\bullet x_{-1} x_{-2} \ldots x_{-\ell}$ is the β-fractional part of x. We usually call it simply the integer and the fractional part of x. The length of the fractional part of x is denoted by $\mathrm{fp}_\beta(x)$. Elements of $\mathrm{Fin}(\beta)$ with vanishing fractional part (i.e. $\mathrm{fp}_\beta(x) = 0$) are called β-integers. The set of β-integers is denoted by $\mathbb{Z}_\beta$.

The sets $\mathbb{Z}_\beta$ and $\mathrm{Fin}(\beta)$ are generally not closed under addition and multiplication. In spite of that it is sometimes useful in computer science to consider these operations in β-arithmetics. That is why it is important to study what fractional parts may appear as a result of addition and multiplication of β-integers.

Definition 1.1. Let $\beta > 1$. We denote

$$\begin{aligned} L_\oplus(\beta) &:= \min\{L \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_\beta,\ x + y \in \mathrm{Fin}(\beta) \Rightarrow \mathrm{fp}_\beta(x+y) \leq L\}, \\ L_\odot(\beta) &:= \min\{L \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_\beta,\ xy \in \mathrm{Fin}(\beta) \Rightarrow \mathrm{fp}_\beta(xy) \leq L\}. \end{aligned}$$

Minimum of an empty set is defined to be $+\infty$.

The aim of this paper is to give some quantitative results for $L_\oplus(\beta)$ and $L_\odot(\beta)$. Let us mention some of the known results. Frougny and Solomyak in [4] showed that $L_\oplus(\beta)$ is finite if β is a Pisot number. A Pisot number β is an algebraic integer such that $\beta > 1$ and all its algebraic conjugates are in modulus smaller than 1. Let us mention that to our knowledge no example is known of a β such that $L_\oplus(\beta)$ or $L_\odot(\beta)$ is infinite.

Results for the special case of quadratic Pisot units are found in [3]. The authors gave exact values for $L_\oplus(\beta)$ and $L_\odot(\beta)$, when $\beta > 1$ is a solution either of equation $x^2 = mx - 1$, $m \in \mathbb{N}$, $m \geq 3$ or of equation $x^2 = mx + 1$, $m \in \mathbb{N}$. In the first case $L_\oplus(\beta) = L_\odot(\beta) = 1$; in the second case $L_\oplus(\beta) = L_\odot(\beta) = 2$.

In this article we provide estimates on $L_\oplus(\beta)$ and $L_\odot(\beta)$ for those algebraic numbers $\beta > 1$ that have at least one of the conjugates in modulus smaller than 1. Other results are valid for Pisot numbers β. The last part of the paper is devoted to quadratic Pisot numbers. We reproduce the results of [3] as a special case.

2 Beta-integers and cut-and-project sequences

The Rényi development of unity plays an important role in the description of properties of sets $\mathbb{Z}_\beta$ and $\mathrm{Fin}(\beta)$. For its definition we introduce the transformation $T_\beta(x) := \{\beta x\}$, for $x \in [0,1]$. The Rényi development of unity is defined as

$$d(1,\beta) := t_1 t_2 \dots t_i \dots , \qquad \text{where} \;\; t_i := [\beta T_\beta^{i-1}(1)] \,.$$

Parry in [6] has showed that $x = x_k x_{k-1} \dots x_1 x_0 \bullet x_{-1} \dots x_{-p}$ is a β-expansion if and only if $x_i x_{i-1} \dots x_{-p}$ is lexicographically smaller than $t_1 t_2 \dots t_i \dots$ for every $-p \le i \le k$.

$\mathrm{Fin}(\beta)$ and $\mathbb{Z}_\beta$ are centrally symmetric sets. While $\mathrm{Fin}(\beta)$ is dense in $\mathbb{R}$, $\mathbb{Z}_\beta$ has no accumulation points. Distances between consecutive points in $\mathbb{Z}_\beta$ take values $\{0 \bullet t_i t_{i+1} \dots \mid i \in \mathbb{N}\}$. It is obvious that if $d(1,\beta)$ is eventually periodic, then $\mathbb{Z}_\beta$ has a finite number of distances between consecutive points. Numbers β with this property are called beta-numbers. Some results and conjectures on beta-numbers are given in [2, 9]; a description of beta-numbers is provided in [8]. Note that every Pisot number β is a beta-number.

The set $\mathbb{Z}_\beta$ of β-integers forms a ring only in the case that β is a rational integer, $\beta > 1$. If β is an algebraic integer of order $q \ge 2$, then $\mathbb{Z}_\beta$ can be naturally embedded into the ring $\mathbb{Z}[\beta]$ defined as

$$\mathbb{Z}[\beta] := \{n_0 + n_1\beta + \dots + n_{q-1}\beta^{q-1} \mid n_i \in \mathbb{Z}\} \,.$$

Note that the ring $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$. In certain cases $\mathbb{Z}[\beta]$ coincides with $\mathrm{Fin}(\beta)$, i.e. $\mathrm{Fin}(\beta)$ is a ring, see [4]. Let us show that for β an algebraic integer, the ring $\mathbb{Z}[\beta]$ is a projection of an integer lattice $\mathbb{Z}^q \subset \mathbb{R}^q$ on a one-dimensional subspace V_1 for a suitable decomposition $V_1 \oplus V_2$ of the space $\mathbb{R}^q$. A similar construction can be found in [1].

Denote $\beta^{(1)} = \beta$, $\beta^{(2)}, \dots, \beta^{(s)}$, the real roots of the minimal polynomial of β and by $\beta^{(s+1)}, \beta^{(s+2)}, \dots, \beta^{(q-1)}, \beta^{(q)}$ the non real conjugates of β. We have ordered the complex roots in such a way that $\overline{\beta^{(s+1)}} = \beta^{(s+2)}, \dots, \overline{\beta^{(q-1)}} = \beta^{(q)}$.

