MIKHAILOV : INTRODUCTION : WHAT IS INTEGRABILITY

! NO UNIVERSAL DEFINITION OF INTEGRABILITY

! —— " —————— OF LAX PAIR !

~~A. DEGASPERIS~~

NOVIKOV : What for integrability ?

KdV hierarchy = theory of isospectral deformations
of Sturm - Liouville operator
("spectral symmetry group")

Is it possible to use this connection for the theory
of fundamental linear operators?

- YES I. 1-D operators periodic problems
or rapidly decreasing b. c.
II. 2-D operators — " —
— assoc. with one energy level

1-D operators :
algebraic assumption $[L, A] = 0$ ⇒ L is a "finite-gap" operator etc.

completely integrable
n-dim. Hamilt. system
(⇒ periodic or quasi-periodic solns)

Quantum elliptic Calogero - Moser model

$$L = \sum_{i=1}^{n+1} \partial_i^2 + g \sum_{i<j} \wp (x_i - x_j)$$

(n+1) particles
total momentum = 0
⇒ dim = n

known to be integrable algebraically

spectrum unknown (⇐ problems with singular potentials

=> in such case no known connection between algebraic theory and spectral theory

Another example
2 D Schrödinger op.

$$L = \partial\bar{\partial} + A\bar{\partial} + B\partial + 2V$$

„magnetic field" potential

$$2H = -\frac{\partial A(z,\bar{z})}{\partial z} + \frac{\partial B(z,\bar{z})}{\partial \bar{z}} \qquad V$$

$L \to fLf^{-1}, \psi \to f\psi$ gauge transformations

Lax pair associated with one energy level

$$\frac{\partial L}{\partial t} = [L, A] + B \cdot L$$

Inverse problems $L\psi = 0$ (one energy level)

Exactly solvable cases:

1. Periodic case: One Fermi level is „algebraic" ($L\psi = 0$ => Riemann surface with finite genus)
2. Rapidly decreasing case: Scattering matrix is trivial for one energy level only

Discrete spectral symmetries

1. Darboux or Backlund transformations for 1D Schrödinger operators $L\psi = \lambda\psi$
2. 2 D Schrödinger equations $L\psi = 0$ ... Laplace transformations

How to use them for the spectral theory

1-D case $L = -\partial^2 + u + c = -(\partial + a)(\partial - a)$

$$L \to \tilde{L} = -(\partial - a)(\partial + a)$$

$L\psi = \lambda\psi \Rightarrow \psi \to \tilde{\psi} = (\partial - a)\psi : \tilde{L}\tilde{\psi} = \tilde{\lambda}\tilde{\psi}$

i.e. almost isospectral transformation (may kill one eigenfunction)

depends on 2 parameters: c
parameter from solving Riccati eqn $u + c = a^2 + \dots$

Def: „Cyclic chains" $L_0 = L \to \tilde{L} = L_1 \to \tilde{L}_1 = L_2 \to \dots$
$\to L_n = L_0$

$$L_n = B_{c_n} \cdot \ldots \cdot B_{c_1}(L_0) = L_0$$

=> Theorem: $\sum c_i = 0 \Rightarrow L_0$ is „finite-gap" operator for $n = 2k+1$

$c = \sum c_i \neq 0$, $\forall i : L_i$ real smooth (no singularities)
$n = 2k+1$
$\Rightarrow L_0 = -\partial^2 + u$ has discrete spectrum in $L_2(\mathbb{R})$ equal to the union of finite number of [illegible] arithmetical progressions

2D Schröd. ops
a) $L = -(\partial + A)(\bar{\partial} + B) + 2V'$
b) $L\psi = 0 \Rightarrow \tilde{\psi} = (\bar{\partial} + B)\psi$, $\tilde{L}\tilde{\psi} = 0$

where $\tilde{L} = -V'(\bar{\partial} + B)(V')^{-1}(\partial + A) + 2V'$

Proof: elementary

The „inverse factorization" $L = -(\bar{\partial} + B)(\partial + A) + 2V''$
$\Longrightarrow$ inverse transformation

gauge invariance (can be written using gauge invariants)

The infinite chain of Laplace transf's is isomorphic to the 2D Toda lattice

$$H_{n+1} = H_n + \frac{1}{2}\partial\bar{\partial}(f_n) \qquad e^{f_{n+1}} = e^{f_n} + H_{n+1}$$

$V_n = \exp(f_n)$, $H_{n+1} - H_n = \frac{1}{2}\Delta(f_n) = (e^{f_{n+1}} - e^{f_n}) - (e^{f_n} - e^{f_{n-1}})$

def. $f_n = g_n - g_{n-1}$ ⇒ ~~[illegible]~~ 2D Toda chain

$\boxed{\text{Discrete spectral symmetries} \to \text{discrete variable}}$

Cyclic Laplace chain $L_0 \to L_1 \to .. \to L_n = L_0$

Let $\forall i$: L_i are smooth, 2-periodic in R^2 (i.e. physical quantities not coef. in L_i are 2-periodic)
⇒ they are topologically trivial and genus of Fermi curve $L\psi = 0$ is finite.

Topologically nontrivial operators (i.e. magnetic flux through the elementary cell is $\neq 0$)
... under certain assumptions 2 integrable highly degenerate levels

math-ph/0003009, math-ph/0009009 (graphs)

KRUSKAL: Painlevé property

1st application: Kowalewska top
⇒ 1st mystery: why complex independent variable, i.e. complex time?

Painlevé property .. nonlinear analog of ~~the~~ Frobenius theory in ~~neighbourhood~~ neighborhood of singularities in linear ODEs

Riccati: $u' = u^2$ $\qquad u = \frac{-1}{t - t_0} \quad t_0 \in R \cup \{\infty\}$
moveable singularity (pole)

Note: $u' = u^3$ ⇒ multivalued solutions → excluded from our analysis

· 2nd order equations

6 Painlevé equations $\quad P_I$: $u'' = 6u^2 + z$, $\quad P_{II}$: $u'' = 2u^3 + zu_i + \alpha_i$, ...
⇒ solutions .. Painlevé transcendents
singular limits allow to go from 6th to 5th ... to 1st
2nd to 6th .. single poles
1st .. double poles

Naive example $\quad u'' = 6u^2 + z + \cancel{c} \quad (P_I)$
↑ can be rescaled ← we remove by shift

Usual theory ⇒ give $\overbrace{u, u'}^{|\,|<+\infty}$ at $z = z_0$ ⇒ $\exists_1$ analytic solution in neighborhood of z_0

What if $u = \infty$ or $u' = \infty$ at z_0?
Assume $u \sim a(z - z_0)^p$, $\quad p \in C$, $\mathrm{Re}\, p < 0$
asymptotic behavior
also assume $u' \sim ap(z - z_0)^{p-1}$
i.e.: $ap(p-1)(z - z_0)^{p-2} \sim 6a^2(z - z_0)^{2p} + \cancel{z}$
↑ small compared to $(z - z_0)^{2p}$

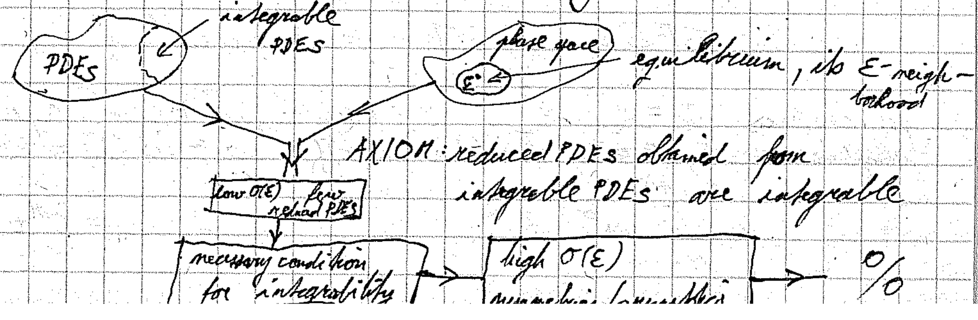
⇒ $p - 2 = 2p$, i.e. $\boxed{p = -2}$ $\wedge$ $6a = 6$, i.e. $\boxed{a = 1}$

⇒ ∃ Laurent series etc. ⇒ ∃ solution with such asymptotics

$$u = \frac{1}{(z - z_0)^2} + k(z - z_0)^2 + \dots + c(z - z_0)^4$$
↑ definite constant ↑ arbitrary const.

DEGASPERIS: Multiscale perturbations

Multiscale expansion and integrability



AXIOM: reduced PDEs obtained from integrable PDEs are integrable

$\rightarrow$ $\boxed{\text{asymptotic integrability}}$

if reduced PDEs is ~~[illegible]~~ not integrable $\Rightarrow$ original PDE non-integrable
is 〃 — $\Rightarrow$???

Example: unharmonic oscillator $\ddot{g} + \omega_0^2 g = c_2 g^2 + c_3 g^3 + \dots$

$$g = g(t, \varepsilon) \quad \text{s.t.} \quad g(0,\varepsilon) = \varepsilon, \quad \dot{g}(0,\varepsilon) = 0$$

$$g(t,\varepsilon) = g\left(t + \frac{2\pi}{\omega(\varepsilon)}, \varepsilon\right)$$

$$\theta = \omega(\varepsilon) t \qquad g(t,\varepsilon) = f(\theta, \varepsilon) \qquad f(\theta,\varepsilon) = f(\theta + 2\pi, \varepsilon)$$

$$\omega^2(\varepsilon) = \omega_0^2 + \varepsilon \gamma_1 + \varepsilon^2 \gamma_2 + \dots, \quad f(\theta,\varepsilon) = \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \dots$$

$\Rightarrow O(\varepsilon): f_1'' + f_1 = 0 \qquad O(\varepsilon^2): f_2'' + f_2 = \{\dots\}$

we don't ~~[illegible]~~ want secular term (resonance) $\Rightarrow \gamma_1 = 0$

similarly $O(\varepsilon^3) \Rightarrow \gamma_2 = -\frac{5c_2^2}{6\omega_0^2} - \frac{3}{4} c_3$

reformulation: $g(t,\varepsilon) = \sum_{m=1}^{n} \sum_{\alpha=-m}^{m} \varepsilon^m f_m^{(\alpha)} e^{i\alpha\theta} + O(\varepsilon^{n+1})$

$$\theta = \omega_0 t + \omega_1 \varepsilon t + \omega_2 \varepsilon^2 t = \sum_{i=0} \omega_i t_i, \quad t_n \equiv \varepsilon^n t$$

$$g(t,\varepsilon) = \sum_{m=1}^{n} \sum_{\alpha=-m}^{m} \varepsilon^m E^{\alpha} \varphi_m^{(\alpha)}(t_1, t_2, \dots) + O(\varepsilon^{n+1})$$

$$E = e^{i\omega_0 t}$$

$$D^{(\alpha)}(\varepsilon) \equiv i\omega_0 \alpha + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^3 \frac{\partial}{\partial t_3} + \dots$$

$$\frac{d}{dt}\left(E^{\alpha} \varphi_m^{(\alpha)}\right) = \dots$$

1+1 dim. dispersive waves

$$Du = F[u, u_x, \dots] \qquad u = u(x,t)$$

for instance $D = \partial_t - \partial_x^2 \qquad F = c u_x^3 + (c_2 u^2 + c_3 u^3 + c_4 u^4 + \dots)_x$

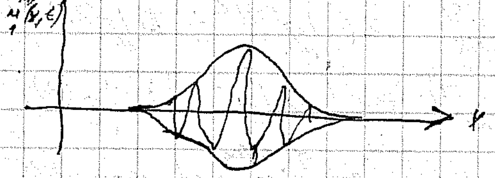
assumptions: $u = u^*$, $E = \exp[i(kx - \omega t)]$, $\omega = \omega(k) = k^3$

$$u = \sum_{\alpha=-\infty}^{+\infty} u^{(\alpha)} E^{\alpha} \qquad \xi = \varepsilon x \qquad t_n = \varepsilon^n t$$

$$u^{(\alpha)} = u^{(\alpha)}(\xi, t_1, t_2, \dots) \qquad u^{(\alpha)} = \sum_{n=1} \varepsilon^n u_n^{(\alpha)}$$

$$u_3^{(0)} = 0 \qquad u_n^{(\alpha)} = 0 \quad \text{for } n < |\alpha|$$

leading $O(\varepsilon)$ term: $u(x,t) = \varepsilon u_1^{(1)} E + \text{c.c.} + O(\varepsilon^2)$



$$\varepsilon = \frac{\Delta k}{k}$$

$$\partial_x\left(E^{\alpha} u^{(\alpha)}\right) = E^{\alpha}\left(i\alpha k + \varepsilon \partial_{\xi}\right) u^{(\alpha)}$$

$$\partial_t\left(E^{\alpha} u^{(\alpha)}\right) = E^{\alpha}\left(\qquad \right) u^{(\alpha)}$$

$$O(\varepsilon^n) = D_0^{(\alpha)} u_n^{(\alpha)} + \dots + D_{n-1}^{(\alpha)} u_1^{(\alpha)} = F_n^{(\alpha)}$$

Definition: the α-th harmonic is at resonance $\Leftrightarrow D_0^{(\alpha)} = 0$

amplitudes of non-resonating harmonics are diff. polynomials of the amplitudes of resonating harmonics.