At first we have to find (possibly) complex vectors

$$\left(\vec{x}^{(1)}\right)^T = (x_0^{(1)}, x_1^{(1)}, \dots, x_{q-1}^{(1)}) \,, \quad \dots \quad , \quad \left(\vec{x}^{(q)}\right)^T = (x_0^{(q)}, x_1^{(q)}, \dots, x_{q-1}^{(q)}) \,,$$

such that for any $\vec{x} = (n_0, n_1 \dots, n_{q-1}) \in \mathbb{R}^q$ we have

(1)

$$\vec{x} = \left(\sum_{i=0}^{q-1} n_i (\beta^{(1)})^i\right) \vec{x}^{(1)} + \left(\sum_{i=0}^{q-1} n_i (\beta^{(2)})^i\right) \vec{x}^{(2)} + \dots + \left(\sum_{i=0}^{q-1} n_i (\beta^{(q)})^i\right) \vec{x}^{(q)} \,.$$

Denote by $\mathbb{X}$ the $q \times q$ matrix with $(\mathbb{X})_{ij} = x_j^{(i)}$. Then (1) holds for each $\vec{x}$ if and only if

$$I_q = \mathbb{V}(\beta^{(1)}, \dots, \beta^{(q)}) \cdot \mathbb{X} \,,$$

where $\mathbb{V}(\beta^{(1)},\ldots,\beta^{(q)})$ is the Vandermonde matrix in variables $\beta^{(1)},\ldots,\beta^{(q)}$,

$$\mathbb{V}(\beta^{(1)},\ldots,\beta^{(q)}) := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \beta^{(1)} & \beta^{(2)} & \ldots & \beta^{(q)} \\ (\beta^{(1)})^2 & (\beta^{(2)})^2 & \ldots & (\beta^{(q)})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (\beta^{(1)})^{q-1} & (\beta^{(2)})^{q-1} & \ldots & (\beta^{(q)})^{q-1} \end{pmatrix}.$$

The determinant of $\mathbb{V}(\beta^{(1)},\ldots,\beta^{(q)})$ is equal to $\Pi_{q\geq i>j\geq 1}(\beta^{(i)}-\beta^{(j)})$. Since all conjugates are distinct, the determinant is non zero.

Using the Cramer rule to compute $x_j^{(i)}$, we obtain that $\vec{x}^{(i)}$ is real if $\beta^{(i)}$ is real, and if $\beta^{(j)}$ and $\beta^{(j+1)}$ are mutually complex conjugated roots then $\vec{x}^{(j)} = \overline{\vec{x}^{(j+1)}}$.

Thus we can define a real basis $\vec{y}^{(1)},\ldots,\vec{y}^{(q)}$ of $\mathbb{R}^q$ in such a way that $\vec{y}^{(i)} = \vec{x}^{(i)}$ if $\vec{x}^{(i)}$ is a real vector, and $\vec{y}^{(j)} = \vec{x}^{(j)} + \overline{\vec{x}^{(j)}}$, $\vec{y}^{(j+1)} = i(\vec{x}^{(j)} - \overline{\vec{x}^{(j)}})$, if $\vec{x}^{(j)}$ and $\vec{x}^{(j+1)} = \overline{\vec{x}^{(j)}}$ are mutually complex conjugated vectors.

Note that the coordinates of a vector $\vec{x} = (n_0, n_1 \ldots, n_{q-1}) \in \mathbb{R}^q$ with respect to the basis $\vec{y}^{(1)},\ldots,\vec{y}^{(q)}$ are

$$\begin{array}{ll} \displaystyle\sum_{p=0}^{q-1} n_p(\beta^{(i)})^p\,, & \text{if } \vec{y}^{(i)} = \vec{x}^{(i)}\,, \\ \Re\left[\displaystyle\sum_{p=0}^{q-1} n_p(\beta^{(j)})^p\right], & \text{if } \vec{y}^{(j)} = \vec{x}^{(j)} + \overline{\vec{x}^{(j)}}\,, \\ \Im\left[\displaystyle\sum_{p=0}^{q-1} n_p(\beta^{(j)})^p\right], & \text{if } \vec{y}^{(j)} = i(\vec{x}^{(j)} - \overline{\vec{x}^{(j)}})\,. \end{array}$$

If we put $V_1 = \mathbb{R}\vec{y}^{(1)}$ and $V_2 = \mathbb{R}\vec{y}^{(2)} + \mathbb{R}\vec{y}^{(3)} + \cdots + \mathbb{R}\vec{y}^{(q)}$, the set $\mathbb{Z}[\beta]$ is the projection of $\mathbb{Z}^q$ on V_1 along V_2.

Projections of crystallographic and non-crystallographic lattices are studied by the theory of cut-and-project sets. Let us recall here a special case of their definition, which will be used here.

Definition 2.1. Let U_1, U_2 be linear subspaces of $\mathbb{R}^d$ such that $\dim U_1 = 1$, $\dim U_2 = d-1$ and $U_1 \oplus U_2 = \mathbb{R}^d$. Denote by π_1 the projection on U_1 along U_2 and by π_2 the projection on U_2 along U_1. Let $\Omega \subset U_2$ be a bounded set with non-empty interior Ω°, such that the closures of Ω and Ω° coincide. If the mapping $\pi_1 : \mathbb{Z}^q \to \pi_1(\mathbb{Z}^d)$ is one-to-one and $\pi_2(\mathbb{Z}^d)$ is dense in V_2, then the set $\Sigma(\Omega) = \big\{\pi_1(x) \mid x \in \mathbb{Z}^d\,,\ \pi_2(x) \in \Omega\big\}$ is called a cut-and-project set with acceptance window Ω.

Basic properties of cut-and-project sets can be found in [5]. For us the most important property is that the set $\Sigma(\Omega)$ is relatively dense and uniformly discrete, i.e. there exists a real increasing sequence $(\alpha_n)_{n\in\mathbb{Z}}$ and constants $r, R > 0$, such that $\Sigma(\Omega) = \{\alpha_n \vec{y} \mid n \in \mathbb{Z}\}$ and $r \leq \alpha_{n+1} - \alpha_n \leq R$

for all $n \in \mathbb{Z}$. In particular, the distances between consecutive points of $\Sigma(\Omega)$ take only finitely many values, i.e. the set $\{\alpha_{n+1} - \alpha_n \mid n \in \mathbb{Z}\}$ is finite.

Let us consider again β to be an algebraic integer of order q and the decomposition $\mathbb{R}^q = V_1 \oplus V_2$ as described above. As shown by Akiyama [1], the projection $\pi_1(\mathbb{Z}^2) = \mathbb{Z}[\beta]$ of $\mathbb{Z}^2$ on V_1 is one-to-one and the projection $\pi_1(\mathbb{Z}^2)$ on V_2 is dense in V_2. For $\alpha \in \mathbb{Q}[\beta]$ we denote $\alpha^{(k)}$ the image of α under the k-th Galois isomorphism $\mathbb{Q}[\beta] \to \mathbb{Q}[\beta^{(k)}]$ induced by the assignment $\beta \to \beta^{(k)}$, i.e. if $\alpha = \sum_{i=0}^{q-1} n_i \beta^i$ for $n_i \in \mathbb{Q}$, then $\alpha^{(k)} = \sum_{i=0}^{q-1} n_i \big(\beta^{(k)}\big)^i$.

We shall focus on specific acceptance windows $\Omega(h) \subset V_2$, for $h > 0$. As the acceptance window $\Omega(h) \subset V_2$ we choose the cartesian product of one-dimensional line-segments $\{t\vec{y}^{(i)} \mid |t| < h\}$ if $\beta^{(i)}$ is real and two-dimensional ellipses $\{t\vec{y}^{(j)} + s\vec{y}^{(j+1)} \mid t^2 + s^2 < h^2\}$ if $\beta^{(j)}$ and $\beta^{(j+1)}$ are complex conjugated. Such an acceptance window $\Omega(h)$ satisfies the assumptions of Definition 2.1.