$\alpha = 0$
- $n = 1$: $0 = 0$
- $n = 2$: $\partial_1 u_1^{(0)} = F_2^{(0)} = 0 \Rightarrow u_1^{(0)} = 0$
- $n = 3$: $\partial_3 u_2^{(0)} = F_3^{(0)} = 2c_2 \partial_{\xi} |u_1^{(1)}|^2$

$\alpha = 1$
- $n = 1$: $0 = 0$
- $n = 2$: $(\partial_1 + 3k^2 \partial_{\xi}) u_1^{(1)} = F_2^{(1)} = 0 \Rightarrow u_2^{(0)} = -\frac{2c_2 |u_1^{(1)}|^2}{3k^2}$
- $n = 3$: $(\partial_1 + 3k^2 \partial_{\xi}) u_2^{(1)} + (\partial_2 - 3ik \partial_{\xi}^2) u_1^{(1)} = F_3^{(1)}$

$\Rightarrow$ ~~$u_2^{(1)} = \frac{c_2}{3k^2} (u_1^{(1)})^2$~~

$$u_2^{(1)} = \frac{c_2}{3k^2} (u_1^{(1)})^2$$

Proposition. $\partial_t v + Lv = w(t)$, $v = v(t)$, $\partial_t w + Lw = 0$
$\Rightarrow$ $w(t)$ is secular, indeed $v_0(t) = t\, w(t)$ is a particular solution

$\Rightarrow \dots \Rightarrow$ NLS as a condition for not having a secular behavior

$$i u_{1 t_2}^{(1)} = \frac{d^2\omega}{dk^2}\left(u_{1\xi\xi}^{(1)} - 2c |u_1^{(1)}|^2 u_1^{(1)}\right)$$

original eqn. not integrable

integrable if $c \in \mathbb{R}$, if for given PDE results NLS for $\operatorname{Im} c \neq 0 \Rightarrow$

ABLOWITZ Discrete NLS Equations

NLS (fiber) optics $\quad iu_z + u_{tt} + 2|u|^2u = 0$

vector NLS systems $\quad \vec{E} = \binom{u}{v} \quad$ 2 coupled eqns. for u, v

$$\Rightarrow \quad i g^{(1)}_t = g^{(1)}_{xx} + 2\left(|g^{(1)}|^2 + B|g^{(2)}|^2\right) g^{(1)}$$

$$i g^{(2)}_t = g^{(2)}_{xx} + 2\left(B|g^{(1)}|^2 + |g^{(2)}|^2\right) g^{(2)}$$

$B = 1 \Rightarrow$ integrable (energy of solitons conserved, phase shift)

even generalisation $\quad i\vec{g}_t = \vec{g}_{xx} + 2\left(\sum_i |g^{(i)}|^2\right)\vec{g}$

is integrable

discrete version $\quad i u_{nz} + \dfrac{u_{n+1} + u_{n-1} - 2u_n}{2h^2} + (NL)_n = 0$

(a) $(NL)_n = |u_n|^2 u_n$ $\quad$ fiber optics, condensed matter physics

(b) $(NL)_n = |u_n|^2 \left(\dfrac{u_{n+1} + u_{n-1}}{2}\right)$ integrable

$$u_n(t) = A e^{-i\Phi}\left(\operatorname{sech}(\theta_n - \hat{\theta}_0)\right)$$
$$\theta_n = \alpha(hn - vt), \; v = \ldots\ldots$$

SHABAT: Isospectral symmetries

$$(*)\begin{cases} \psi_{xx} = U(x,t,\lambda)\,\psi & U \text{ polynomial in } \lambda \\ \psi_t + \lambda_t \psi_\lambda = A(x,\lambda)\,\psi_x + B(x,\lambda)\,\psi & A(x,\lambda) \text{ "symmetry" polynomial in } \lambda \end{cases}$$

? $\exists\, \psi(x,t,\lambda)$ s.t. ⤴, i.e. compatibility of eqns above?

most interesting cases $U = u - \lambda$, $U = \lambda^2 + U_1(x)\lambda + U_0(x)$

$\phi \equiv \log\psi \quad$ e.g. $\psi_{xx} = (u - \lambda)\psi \Rightarrow u = \phi_{xx} + \phi_x^2 + \lambda$

later we will use $A = a_1(x)\lambda + a_0(x)$

Compatibility conditions for $(*)$

$$2(U_t + \lambda_t U_\lambda) = 4UA_x + 2U_xA - A_{xxx}$$

Derivation: $\quad \psi_{xt} + \lambda_t\psi_{x\lambda} = A_x\psi_x + A(U\psi) + B_x\psi + B\psi_x \quad / \partial_x$
$\qquad = \tilde{A}\psi_x + \tilde{B}\psi$

$$\Rightarrow \psi_{xxt} + \lambda_t\psi_{xx\lambda} = \tilde{\tilde{A}}\psi_x + \tilde{\tilde{B}}\psi$$
$$= \lambda_t U_\lambda\psi + \overbrace{U\psi_t + \lambda_t U_\lambda\psi + \lambda_t U\psi_\lambda}^{=A\psi_x + B\psi}$$

comparing coeffs. of ψ, ψ_x

$$\Longrightarrow 2B_x = -A_{xx},\; B_{xx} + 2A_xU + AU_x = U_t + \lambda_t U_\lambda$$

Ex 1: $\quad A = a(x) \qquad U = u - \lambda$

$\Rightarrow$ scaling transf. $\quad \psi(x,\lambda) = \hat{\psi}(\hat{x},\hat{\lambda}),\; \hat{x} = x/q,\; \hat{u} = q^2u,$
$\qquad \hat{\lambda} = q^2\lambda$

$$2(u_t - \lambda_t) = 4(u - \lambda)a_x + 2au_x - a_{xxx}$$

assume $\lambda_t = \varepsilon_1\lambda + \varepsilon_0 \qquad \Rightarrow \boxed{a_{xx} = 0}$

if $\varepsilon_1 \neq 0 \Rightarrow$ without loss of generality $A = x$

$\Rightarrow \ldots \Rightarrow \psi_t + 2\lambda\psi_\lambda = x\psi_x,\; u_t = 2u + xu_x$

Characteristics: $\quad \dfrac{dt}{1} = \dfrac{dx}{-x} = \dfrac{d\lambda}{2\lambda} = \dfrac{du}{2u} \Rightarrow$ symmetry

Ex2: $\quad \lambda_t = 0 \qquad A = a_1(x)\lambda + a_0(\lambda) \overset{\text{comp.}}{\underset{\text{cond.}}{\Longrightarrow}} A = 4\lambda + 2u$

$$\Rightarrow \phi_t + \phi_{xxx} - 6\lambda\phi_x = 2\phi_x^3 \qquad MKdV$$

Ex 3: $\quad A = \lambda^{-1}a(x) \qquad \lambda\psi_t = a\psi_x - \frac{1}{2}a_x\psi \Rightarrow u_t = -2a_x$
$\qquad u = \phi_{xx} + \phi_x^2 \quad \Rightarrow \quad \phi_{xy} + e^{2\phi} = \alpha e^{-2\phi} \quad$ SG in exp form

We have reduced comp. cond. to one PDE, it is solvable due to theorems of PDEs.

Stationary case $\psi_t = 0$ together with $\lambda_t = 0$

$$\Rightarrow \frac{d}{dx}\left(-2AA_x + A_x^2 + 4UA^2\right) = -A_{xxx} + 4UA_x + 2U_xA = 0$$

$\lambda_t = 0 \Rightarrow$ ODEs (Painlevé I .. Painlevé V depending on coeffs. $\lambda_t = \varepsilon_1\lambda^2 + \varepsilon_2\lambda + \varepsilon_0$)

assume $\psi_{t_j} + \lambda_{t_j} \psi_x = A_j(x,\lambda)\psi_x + B_j(x,\lambda)\psi$

$\psi_{t_1 t_2} - \psi_{t_2 t_1} \Rightarrow A = D_2(A_1) - D_1(A_2) + A_1 A_{2x} - A_2 A_{1x}$

$\lambda_{t_j} = K_j(\lambda)$ $\qquad \lambda_t = K_{12} K_2 - K_{22} K_1$ Lie bracket „$A = \{A_1, A_2\}$"

Theorem:

$-2AA_{xx} + A_x^2 + 4uA^2 = \alpha(\lambda)$, formal solution $A = 1 + \sum \lambda^{-k} a_k$ $(a_k(u,\dots))$

$A_m = \lambda^m(1 + \dots + \lambda^{-m})$

$\Rightarrow$ basis of all isospectral symmetries, they commute in the sense of Lie bracket

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DEGASPERIS:

Summary $\quad Du = F[u]$

$$u = \sum_{m=1} \sum_{\alpha=-\infty}^{+\infty} \varepsilon^m E^\alpha u_m^{(\alpha)}(\xi, t_m)$$

harmonics $e^{i(kx-\omega t)}$

restriction: $u_m^{(\alpha)}$ bounded !
$\Rightarrow$ further conditions

$\omega = \omega(k) : DE = 0$

~~Linear case~~ $\quad D = \partial_t - i\omega(-i\partial_x) \qquad D(E^\alpha u_m^{(\alpha)}) = E D^{(\alpha)} u_m^{(\alpha)}$

$$D^{(\alpha)} = (-i\omega\alpha + i\omega(\alpha k)) + \varepsilon(\partial_{t_1} + v\partial_\xi)$$
$$+ \varepsilon^2(\partial_{t_2} - i\partial_\xi^2) + \varepsilon^3(\dots) + \dots, \quad v = \frac{d}{dk}\omega(k)$$

$\Rightarrow$ up to $\quad t \sim \varepsilon^{-1}$ .... propagation with v, no variation in shape
$\qquad t \sim \varepsilon^{-2}$ ... Schröd. type dispersion
$\qquad$ ......

$\Rightarrow$ approximate solution

Remark: $\quad u_1^{(0)} = 0$
$\qquad u_m^{(\alpha)} = 0 \quad$ for $\quad m < |\alpha|$

Necessary conditions for integrability

$u = u(x,t)$ (real case) $\qquad Du = \frac{\partial^h}{\partial x^h} F(u, u_x, u_{xx}, \dots)$

$D = \frac{\partial}{\partial t} + i\omega\left(-i\frac{\partial}{\partial x}\right) \qquad \omega(k) = \sum_{m=0} R_{2m+1} k^{2m+1}$

$De^{iz} = 0, \qquad z = kx - \omega(k)t$

$$F = \sum_{m=2}^{\infty} \sum_{\substack{0 \le j_1 \le j_2 \le \\ \le \dots \le j_m}} c^{(m)}_{j_1 \dots j_m} u^{(j_1)} u^{(j_2)} \dots u^{(j_m)}, \qquad u^{(j)} = \left(\frac{\partial}{\partial x}\right)^j u$$

(weak nonlinearity)

$\Rightarrow$ systematic derivation of reduced PDEs

$$u(x,t) = \varepsilon\left[\varepsilon^{\gamma_0}\psi_0 + (\psi_1 e^{iz} + c.c) + \varepsilon^{\gamma_2}(\psi_2 e^{2iz} + c.c.) + \dots\right]$$

assume $\gamma_n \ge 0 \qquad \psi_n = \psi_n(\xi, \tau), \ \xi = \varepsilon^p(x - vt),$
$\qquad \tau = \varepsilon^{2p} t, \quad v = v(k) = \frac{d}{dk}\omega(k), \ p > 0$

$$\frac{\partial}{\partial t}(\psi_n e^{inz}) = e^{inz}\left(-in\omega - v\varepsilon^p\frac{\partial}{\partial \xi} + \varepsilon^{2p}\frac{\partial}{\partial \tau}\right)\psi_n$$
$$\frac{\partial}{\partial x}(\text{—}''\text{—}) = e^{inz}\left(ink + \varepsilon^p\frac{\partial}{\partial \xi}\right)\psi_n$$

$$D(\psi_n e^{inz}) = e^{inz} D_n \psi_n$$

$$\varepsilon^{\gamma_n} D_n \psi_n = \left(ink + \varepsilon^p\frac{\partial}{\partial \xi}\right)^h \sum_{m=2} \varepsilon^{m-1} f_n^{(m)}$$

$$f_n^{(m)} = \sum \varepsilon^{\sum \gamma_j} g(n_1, \dots, n_m)\, \psi_{n_1}\psi_{n_2} \dots \psi_{n_m}$$

$$\Rightarrow D_n = iA_n^{(0)} + \varepsilon^p A_n^{(1)}\frac{\partial}{\partial \xi} + \varepsilon^{2p}\left(\frac{\partial}{\partial t} - iA_n^{(2)}\frac{\partial^2}{\partial \xi^2}\right) + O(\varepsilon^{3p})$$

No resonance $\quad A_n^{(0)} \ne 0 \qquad$ Weak resonance $\quad A_n^{(0)} = 0, A_n^{(1)} \ne 0$
Strong resonance $\quad A_n^{(0)} = A_n^{(1)} = 0, \quad A_n^{(2)} \ne 0$

it is sufficient to consider harmonics up to 2 $\Rightarrow$ five equations according to different combination of resonances of different type

$\Rightarrow$ 10 types of reduced PDEs (or systems: 1 evol. PDE + auxiliary conditions)

some of them don't pass Painlevé test (i.e. non-integrable)

Remark: Reduction conserves usually, but not always, interesting properties like integrals of motion, symmetries etc.
(⇒ possibility to go from KdV to NLS deriving integrals of motion ~~etc~~, Lax pair etc.)