The point $\alpha\vec{y}^{(1)}$ belongs to $\Sigma\big(\Omega(h)\big)$ if and only if $\alpha \in \mathbb{Z}[\beta]$ and $|\alpha^{(k)}| < h$ for $k = 2, 3, \dots, q$. In other words, we have the following proposition.

Proposition 2.2. *Let β be an algebraic integer of order q. If $h > 0$, then the set*

$$\Sigma(h) = \{\alpha \in \mathbb{Z}[\beta] \mid |\alpha^{(k)}| < h\,,\ k = 2, \dots, q\}$$

is relatively dense and uniformly discrete and the distances in $\Sigma(h)$ take only finitely many values.

In the following sets $\Sigma(h)$ are called the cut-and-project sequences. In the case that β is a Pisot number, we show the relation between cut-and-project sequences and β-integers $\mathbb{Z}_\beta$.

Proposition 2.3. *Let β is a Pisot number of order q. Denote by $\ell = [\beta] \max\{(1 - |\beta^{(i)}|)^{-1} \mid i = 2, 3, \dots, q\}$. Then*

$$\mathbb{Z}_\beta \subset \Sigma(\ell)\,, \qquad \mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \Sigma(2\ell)\,, \qquad \mathbb{Z}_\beta\mathbb{Z}_\beta \subset \Sigma(\ell^2)\,.$$

Proof. Let $x \in \mathbb{Z}_\beta$, i.e. $x = \pm\sum_{i=0}^{n} x_i \beta^i$, for some n, then

$$|x^{(j)}| \ \leq\ \sum_{i=0}^{n} [\beta] \Big|\beta^{(j)}\Big|^i \ <\ [\beta] \frac{1}{1 - |\beta^{(j)}|} \leq \ell\,, \qquad \text{for } j = 2, \dots, q\,.$$

The statement follows easily. ∎

3 Sufficient conditions for finiteness of $L_\oplus$ and $L_\odot$

In this section we provide sufficient conditions on β so that $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite. First we demonstrate Theorem 3.1 stating that $L_\oplus(\beta)$ and

$L_\odot(\beta)$ are finite for a Pisot β. The statement for $L_\oplus$ has been proven in [4], however, we provide a different and simpler proof. We further show that this condition is not necessary. Theorem 3.3 provides a different sufficient condition together with bounds $L_\oplus(\beta)$ and $L_\odot(\beta)$. In the next section we apply Theorem 3.3 to the case of quadratic Pisot numbers.

Theorem 3.1. *Let β be a Pisot number. Then $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite.*

Proof. Let $x, y \in \mathbb{Z}_\beta$. For determination of $L_\odot(\beta)$ it suffices to consider $x, y > 0$. Let us denote $z_0 = \max\{z \in \mathbb{Z}_\beta \mid z \le xy\}$ and $r := xy - z_0$. Since distances in $\mathbb{Z}_\beta$ are bounded by 1, we have $0 \le r < 1$. Therefore obviously the remainder r is the fractional part of the β-expansion of xy, i.e. $xy \in \mathrm{Fin}(\beta)$ if and only if $r \in \mathrm{Fin}(\beta)$. Since $\ell > 1$, we have $\Sigma(\ell) \subset \Sigma(\ell^2)$ and according to Proposition 2.3 both xy and z_0 belong to $\Sigma(\ell^2)$.

According to Proposition 2.2 distances in $\Sigma(\ell^2)$ take only finitely many values, say $f_1, \dots, f_T$. The gap r between z_0 and xy must be composed from these distances. Therefore $1 > r = xy - z_0 = \sum h_i f_i$, where $h_i \in \mathbb{N}_0$. Fractional parts of all results of multiplication xy belong to the set

$$F := \left\{ \sum_i h_i f_i < 1 \;\middle|\; h_i \in \mathbb{N}_0 \right\},$$

which is finite and therefore

$$L_\odot(\beta) \le \max\{\mathrm{fp}_\beta(r) \mid r \in F \cap \mathrm{Fin}(\beta)\}.$$

To derive the finiteness of $L_\oplus(\beta)$ one uses an analogous argument. □

A simple consequence of the above proof is that $\mathbb{Z}_\beta$ is a Meyer set.

Corollary 3.2. *Let β be a Pisot number. Then there exists a finite set F such that*

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F, \qquad \mathbb{Z}_\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F.$$

Theorem 3.1 gives a sufficient condition for finiteness of $L_\oplus(\beta)$ and $L_\odot(\beta)$. An upper bound on the value of $L_\oplus(\beta)$ is determined in [10] using some complicated techniques. However, their result applies only to a class of Pisot numbers. The condition that β is Pisot is however not necessary. In the following theorem we provide a similar estimate on $L_\oplus(\beta)$ with less restrictive criteria for β. Moreover, we determine the upper bound for $L_\odot(\beta)$.

Theorem 3.3. *Let $\beta > 1$ be an irrational algebraic number such that at least one among its conjugates, say β', is in modulus smaller than 1. Denote*

$$\begin{aligned} H &= \sup\{|z'| \mid z \in \mathbb{Z}_\beta\} \\ K &= \inf\{|z'| \mid z \in \mathbb{Z}_\beta\,,\ z \notin \beta\mathbb{Z}_\beta\} \end{aligned}$$

If $K > 0$, then $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite and

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus(\beta)} < \frac{2H}{K} \tag{2}$$

$$\left(\frac{1}{|\beta'|}\right)^{L_\odot(\beta)} < \frac{H^2}{K} \tag{3}$$

Proof. Let $x, y \in \mathbb{Z}_\beta$ and $x + y \in \mathrm{Fin}(\beta)$, $x + y = \sum_{i=-L}^{k} a_i\beta^i$, $a_{-L} \geq 1$. Then $\beta^L(x + y) \in \mathbb{Z}_\beta$ and $\beta^L(x + y) \notin \beta\mathbb{Z}_\beta$. Thus

$$K \leq |\beta'|^L|x' + y'| \leq |\beta'|^L(|x'| + |y'|) < 2H|\beta'|^L\,,$$

which implies (2). Note that the supremum H is never attained, i.e. $|z'| < H$ for all $z \in \mathbb{Z}_\beta$. The proof is similar for multiplication. □

Remark 3.4.