No known generalization to lattice calculations.

SOKOLOV: Symmetries and first integrals of finite-dimensional dynamical systems

(1) $\frac{du_i}{dt} = F_i(u_1, \dots, u_n) \qquad i = 1, \dots, n$

$G(u_1, \dots, u_n) \Rightarrow \frac{dG}{dt} = \sum_{k=1}^{n} F_k(u_i) \frac{\partial G}{\partial u_k} \overset{!}{=} X_F(G)$

Point transformations $\hat{u}_i = \phi_i(u_1, \dots, u_n)$, assume $\det J = \det \frac{\partial \hat{u}_i}{\partial u_j} \neq 0$

$\Rightarrow u_i = \hat{\phi}_i(\hat{u}_1, \dots, \hat{u}_n)$ locally

$\sigma : f(u_1, \dots, u_n) \longrightarrow f(\hat{\phi}_1(\hat{u}_1, \dots, \hat{u}_n), \dots, \hat{\phi}_n(\hat{u}_1, \dots, \hat{u}_n))$

A operator $\quad \sigma(A) = \sigma \circ A \circ \sigma^{-1}$
$(\Rightarrow \sigma(A \circ B) = \sigma(A) \circ \sigma(B))$

Lemma: $X = \sum X_k \frac{\partial}{\partial u_k} \Rightarrow \sigma(X) = \sum_k \hat{X}_k \frac{\partial}{\partial \hat{u}_k} \qquad \hat{X}_k = \sigma(X(\phi_k))$

Theorem 1: $u^0 = (u_1^0, \dots, u_n^0)$ s.t. $X_F|_{u^0} \neq 0 \Rightarrow \exists$ transf. in an open ~~neighbourhood~~ neighbourhood s.t. $\hat{X}_F = \frac{\partial}{\partial \hat{u}_1}$ (see Olver)

Not constructive, no one knows how to construct $\hat{u}_i$.

If $\exists X^1, \dots, X^n : [X^i, X^j] = 0, \ \forall i, j \Rightarrow$ known transf. (later)

Linearization of nonlinear ODE: $\bar{u} = u + \epsilon v$, suppose u solution of (1)

$\Rightarrow v_t = F_* v, \quad F_{*ij} = \frac{\partial F_i}{\partial u_j}$ ... Frechet derivative

∀ vector field X we can construct corresponding matrix X_* under transformation $\hat{u}_i = \phi_i(u_j)$

$\hat{X}_* = J X_* J^{-1} + X(J) J^{-1} \qquad J = \frac{\partial \hat{u}_i}{\partial u_j}$

Def: $I = I(u_1, \dots, u_n)$ a first integral of (1)
$\Leftrightarrow \frac{dI}{dt} = X_F(I) = 0$

Def: First integrals $\phi_k(u_j)$ are functionally independent
$\Leftrightarrow \operatorname{rank} \frac{D(\phi_1, \dots, \phi_m)}{D(u_1, \dots, u_n)} = \left| \begin{matrix} \frac{\partial \phi_1}{\partial u_1} \cdots \\ \vdots \\ \frac{\partial \phi_m}{\partial u_1} \cdots \end{matrix} \right| = \underline{\underline{m}}$ (maximal rank)

Theorem 1 $\Rightarrow \exists$ locally $n-1$ functionally independent first integrals

Symmetry (infinitesimal symmetry)

$\sum_k G_k \frac{\partial}{\partial u_k} = X_G$ vector field s.t. $[X_G, X_F] = 0$

equivalent condition: (1) and $\frac{du_i}{d\tau} = G_i(u_1, \dots, u_n)$ are compatible

i.e. $\underbrace{\exists}_{\forall u_0} u(t, \tau) : u(0,0) = u_0$ common solution of (1) &
$\Rightarrow$ we ~~[illegible]~~ have ∀ $u(t)$ a whole family of solutions depend. on 1 param.

equiv. formulation: $\frac{d\vec{G}}{dt} = F_*(\vec{G})$ or $F_*(\vec{G}) - G_*(\vec{F}) = 0$

$\Rightarrow \vec{G} = (G_1, \dots, G_n)$ satisfies linearization of (1)

Integration by quadratures

(1) $I_1, \dots, I_{n-1}$ functionally indep. first integrals

$\Rightarrow \hat{u}_1 = \phi_1, \quad \hat{u}_2 = I_1, \dots, \hat{u}_n = I_{n-1}$

$\Rightarrow \frac{d\hat{u}_1}{dt} = f_1(\hat{u}_1, \dots, \hat{u}_n) \qquad \frac{d\hat{u}_2}{dt} = \dots = \frac{d\hat{u}_n}{dt} = 0$

(2) $n-1$ symmetries either commuting or forming a solvable Lie algebra (S. Lie):

$X^i = X_{G^i}$ s.t. $\det \begin{pmatrix} F_1 & F_2 & \dots & F_n \\ G_1^1 & G_2^1 & \dots & G_n^1 \\ \vdots & & & \vdots \end{pmatrix} \neq 0$

$[\overset{''}{X_i}, \overset{''X\partial}{X_j}]=0 \quad \forall i,j$ $\Rightarrow$ (1) can be locally solved in quadratures

Theorem 2: (1) with k symmetries, $(n-k+1)$ first integrals functionally indep.

$$\text{rank}\begin{pmatrix} F_1 & F_2 & \cdots & F_n \\ G_1^1 & G_2^1 & \cdots & G_n^1 \\ \vdots & & & \\ G_1^k & G_2^k & \cdots & G_n^k \end{pmatrix} \text{ maximal } \wedge [X_i, X_j]=0$$

$\wedge \; X_i(I_j)=0 \Rightarrow$ (1) can be integrated by quadratures (locally)

in fact Liouville theorem is a special case of this theorem (symmetries of the form $\frac{du_i}{d\tau_j} = \{u_i, I_j\}$, $j=1,..,n-1$

i.e. $(n-1)$ symmetries + $(n-1)$ first integrals + hamiltonian $= 2n-1$)

Ex: Rigid body dynamics $\{p_i, p_j\} = \epsilon_{ijk} p_k$, $\{p_i, g_j\} = \epsilon_{ijk} g_k$, $\{g_i, g_j\}=0$

$$H = I_1 = a p_1^2 + b p_2^2 + c p_3^2 + 2x g_1 + 2y g_2 + 2z g_3$$

two integrals $I_1 = g_1^2 + g_2^2 + g_3^2$, $I = p_1 g_1 + p_2 g_2 + p_3 g_3$ are Casimirs $\Rightarrow$ trivial symmetries $\Rightarrow$ one needs one more integral I_4

$I_4 = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 \overset{BÚNO}{\Rightarrow} a=b$, $x=y=0 \Rightarrow$ Lagrange top

I_4 $\underbrace{\text{quadratic in } p \text{ and linear in } g}_{\text{weight } 2}$ $\Rightarrow$ $x=y=z=0$ $I_4 = p_1^2 + p_2^2 + p_3^2$ .. Euler top

in weight 3 no solution, in weight 4 case:

$$I_4 = (p_1^2 - p_2^2 - 2x g_1 + 2y g_2)^2 + 4(p_1 p_2 - y g_1 - z g_2)^2$$

Kowalewski top

Rigid body in ideal fluid .. Kirchhoff equations

$$H = P^T A P + P^T B Q + Q^T C Q$$

4 known integrable cases (one new by Sokolov similar to Kowalewski case)

MASON: Anti-Self-Duality & Integrability

Key equations : (1) Anti-self-dual YM eqns.
(2) Condition that a 4-dim.^{l} conformal structure has A.S.D. Weyl tensor

What are A.S.D. YM eqns?

Needed: vector bundles, connections and curvature

M manifold, x^a local coords. $(a=1,..,n)$

Def: Vector bundle E is a smooth assignment of an abstract vector space V_p to each $p \in M$ ($\dim V_p = n$)

a section s of E .. smooth choice of $s(p) \in V_p$, $\forall p \in M$

a gauge .. a smooth choice of basis of V_p, $\forall p \in M$

$\hookrightarrow$ representation $s(p)$ by $\begin{pmatrix} s_1(p) \\ \vdots \\ s_n(p) \end{pmatrix}$, n smooth functions

Gauge transformation ... a change of basis $\Leftrightarrow$ $n \times n$ p-dependent smooth matrix $\Lambda(p)$

Often we choose Hermitian metric on each V_p & require $\Lambda \in SU(n)$, more generally $\Lambda \in G$ = gauge group

Def: Connection ... derivative operator $s \to D_a s$ $\quad \underset{\llcorner \text{ repr. of } s \text{ in given choice of gauge}}{\underline{s}} \to D_a \underline{s} = \frac{\partial}{\partial x^a}\underline{s} + A_a \underline{s}$

Under gauge transform $A_a \to \hat{A}_a = \Lambda A_a \Lambda^{-1} - \frac{\partial \Lambda}{\partial x^a}\Lambda^{-1}$ covariance ($\Rightarrow \hat{D}_a \hat{\underline{S}} = \Lambda D_a \underline{S}$)

This is the basic dependent variable of a YM theory describing weak, strong interactions, EM etc.

Def: Curvature $F_{ab} = [D_a, D_b] \in$ ~~gl~~(n) or Lie G

gauge transf. $\hat{F}_{ab} = \Lambda F_{ab} \Lambda^{-1}$

Proposition : $F_{ab}=0 \Leftrightarrow \exists$ gauge s.t. $D_a = \frac{\partial}{\partial x^a}$

In 4-dimensions work on $\mathbb{C}^4$ with metric $ds^2 = \eta_{ab} dx^a dx^b$ volume form $dVol = \epsilon_{abcd} dx^a \wedge dx^b \wedge dx^c \wedge dx^d$ $\quad \eta_{ab}$ constant matrix

Define $\quad {}^*F_{ab} = \frac{1}{2}\epsilon_{abcd}\eta^{ce}\eta^{df}F_{ef}$

A.S.D. ~~Eq~~ eqns : $F_{ab} = - {}^{*}F_{ab}$

Properties: A.S.D. YM soln $\Longrightarrow$ soln of YM eqns.
(Bianchi: $D_a F_{bc}$ + cyclic perms $= 0 \Leftarrow$ Jacobi id. $[D,[D,D]]=0$
$\Updownarrow$ $\Longleftrightarrow D_a {}^{*}F^{ab} = 0$
if $F_{ab} = - {}^{*}F_{ab}$ then $D_a F^{ab} = 0$)

Proposition: $^{*}({}^{*}F_{ab}) \underset{\uparrow}{=} F_{ab}$
Euclidean or (2,2) signature only

A.S.D. eqns are 3 matrix eqns on 4 matrix unknowns A_a (connection components)

But we regard A_a and $\hat{A}_a = \Lambda A_a \Lambda^{-1} - \frac{\partial \Lambda}{\partial x^a} \Lambda^{-1}$ are equivalent $\Longrightarrow$ really 3 unknowns

Proposition: This is an integrable system.

Proof: introduce coords $(\omega, \tilde{\omega}, z, \tilde{z})$ on $\mathbb{C}^4$ s.t. $ds^2 = d\omega\, d\tilde{\omega} - dz\, d\tilde{z}$
Euclidean signature ~~so~~ $\tilde{\omega} = \overline{\omega}$, $\tilde{z} = -\overline{z}$
(2,2) —"— $\tilde{\omega} = \overline{\omega}$, $\tilde{z} = \overline{z}$ or $\omega, \tilde{\omega}, z, \tilde{z}$ are real

Lax pair $L = D_\omega - \xi D_{\tilde{z}}$ $\qquad M = D_z - \xi D_{\tilde{\omega}}$

$$[L,M]=0 \Longleftrightarrow \text{A.S.D. YM} \Longleftrightarrow F_{\omega z} = F_{\tilde{\omega}\tilde{z}} = F_{\omega\tilde{\omega}} - F_{z\tilde{z}} = 0$$

Theorem: [Ward] The ASD YM eqns satisfy a form of the Painlevé property.

Theorem: The ASDYM eqns are bi-Lagrangian (formally bi-Hamiltonian) with a recursion operator, conserved quantities & associated hierarchy.