1. Using the same inequalities as in the proof of Proposition 2.3 we obtain

$$H \leq [\beta]\frac{1}{1 - |\beta'|}\,.$$

2. If $\beta' \in (0, 1)$, then $K = 1$. Indeed, for $z = \sum_{i=0}^{n} z_i\beta^i$, $z_0 \neq 0$, one has

$$z' = \sum_{i=0}^{n} z_i(\beta')^i \geq z_0 \geq 1\,.$$

Corollary 3.5. *Let $\beta > 1$ be an algebraic integer such that at least one of its conjugates, say β', belongs to $(0, 1)$. Then*

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus(\beta)} < \frac{2[\beta]}{1 - \beta'} \qquad \textit{and} \qquad \left(\frac{1}{|\beta'|}\right)^{L_\odot(\beta)} < \frac{[\beta]^2}{(1 - \beta')^2}\,.$$

4 Theorem 3.3 for quadratic Pisot numbers

So far we have been interested in results for $L_\oplus(\beta)$ and $L_\odot(\beta)$ for general algebraic integers β. From now on we shall focus on quadratic Pisot numbers. In the quadratic case the Pisot condition implies that β is a solution of an equation

$$\begin{aligned} x^2 &= mx - n\,, \quad m, n \in \mathbb{N}\,, \quad m \geq n + 2\,,\\ x^2 &= mx + n\,, \quad m, n \in \mathbb{N}\,, \quad m \geq n\,. \end{aligned}$$

We shall try to apply Theorem 3.3 on such β and derive the corresponding bounds on $L_\oplus(\beta)$ and $L_\odot(\beta)$. It will be seen that the situation drastically differs for the two types of quadratic equations.

Note that for $n = 1$, the root β is a quadratic Pisot unit. For such β the values of $L_\oplus(\beta)$ and $L_\odot(\beta)$ have been determined in [3].

Let us now study the case $\beta > 1$ being the solution of the equation $x^2 = mx - n$, $m, n \in \mathbb{N}$, $m \geq n + 2$. Note that $[\beta] = m - 1$, thus the digits in β-expansions are $0, 1, 2, \ldots, m - 1$. The conjugate β' of β satisfies $\beta' \in (0, 1)$, and the β-development of unity is $d(1, \beta) = (m-1)(m-n-1)^\omega$. For $z \in \mathbb{Z}_\beta$, $z = \sum_{i=0}^n z_i \beta^i$ we have

$$
\begin{aligned}
z' = \sum_{i=0}^n z_i(\beta')^i &< (m-1) + (m-2)\beta' + (m-2)\beta'^2 + \cdots = \\
&= 1 + (m-2)\frac{1}{1-\beta'} = \frac{\beta(\beta-1)}{\beta - n} = H\,.
\end{aligned} \tag{4}
$$

Clearly, $\frac{\beta(\beta-1)}{\beta-n}$ above is the desired supremum H of Theorem 3.3, since we can construct a sequence of numbers

$$z_n = (m-1)\beta^0 + \sum_{i=1}^n (m-2)\beta^i \;\in\; \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta\,,$$

such that $\lim_{n\to\infty} |z_n'| = H$. For the relation (4) we have considered the admissibility of sequences of digits in β-expansions. According to Remark 3.4 we have $K = 1$, and hence we can use Theorem 3.3 to derive results for $L_\oplus(\beta)$ and $L_\odot(\beta)$.

Proposition 4.1. *Let $\beta^2 = m\beta - n$, $m \geq n + 2$. Then*

$$L_\oplus(\beta) \leq 3m \ln m \qquad \textit{and} \qquad L_\odot(\beta) \leq 4m \ln m\,.$$

In particular, if $n = 1$, then $L_\oplus(\beta) = L_\odot(\beta) = 1$.

Proof. Since $K = 1$ and $H = \frac{\beta(\beta-1)}{\beta-n} = \frac{(\beta-1)^2}{m-n-1}$ we can estimate

$$\left(\frac{m-1}{n}\right)^{L_\oplus} < \left(\frac{\beta}{n}\right)^{L_\oplus} = \left(\frac{1}{\beta'}\right)^{L_\oplus} < \; 2\frac{(\beta-1)^2}{m-n-1} \; < \; 2\frac{(m-1)^2}{m-n-1}\,.$$

For $n = 1$ we obtain directly $L_\oplus \leq 1$. For general $n \leq m - 2$ we estimate the left hand side of the inequality by

$$\left(\frac{m-1}{n}\right)^{L_\oplus} \geq \left(\frac{m-1}{m-2}\right)^{L_\oplus} > \; e^{\frac{1}{m}L_\oplus}\,,$$

where we have used $(1 + \frac{1}{k})^{k+1} > e$ for $k \in \mathbb{N}$. The right hand side of the inequality is estimated by m^3. Altogether we get $L_\oplus(\beta) \leq 3m \ln m$. The estimate for $L_\odot(\beta)$ is derived analogically, the first step for $n = 1$ being

$$\beta^{L_\odot} = \left(\frac{1}{\beta'}\right)^{L_\odot} < \; \left(\frac{\beta(\beta-1)}{\beta-1}\right)^2 = \beta^2 \qquad \Longrightarrow \qquad L_\odot \leq 1\,.$$

In order to show that for $n = 1$ we have $L_\oplus(\beta) = L_\odot(\beta) = 1$ it suffices to realize that

$$\Big((m-1)+(m-1)\Big)_\beta = \Big(2\cdot(m-1)\Big)_\beta = \left(\beta + (m-2) + \frac{1}{\beta}\right)_\beta = 1(m-2)\bullet 1$$

$\square$

Let us now study the case of $\beta > 1$ solution of the equation $x^2 = mx+n$, $m, n \in \mathbb{N}$, $m \geq n$. Note that $[\beta] = m$, therefore the digits in the β-expansion are $0, 1, 2, \ldots, m$. The β-development of unity is $d(1, \beta) = mn$. Now the conjugate β' of β satisfies $\beta' \in (-1, 0)$. If $w \in \mathbb{Z}_\beta$, $w = \sum_{i=0}^n w_i\beta^i$, we have

$$\begin{aligned} \cdots + m\beta'^3 + m\beta' \ &< \ w' \ < \ m + m\beta'^2 + m\beta'^4 + \cdots \\ -1 \ &< \ w' \ < \ \frac{m}{1-\beta'^2} = \frac{\beta^2 m}{m\beta + n - n^2} = H\,. \end{aligned}$$

Unfortunately, in this case $K = 0$ for all $n \in \mathbb{N}$ except $n = 1$. Therefore only for $n = 1$ can we use Theorem 3.3 to find values of $L_\oplus(\beta)$ and $L_\odot(\beta)$. In this case for $z \in \mathbb{Z}_\beta$, $z = \sum_{i=0}^n z_i\beta^i$ with $z_0 \neq 0$, we have

$$\begin{aligned} z' &\geq z_0 + z_1\beta' + z_3\beta'^3 + z_5\beta'^5 + \cdots \geq \\ &\geq 1 + (m-1)\beta' + m\beta'^3 + m\beta'^5 + \cdots = \\ &= 1 - \beta' + \frac{m\beta'}{1-\beta'} = -\beta' = \frac{1}{\beta} = K\,. \end{aligned}$$

Note that H is equal to β for $n = 1$. Using (2) and (3) we obtain for $m \geq 2$

$$\left.\begin{array}{lcccc} \beta^{L_\oplus} & < & 2\beta^2 & < & \beta^3 \\ \beta_{L_\odot} & < & \beta^3 & & \end{array}\right\} \Longrightarrow \begin{array}{lcl} L_\oplus(\beta) & \leq & 2 \\ L_\odot(\beta) & \leq & 2 \end{array}$$

To prove that $L_\oplus(\beta) = L_\odot(\beta) = 2$ we calculate

$$(m+m)_\beta = (2\cdot m)_\beta = \left(\beta + (m-1) + \frac{m-1}{\beta} + \frac{1}{\beta^2}\right)_\beta = 1(m-1)\bullet(m-1)1$$

For $m = 1$, i.e. β the golden ratio, it does not hold that $2\beta^2 < \beta^3$. A slightly finer discussion is necessary to obtain the exact bound on the number of fractional digits of the addition $x + y$.