ASDYM eqns are invariant under conformal group (i.e. transf. of $\mathbb{C}^4$ preserving metric up to scale)
$$SO(6,\mathbb{C})/\mathbb{Z}_2 = SL(4,\mathbb{C})/\mathbb{Z}_2 \times \mathbb{Z}_2$$

Reductions will also be integrable (w.r.t. symmetries)

Ex: $G = SL(2,\mathbb{C})$, $(w, \tilde{w}, z, \tilde{z})$ ~~etc~~ real ((2,2) signature)
connection A_a independent of $\tilde{z}$ (invariant under null translation)
$\hookrightarrow$ answer: two cases $A_{\tilde{z}}$ can be reduced to normal form
$A_{\tilde{z}} = c \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $A_{\tilde{z}} \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(1) $A_w = \begin{pmatrix} 0 & \tilde{\psi} \\ \psi & 0 \end{pmatrix}$ $\quad A_z = \frac{1}{4}\begin{pmatrix} -v & 2\psi_y \\ -2\psi_y & v \end{pmatrix} \Longrightarrow i\psi_t = \psi_{xy} + v\psi$
$w = x,\ \tilde{w} = y,\ z = t$ $\qquad v_x = 2\,(|\psi|^2)_{,y}$
2+1 NLS, further symmetry
$\partial_x - \partial_y \Longrightarrow$ 1+1 NLS

(2) $q_{xt} - q_{xxxy} - 8 q_x q_{xy} - 4 q_y q_{xx} = 0$ $\Longrightarrow$ potential form of KdV

Reductions of ASDYM $\Longrightarrow$ partial classification and ordering of integrable systems

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KRUSKAL : Painlevé property

Example: $u'' = 2u^3 + zu + \alpha(z)$ $\qquad$ α analytic in z

z^0 chosen point $\qquad z = z^0 + \varphi \qquad \varphi << 1$

bad points $|u| \to +\infty$ otherwise use usual theorems on ODEs
balance of dominant terms $u'' = 2u^3 + ..$ $\quad 2u'u'' = 2u'u^3 + ..$
$$\Longrightarrow u'^2 = u^4 + \int 2\, dz\, z u u' + 2\int dz\, \alpha u' + K$$
$$\Longrightarrow \frac{u'}{u^2} = \pm\left[1 + \frac{2}{u^4}\int dz\ z u u' + \frac{2}{u^4}\int dz\, \alpha u' + \frac{2K}{u^4}\right]^{1/2}$$
$$\simeq \pm[1 + \dots] \quad \Longrightarrow -\frac{1}{u} = \pm\int_{z_0}^{z} dy\,[1+\dots]$$
$$\Longrightarrow -\frac{1}{u} = \pm(z - z_0) + \dots \qquad u = \frac{\mp 1}{z - z_0} + \dots = \left\{\mp\int_{z_0}^{z} dy\,[1+..]\right\}^{-1}$$

from analysis $u = \frac{\mp 1}{z - z_0} + c\cdot(z - z_0)$, no constant term

$\alpha' \neq 0 \Rightarrow$ log behaviour $\Longrightarrow$ we need $\alpha' = 0$ to pass Painlevé test

usual approach – put in Laurent series and check whether the eqns can be satisfied, if not $\Longrightarrow$ there should be some logarithm
if there is no indication of log in power series, there still may be some branching in exponentially ~~smaller~~ terms (or essential singularity)

Ex: $u'' = u'^2 \frac{2u - 1}{u^2 + 1}$ $\qquad$ simple poles, but $u = \tan \ln(Az + B)$
$\Longrightarrow$ essential singularity (cluster of poles)

$\Longrightarrow$ Limit point of poles usually causes problems.

NOVIKOV :

Bäcklund transf. (Darboux) L linear op.

strong factorization a) $L = QQ^{+} + c$, $c \in \mathbb{R}$

(self-adjoint factorization)

b) $L = Q_1 Q_2 + c$ (general factorization)

Backlund transf. a) $L \to \tilde{L} = Q^{+}Q + c$, $\psi \to \tilde{\psi} = Q^{+}\psi$

b) $L \to \tilde{L} = Q_2 Q_1 + c$, $\psi \to \tilde{\psi} = Q_2 \psi$

„almost isospectral transf."

Laplace transf. weak factorization

a) $L = QQ^{+} + V$ self. adjoint

b) $L = Q_1 Q_2 + V$ general

Ex 1: 1-D difference op. $n \in \mathbb{Z}$ $T: n \to n+1$, $T^{-1}: n \to n-1$

Second order ops. $L = \tilde{c}_n T^{-1} + V_n + c_n T$

1st idea of discretization $\partial_x \to$ covariant shift $c_n T$ - needed for integrability (in numerical usually $\tilde{c}_n = c_n = 1$)

i.e. $(L\psi)_n = \tilde{c}_n \psi_{n-1} + V_n \psi_n + c_n \psi_{n+1}$

self - adjoint $T^{+} = T^{-1} \Rightarrow \tilde{c}_n = c_{n-1}$

Factorization : $Q = a_n + b_n T$, $Q^{+} = \cdot -$

$\Rightarrow$ Riccati eqns.

I) $QQ^{+} = Q^{+}Q + \alpha$ $Q^{+} = 1 + \sqrt{a + bn}\, T$ "discrete oscillator"

$a/b \in \mathbb{Z} \Rightarrow$ well-defined op. on half-line $\mathbb{Z}^{+}$: $n \geq -\overset{\overset{n_0}{\shortparallel}}{a/b}$

$\{\psi_n\} \in \mathcal{L}_2(\mathbb{Z}^{+})$

ground state $\psi_{0n}^{2} = \frac{1}{(k-1)!}\alpha^{-(k-1)}$, $k = n + n_0 > 0$

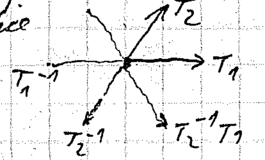
$\vec{\psi}_{jn} = Q^{j}\, \vec{\psi}_{0n} = P_j(n + n_0)\, \psi_{0n}$

Charlier polynomials

2D case : 2 basic shifts T_1, T_2

2nd idea of discretization „elliptic case" above „hyperbolic case"

real self-adjoint 2D Schröd op., we want to have Laplace transf. in discrete case $\Rightarrow$ we must use equilateral triangle lattice



6 neighbours

SHABAT :