In the above considerations we are not able to derive any estimates on $L_\oplus(\beta)$ and $L_\odot(\beta)$ if β is a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n \geq 2$. Therefore in the rest of the paper we focus on such quadratic Pisot numbers. At first we give an estimate on $L_\odot(\beta)$ using $L_\oplus(\beta)$ and then we determine the value of $L_\oplus(\beta)$.

5 Relation of $L_\oplus$ and $L_\odot$ for quadratic Pisot numbers

In Section 2 we have shown that $\mathbb{Z}_\beta$ can be embedded into a cut-and-project sequence with a suitably chosen window. In our case β is a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n \geq 2$. Therefore we chose $\Sigma(H)$, where $H = \frac{m}{1-\beta'^2}$. We show that a cut-and-project set with arbitrary window can be embedded into a finite union of shifted copies of $\mathbb{Z}_\beta$ where the shifts belong to $\mathbb{Z}[\beta]$. In fact, a product xy of $x, y \in \mathbb{Z}_\beta$ can be expressed as a sum of a β-integer and a small rational integer and therefore we can find an upper estimate of $L_\odot(\beta)$ using $L_\oplus(\beta)$. Similar result can be proven also for non quadratic Pisot β. The demonstration is however rather technical.

Theorem 5.1. *Let $\beta > 1$ be a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$, and let $h > 0$. Then there exists $p \in \mathbb{N}$, such that*

$$\Sigma(h) \quad \subset \quad \mathbb{Z}_\beta + \{-p, -p+1, \ldots, -1, 0, 1, \ldots, p-1, p\},$$

where

$$p \leq h - \beta' H = h - \beta' \frac{m}{1-\beta'^2}.$$

Proof. Since β is a quadratic integer, we can rewrite every power β^k as a integer combination of 1 and β. Let us define F_k, G_k by

$$\beta^k = F_k \beta + G_k.$$

Since $\beta^{k+1} = \beta(F_k\beta + G_k) = F_k m\beta + F_k n + G_k \beta$, the sequences $(F_k)_{k\in\mathbb{N}_0}$, $(G_k)_{k\in\mathbb{N}_0}$ satisfy $F_{k+1} = mF_k + G_k$, $G_{k+1} = nF_k$, which gives a recurrence relation

$$F_{k+2} = mF_{k+1} + nF_k, \qquad \text{where} \quad F_0 = 0, \quad F_1 = 1.$$

It is easy to see that every $x \in \mathbb{N}$ can be written in the form $x = \sum_{i=1}^{j} c_i F_i$, where $c_i \in \{0, 1, \ldots, m\}$ and $c_i c_{i-1}$ is lexicographically smaller than mn. The coefficients $c_j c_{j-1} \ldots c_1$ can be found by the so-called 'greedy algorithm'. Thus j is a number for which $F_j \leq x < F_{j+1}$ and $c_j := \left[xF_j^{-1}\right]$. We obtain coefficients c_i, $i < j$, by applying the same steps to the integer $\tilde{x} = x - c_j F_j$.

Let $z \in \Sigma(h)$, i.e. $z = a + b\beta$ and $|z'| < h$. Since both $\Sigma(h)$ and $\mathbb{Z}_\beta$ are symmetric with respect to the origin, it suffices to show the statement for $b \geq 0$. Let us express $b = \sum_{i=1}^{j} c_i F_i$. Then

$$z = \sum_{i=1}^{j} c_i(F_i\beta + G_i) - \sum_{i=1}^{j} c_i G_i + a = z_1 + z_2, \tag{5}$$

where $z_2 := a - \sum_{i=1}^{j} c_i G_i \in \mathbb{Z}$ and $z_1 := \sum_{i=1}^{j} c_i \beta^i \in \beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. Applying the Galois automorphism to the equality $z = z_1 + z_2$ gives $z_2 = z' - z_1'$. Since $|z'| < h$ and $|z_1'| < -\beta' H$, the integer z_2 belongs to the interval $(-h + \beta' H, h - \beta' H)$. □

Corollary 5.2.

$$\mathbb{Z}_\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{-p, \dots, p\}\,, \qquad \textit{where} \ \ p \le \frac{(m+2)^4}{4}\,.$$

Proof. Since $\mathbb{Z}_\beta \subset \Sigma(H)$, we have $\mathbb{Z}_\beta\mathbb{Z}_\beta \subset \Sigma(H^2)$. The proof will be completed if we verify that $H^2 - \beta' H \le \frac{1}{4}(m+2)^4$. Let us first show that

$$\frac{1}{1-\beta'^2} < \frac{m+3}{2}\,. \tag{6}$$

We have $-\beta' = \frac{n}{\beta}$, thus for $n \le m-1$

$$1-\beta'^2 = 1 - \frac{n^2}{\beta^2} > 1 - \frac{n^2}{m^2} \ge 1 - \frac{(m-1)^2}{m^2} = \frac{2m-1}{m^2} \ge \frac{2}{m+3}\,.$$

For $n = m$ the inequality (6) is verified directly using $\beta' = \frac{1}{2}(m - \sqrt{m^2+4m})$. Therefore

$$\begin{aligned} H^2 - \beta' H \ &\le\ H^2 + H = \frac{m^2}{(1-\beta'^2)^2} + \frac{m}{1-\beta'^2} < \\ &<\ \frac{m^2(m+3)^2}{4} + \frac{m(m+3)}{2} = \\ &=\ \frac{1}{4}m(m+1)(m+2)(m+3)\ \le\ \frac{1}{4}(m+2)^4\,. \end{aligned}$$

□

The above corollary states that a product of two β-integers can be written as a sum of a β-integer and a rational integer. Let us derive the number of fractional digits of the β-expansion of a rational integer p.

Lemma 5.3. *Let $p \in \mathbb{N}$. Then*

$$\mathrm{fp}_\beta(p) \le (1 + \log_2 p) L_\oplus(\beta)\,.$$

Proof. The proof is based on a simple observation that

$$\mathrm{fp}_\beta(x+y) \le \max\big\{\mathrm{fp}_\beta(x), \mathrm{fp}_\beta(y)\big\} + L_\oplus(\beta)\,, \tag{7}$$

which in particular gives $\mathrm{fp}_\beta(2x) \le \mathrm{fp}_\beta(x) + L_\oplus(\beta)$. Applying the latter k-times we obtain $\mathrm{fp}_\beta(2^k) \le kL_\oplus(\beta)$. We use mathematical induction on j to prove that if p has a binary expansion $p = \sum_{i=0}^{j} a_i 2^i$ then $\mathrm{fp}_\beta(p) \le (j+1)L_\oplus(\beta)$. Using the hypothesis for $p = \sum_{i=0}^{j} a_i 2^i = 2^j + \sum_{i=0}^{j-1} a_i 2^i$ we obtain

$$\begin{aligned} \mathrm{fp}_\beta(p) &\le \max\left\{\mathrm{fp}_\beta(2^j), \mathrm{fp}_\beta\Big(\sum_{i=0}^{j-1} a_i 2^i\Big)\right\} + L_\oplus(\beta) \le \\ &\le \max\{jL_\oplus(\beta), jL_\oplus(\beta)\} + L_\oplus(\beta) = (j+1)L_\oplus(\beta)\,. \end{aligned}$$

The statement of the lemma follows easily from the fact that $j \le \log_2 p$. □

The following theorem is a simple consequence of Corollary 5.2 and Lemma 5.3.

Theorem 5.4. *Let $\beta > 1$ be a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$. Then*

$$L_{\odot}(\beta) \leq 4L_{\oplus}(\beta)\log_2(m+2)\,.$$

Proof. Let $x, y \in \mathbb{Z}_\beta$. Using Corollary 5.2, we have $\mathrm{fp}_\beta(xy) = \mathrm{fp}_\beta(z+p)$ for some $z \in \mathbb{Z}_\beta$ and $p \in \mathbb{N}$, $p \leq \frac{1}{4}(m+2)^4$. Now due to (7)

$$\mathrm{fp}_\beta(z+p) \leq \ \mathrm{fp}_\beta(p) + L_{\oplus}(\beta) \ \leq \ (2 + \log_2 p)L_{\oplus}(\beta) \ \leq$$

$$\leq \left(2 + \log_2 \frac{(m+2)^4}{4}\right) L_{\oplus}(\beta)\,.$$

The statement of the theorem follows easily.

□

6 $L_{\oplus}$ for quadratic β

In this section we obtain an upper bound to $L_{\oplus}(\beta)$. This is done in two steps: first we find an upper bound to $\mathrm{fp}(\mathrm{x} + \mathrm{y})$ where x is an arbitrary β-integer and y is a β-integer of a specific form. Then we show that any β-integer can be written as a finite sum of numbers in this specific form. An upper bound to $L_{\oplus}(\beta)$ is obtained by combining both results.

Let $\beta > 1$ be a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$. Let $(x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots x_{-p}$ be a β-representation of x, i.e. $0 \leq x_i \leq m$. The β-representation $(x)_\beta$ is a β-expansion of x if and only if $x_i x_{i-1}$ is lexicographically smaller than $mn = d(1, \beta)$ for every i.

The following lemma is an easy consequence of the result of Frougny and Solomyak in [4]. It is mentioned here in order to make the article self-contained.

Lemma 6.1. *Let $(x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots x_{-p}$ be a β-representation of x. Then $\mathrm{fp}_\beta(x) \leq p$.*

Proof. If the representation is already in the form of a β-expansion, then $\mathrm{fp}_\beta(x) = p$. Otherwise we can find the largest j such that $x_j x_{j-1}$ is lexicographically bigger or equal to mn. Since $x_i \leq m$ for all i, necessarily $x_j = m$ and $x_{j-1} \geq n$. Since j was the largest index with this property, $x_{j+1} < m$. Therefore we can define a new representation of x as $(x)_\beta = \tilde{x}_k \tilde{x}_{k-1} \ldots \tilde{x}_1 \tilde{x}_0 \bullet \tilde{x}_{-1} \tilde{x}_{-2} \ldots \tilde{x}_{-p}$ where $\tilde{x}_j := x_j - m$, $\tilde{x}_{j-1} := x_{j-1} - n$, $\tilde{x}_{j+1} := x_{j+1} + 1$, and $\tilde{x}_i = x_i$ otherwise. In the new representation the sum of digits is strictly smaller than in the previous one. This procedure can be repeated and in finitely many steps we obtain the β-expansion of x. The result follows easily, since in each step the number of digits in the fractional part of the representation does not increase. □

Let us first determine a lower bound to $L_\oplus(\beta)$. It suffices to find a single example of addition with specified fractional part length. We use the following example.

Example 6.2. Consider $x = m\sum_{i=0}^{k-1}\beta^{2i}$. Then it can be shown by induction on k that

$$x + x = \sum_{i=0}^{k-1}\left(A_{k-i}\beta + B_{k-i}\right)\beta^{2i} + \sum_{i=0}^{k-1}\left(\frac{a_{k-i}}{\beta} + \frac{b_{k-i}}{\beta^2}\right)\beta^{-2i},$$

where the coefficients A_i, B_i, a_i and b_i, $i \in \mathbb{N}$, are defined by

$$\begin{aligned} A_i &= i(m-n+1) - m + n \\ B_i &= 2m - n - i(m-n+1) \\ a_i &= i(m-n+1) - 1 \\ b_i &= m + 1 - i(m-n+1) \end{aligned}$$

Formally written, we have

$$x + x = A_1B_1A_2B_2\ldots A_kB_k \bullet a_kb_k\ldots a_2b_2a_1b_1$$

The above expression is a β-expansion if and only if all the coefficients A_i, B_i, a_i and b_i take values in $\{0,1,\ldots,m\}$, for $i = 1,\ldots,k$. This implies the following conditions on k,

$$\begin{aligned} k(m-n+1) &\leq m+1 \\ (k-1)(m-n+1) &\leq m-1 \end{aligned}$$

For $m = n$ the latter condition is stronger and the maximal k satisfying it is $k = m$. If on the other hand $m > n$, the first condition is stronger and the maximal $k \in \mathbb{N}$ satisfying it is

$$k_0 := \left[\frac{m+1}{m-n+1}\right].$$

Corollary 6.3. *Let β be the larger solution of $x^2 = mx + n$, $m, n \in \mathbb{Z}$, $m \geq n > 0$. Then*

$$L_\oplus(\beta) \geq \begin{cases} 2m & \text{if } m = n \\ 2k_0 & \text{if } m > n. \end{cases}$$

From now on we focus on determining the upper bound for $L_\oplus(\beta)$.

Lemma 6.4. *Let $x, y \in \mathbb{Z}_\beta$, $x, y \geq 0$, with β-expansions*

$$\begin{aligned} (x)_\beta &= x_\ell x_{\ell-1}\ldots x_1x_0\bullet \\ (y)_\beta &= y_ky_{k-1}\ldots y_1y_0\bullet \end{aligned}$$

where $y_i \leq m-n+1$ *for* $i = 0, 1, \dots, k-2, k-1$. *Then the* β*-expansion of* $x+y$ *is equal to*

$$(x+y)_\beta = z_r z_{r-1} \dots z_1 z_0 \bullet z_{-1} z_{-2}$$

where

$$\frac{z_{-1}}{\beta} + \frac{z_{-2}}{\beta^2} \quad \in \quad \left\{ 0 \; , \; \frac{n}{\beta} \; , \; \frac{m-n}{\beta} + \frac{n}{\beta^2} \left(= 1 - \frac{n}{\beta} \right) \right\} .$$

Proof. We make use of the relation $m+p = \beta + p - 1 + (m-n)\beta^{-1} + n\beta^{-2}$, for $p \leq m$, i.e. $(m+p)_\beta = 1(p-1) \bullet (m-n)n$. Symbolically it may be rewritten as

$$\begin{array}{rr|rr} & m & & \\ + & p & & \\ \hline 1 & (p-1) & (m-n) & n \end{array} \tag{8}$$

In the proof of the lemma we proceed by induction on the values of y. Let $y = y_0 \leq m-n+1$. Then according to (8), the β-representation of $x+y$ is

$$(x+y)_\beta = \begin{cases} x_\ell \dots x_1 (x_0 + y_0) \bullet \, , & \text{if } \; x_0 + y_0 \leq m \\ x_\ell \dots (x_1+1)(x_0+y_0-m-1) \bullet (m-n)n \, , & \text{if } \; x_0 + y_0 > m \end{cases}$$

Note that $x_1 + 1 \leq m$ in the second case, since $x_1 = m$ implies $x_0 \leq n-1$, and thus $x_0 + y_0 \leq n - 1 + m - n + 1 = m$, which is a contradiction.

Now assume that the statement holds for all $\tilde{y} < y$ satisfying the conditions of the lemma. Suppose that there exists an index i such that $y_i > 0$ and $x_i < m$. Then $x+y = \tilde{x} + \tilde{y}$, where according to Lemma 6.1 $\tilde{x} = x + \beta^i \in \mathbb{Z}_\beta$ and $\tilde{y} = y - \beta^i$ satisfies the conditions of the lemma. We may thus use the induction hypothesis.

Suppose that $y_i > 0$ implies $x_i = m$ for all $i \leq k$. Since $x_\ell x_{\ell-1} \dots x_1 x_0$ is an expansion, $x_i = m$ implies $x_{i-1} \leq n-1 < m$. Thus $y_i > 0$ implies $y_{i-1} = 0$. Since $y_k > 0$, we have $x_k = m$ and $x_{k+1} < m$. Without loss of generality we can consider only the case when $l \leq k+1$. Therefore we have the following situation

$$\begin{array}{cccccccc|} x_{k+1} & m & x_{k-1} & x_{k-2} & \dots & x_1 & x_0 \\ & y_k & 0 & y_{k-2} & \dots & y_1 & y_0 \\ \hline \end{array}$$

Let j be the smallest integer among $\{1, 2, \dots, [k/2]\}$ such that $y_{k-2j} < m-n+1$. Then

$$\begin{array}{cccccccccccc|} x_{k+1} & m & x_{k-1} & m & \dots & x_{k-2j+3} & m & x_{k-2j+1} & x_{k-2j} & x_{k-2j-1} & \dots & x_0 \\ & y_k & 0 & m-n+1 & \dots & 0 & m-n+1 & 0 & y_{k-2j} & y_{k-2j-1} & \dots & y_0 \\ \hline \end{array}$$

We may check by elementary algebra using the relation $\beta^2 = m\beta + n$ that

$$
\begin{aligned}
(9)\qquad & m\beta^k + (m-n+1)\sum_{i=1}^{j-1}\beta^{k-2i} = \\
& = \beta^{k+1} - \beta^k + (m-n+1)\beta\sum_{i=1}^{j-1}\beta^{k-2i} + (m-n)\beta^{k-2j+1} + n\beta^{k-2j}\,.
\end{aligned}
$$

Using this relation, we may write the sum $x + y = \tilde{x} + \tilde{y}$ as

$$
\begin{array}{cccccccccccc|}
(x_{k+1}{+}1) & (y_k{-}1) & \widetilde{x}_{k-1} & m & \dots & \widetilde{x}_{k-2j+3} & m & \widetilde{x}_{k-2j+1} & x_{k-2j} & x_{k-2j-1} & \dots & x_0 \\
 & & & & & & & & (y_{k-2j}{+}n) & y_{k-2j-1} & \dots & y_0 \\
\hline
\end{array}
$$

where $\widetilde{x}_{k-2i+1} = x_{k-2i+1} + m - n + 1$ for $i = 1, 2, \dots, j-1$ and $\widetilde{x}_{k-2j+1} = x_{k-2j+1} + m - n$. The first row represents the summand $\tilde{x}$, the second row the summand $\tilde{y}$. Due to (9) we have $x + y = \tilde{x} + \tilde{y}$. Obviously $\tilde{x}, \tilde{y} \in \mathbb{Z}_\beta$, the digits of $\tilde{y}$ are $\leq m - n + 1$, except its first non zero digit from the left. We have $\tilde{y} < y$ and thus we may use the induction hypothesis.

There remains to solve the case where $y_{k-2i} = m - n + 1$ for all $i \in \{1, 2, \dots, [k/2]\}$. Then either

$$y = y_k\ 0\ (m-n+1)\ 0\ (m-n+1)\dots\ 0\ (m-n+1)\,,$$

or

$$y = y_k\ 0\ (m-n+1)\ 0\ (m-n+1)\ \dots\ 0\ (m-n+1)\ 0\,,$$

i.e.

$$y = y_k\beta^k + (m-n+1)\sum_{i=1}^{[k/2]}\beta^{k-2i}$$

for k even or odd. We may deduce from the relation (9) that the results of the addition $x + y$ has fractional part $1 - \frac{n}{\beta}$ and $\frac{n}{\beta}$ respectively. This completes the proof. $\square$

Lemma 6.5. *Let $x, y \in \mathbb{Z}_\beta$, $x > y \geq 0$. Then*

$$
x - y = \begin{cases}
z \\
z + (n-1)\displaystyle\sum_{i=0}^{k}\beta^{2i} + 1 & \text{with } z \in \mathbb{Z}_\beta,\ z \geq 0,\ k \geq 0\,. \\
z + (n-1)\displaystyle\sum_{i=1}^{k}\beta^{2i-1} + \frac{n}{\beta}
\end{cases}
$$

Proof. First note that for every $x \in \mathbb{Z}_\beta$ there exists a β-representation $(x)_\beta = x_\ell \dots x_1 x_0\bullet$ such that $x_i + x_{i-1} > 0$ for all $0 < i \leq \ell$, i.e. the β-representation is 'dense'. The dense form can be found by the following procedure: Find the first pair of zeros from the left, say $x_i = x_{i-1} = 0$,

$x_{i+1} > 0$. Put $\tilde{x}_{i+1} = x_{i+1} - 1$, $\tilde{x}_i = m$, $\tilde{x}_{i-1} = n$, and $\tilde{x}_j = x_j$ for all other $0 \le j \le \ell$. The new β-representation $(x)_\beta = \tilde{x}_\ell \dots \tilde{x}_1 \tilde{x}_0 \bullet$, has strictly lower number of vanishing coefficients. Thus the procedure is finite.

The proof of the lemma is done by induction on the value of y. Without loss of generality we may assume that both $(x)_\beta = x_\ell \dots x_1 x_0 \bullet$ and $(y)_\beta = y_k \dots y_1 y_0 \bullet$ are written in their dense form.

Assume that there is an index i such that both x_i and y_i are non-zero. Then $x - y = \tilde{x} - \tilde{y}$, where $\tilde{x} = x - \beta^i$ and $\tilde{y} = y - \beta^i$. Clearly, $\tilde{x}, \tilde{y} \in \mathbb{Z}_\beta$ and $\tilde{y} < y$, thus we may use the induction hypothesis.

Assume that $y_i > 0$ implies $x_i = 0$ for all indices i. Since $x_i + x_{i-1} > 0$, we have $y_{i-1} = 0$. Since $y_k > 0$, we have $x_k = 0$ and $x_{k+1} > 0$. Without loss of generality we consider $l = k + 1$ and $x_{k+1} = 1$. Since both x and y are in their dense form, the remaining cases are as follows. First assume that the maximal index k such that y_k is non-zero, is even. We have $x - y$ equal to

$$
\begin{array}{cccccccc|c}
1 & 0 & x_{k-1} & 0 & x_{k-3} & \dots & x_1 & 0 & \\
- & y_k & 0 & y_{k-2} & 0 & \dots & 0 & y_0 & \\
\hline
1 & 0 & x_{k-1} & 0 & x_{k-3} & \dots & x_1 & 0 & \\
-\,1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \\
+ & m & (n-1) & m & (n-1) & \dots & (n-1) & m & n \\
- & y_k & 0 & y_{k-2} & 0 & \dots & 0 & y_0 & \\
\hline
 & (m-y_k) & x_{k-1} & (m-y_{k-2}) & x_{k-3} & \dots & x_1 & (m-y_0) & \\
+ & & (n-1) & 0 & (n-1) & \dots & (n-1) & 0 & n \\
\hline\hline
\end{array}
$$

which corresponds to the statement of the lemma. For k odd we may write similarly that $x - y$ equals to

$$
\begin{array}{ccccccccc|}
1 & 0 & x_{k-1} & 0 & x_{k-3} & \dots & x_2 & 0 & x_0 \\
- & y_k & 0 & y_{k-2} & 0 & \dots & 0 & y_1 & 0 \\
\hline
 & (m-y_k) & x_{k-1} & (m-y_{k-2}) & x_{k-3} & \dots & x_2 & (m-y_1) & x_0 \\
+ & & (n-1) & 0 & (n-1) & \dots & (n-1) & 0 & n \\
\hline\hline
\end{array}
$$

which is of the desired form. $\square$

Theorem 6.6. *Let β be the larger solution of $x^2 = mx + n$, $m, n \in \mathbb{Z}$, $m \ge n > 0$. Then*

$$L_\oplus(\beta) = 2m \qquad \textit{if} \qquad m = n$$

and

$$2\left\lfloor \frac{m+1}{m-n+1} \right\rfloor \;\le\; L_\oplus(\beta) \;\le\; 2\left\lceil \frac{m}{m-n+1} \right\rceil \qquad \textit{if} \qquad m > n\,.$$

Proof. Let $x, y \in \mathbb{Z}_\beta$, $xy > 0$. Every y may be splitted into a sum $y = y_{(1)} + \dots + y_{(s)}$, for some s, where the summands $y_{(i)}$ have digits $\le m - n + 1$, and thus satisfy the assumptions of Lemma 6.4. We can always choose $y_{(i)}$ in such a way that the sum has at most

$$s_0 := \left\lceil \frac{m}{m-n+1} \right\rceil ,$$

non-vanishing summands. Lemma 6.4 then implies that $\mathrm{fp}_\beta(x+y) \leq 2s_0$.

Now let $xy < 0$, without loss of generality $x > -y$. Then according to Lemma 6.5 $x+y$ can be written either as $z+w$ for some $0 \leq z, w \in \mathbb{Z}_\beta$, or $x+y = z+(n-1)\sum_{i=1}^{k} \beta^{2i-1} + \frac{n}{\beta}$ for $0 \leq z \in \mathbb{Z}_\beta$. The sum $(n-1)\sum_{i=1}^{k} \beta^{2i-1}$ can be written as addition of $\lceil \frac{n-1}{m-n+1} \rceil = s_0 - 1$ summands with digits $\leq m-n+1$. Therefore

$$\mathrm{fp}_\beta \left(z + (n-1) \sum_{i=1}^{k} \beta^{2i-1} \right) \leq 2(s_0 - 1) \, .$$

Adding $\frac{n}{\beta}$ to the result only two more fractional digits may arise, cf. Lemma 6.4.

Thus the proof for the upper bound to $L_\oplus(\beta)$ is finished. The lower bound to $L_\oplus(\beta)$ is given by Corollary 6.3. □

Last two sections were devoted to the study of arithmetics on β-expansions for $\beta > 1$, a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$. This is the case where Theorem 3.3 does not provide us with any results, since $K = 0$. Let us comment on the results obtained in Sections 5 and 6:

1. The lower and upper bound for $L_\oplus(\beta)$ found in Theorem 6.6 differ at most by 2. They coincide if and only if

   $$m - n + 1 \ \text{ divides } \ m \ \text{ or } \ m + 1.$$

   Based on observation, we conjecture that for $m > n$ we actually have $L_\oplus(\beta) = 2k_0$. We also note that for $m > n$ the results of subtraction $x - y$, where $x, y > 0$, has lower number of fractional digits than addition, more precisely, $\mathrm{fp}_\beta(x - y) \leq 2k_0 - 1$.

2. According to Theorem 5.4 we may use the bound to $L_\oplus(\beta)$ to derive an upper estimate to $L_\odot(\beta)$. For example for $m = n$ this gives

   $$L_\odot(\beta) \leq 8m\big(\log_2(m+2)\big) \, .$$

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