Darboux transf. $L = D^{m} + u_1 D^{m-1} + \ldots + u_{m-1} D + u_m$

$L\psi = \lambda\psi$ $\hat{\psi} = a(x)\psi_x + b(x)\psi$ $a(\hat{u}_1 - u_1) + m a_x = 0$

by scaling we can get $u_1 = 0 \Rightarrow \hat{\psi} = \psi_x - f\psi$, $f = D \log \varphi$, $L\varphi = \mu\varphi$

Def: $ML = \hat{L}M$ $M = D^{m} + \ldots +$, $\hat{\psi} = M\psi$, $\hat{L}\hat{\psi} = \lambda\hat{\psi}$

Ex1: $M = D - f$ $V = \operatorname{Ker} M$ $\Rightarrow \dim V = 1 \Rightarrow L\varphi = \mu\varphi$, $\varphi \in V$

Theorem: $\Rightarrow M = (D - f_m) \cdots (D - f_1)$

Dressing chain: $\hat{L}_1 = \hat{L}, \hat{L}_2, \ldots, \hat{L}_{n+1} = \hat{L}$

$(D - f_j) L_j = L_{j+1}(D - f_j)$ $f_j = D \log \varphi_j$, $L_j \varphi_j = \mu\varphi_j$

$ML = \hat{L}M$, $\hat{\psi} = M\psi$ $\hat{L}_\varepsilon = \hat{L} + \varepsilon$

$\hat{L}_\varepsilon \hat{\psi} = (\lambda + \varepsilon)\hat{\psi}$

invariant potentials $g_{j+2} = g_j$ (2-periodic case)

MASON : Twistor theory

? 1-1 correspondence : Solutions to physical eqns : YM, Einstein

$\Updownarrow$

Deformed complex structures on twistor space

? twistor space a correct ~~formulation~~ geometric arena for formulation of quantum gravity

Known correspondence : solutions ASDYM $\Leftrightarrow$ structures on twistor space

$\hookrightarrow$ classification of integrable systems .. reductions of solns of ASDYM

Holomorphic vector bundles on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ (Riemann sphere)

coords : λ on $\mathbb{C}$, $\frac{1}{\lambda}$ on $\mathbb{C} \cup \{\infty\} \setminus \{0\}$

Def: - . - - - is one such $\exists$ family of local bases s.t. gauge transf. ($=$ patching / clutching function) are holomorphic.

Def: holomorphic

Ex3 E .. $\dim E = 1$ $\mathbb{C}$-line bundle
patching function : $U' \to U$: λ^p defined on $U' \cap U$
p.. topological invariant .. Chern class p .. $\mathcal{O}(p)$

$$\left(\begin{array}{l} s \text{.. section of } \mathcal{O}(p) \text{ over } U \\ s' \text{.. } -\text{''}- \ \mathcal{O}(p) \text{ over } U' \end{array} \Rightarrow s = s' \cdot \lambda^p \right)$$

These are the only line bundles on $\mathbb{C}P^1$

For a vector bundle: patching function will be $P(\lambda) \in GL(n, \mathbb{C})$ defined on $U \cap U' = \mathbb{C} \setminus \{0\}$

Theorem [Birkhoff]: Given $P(\lambda)$ defined on $|\lambda| = 1 \Rightarrow \exists\, G(\lambda), G'(\lambda')$ $|\lambda| < 1, |\lambda'| < 1$ holomorphic on U, U' in $GL(n, \mathbb{C})$ and integer $p_1, \dots, p_n$ such that $P(\lambda) = G(\lambda)^{-1} \operatorname{diag}(\lambda^{p_1}, \dots, \lambda^{p_n})\, G'(\lambda')$

For holomorphic vector bundles it means we can absorb G into basis on U, G' ~~ex~~ into basis on U'. Now $\tilde{P} = \begin{pmatrix} \lambda^{p_1} & 0 \\ 0 & \ddots \ \lambda^{p_n} \end{pmatrix}$

Theorem: All holom. v.b. on $\mathbb{C}P^1$ are $\bigoplus_{i=1}^{n} \mathcal{O}(p_i)$

- Σp_i is a topological invariant but individual pieces are not
- Holomorphically p_i's are invariant (up to order)

Σp_i .. Chern class = winding number of $\det P$

- if $\Sigma p_i = 0$ generic choice of $P(\lambda)$ will have all $p_i = 0$

Ex: factorize $\begin{pmatrix} \lambda & \alpha \\ 0 & \lambda^{-1} \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $\alpha \neq 0$

Defn: Bogomolny eqns on $\mathbb{C}^3$ with coords x^i, $i = 0, 1, 2$
$$ds^2 = (dx^1)^2 + dx^0 dx^2 + dx^2 dx^0 = \eta_{ij}\, dx^i dx^j$$
Equations on $(E, D, \Phi : E \to E)$ $\quad (D_i = \partial_i + A_i)$
$E \downarrow \mathbb{C}^3$, Φ ← Higgs field $\quad F_{ij} = \varepsilon_{ijk}\, \eta^{kl} D_l \Phi$

Lax pair: $L = (D_1 - \Phi) - \lambda D_0$
$M = D_2 - \lambda (D_1 + \Phi)$

Twistor geometry arises from $l = \partial_1 - \lambda \partial_0$ $\quad \partial_i = \frac{\partial}{\partial x_i}$
$m = \partial_2 - \lambda \partial_1$

Fix $\lambda \Rightarrow \{l, m\}$ span a null two plane with normal $V^i = m^i - \lambda l^i$, ← simultaneously tangent
$\eta_{ij} V^i V^j = 0$

Definition: A minitwistor Z is just such a null 2-plane.
a $\underbrace{\text{minitwistor space}}_{\mathbb{M}\mathbb{T}}$ is the space of these.

2-plane can be written $\mu = x^0 + \lambda x^1 + \lambda^2 x^2$
$\Rightarrow \lambda, \mu$ coords. on $\mathbb{M}\mathbb{T}$ on patch $\tilde{U}$

Near $\lambda = \infty$ we have $\underbrace{(\lambda', \mu') = \left(\frac{1}{\lambda}, \frac{\mu}{\lambda^3}\right)}_{\text{on patch } \tilde{U}'}$, $\mu' = \lambda'^2 x^0 + \lambda' x^1 + x^2$

$\lambda \in \mathbb{C}P^1$ $\quad \mathbb{M}\mathbb{T}$ = total space $\mathcal{O}(2)$

Fix x^i, a point $x^i \leftrightarrow$ a section $\mu(x, \lambda) = x^0 + \lambda x^1 + \lambda^2 x^2$ .. $L(x^i)$
of $\mathcal{O}(2) \to \mathbb{C}P^1$

Ward transform of a soln (E, D, Φ) to the Bogol. eqns on $\mathbb{C}^3$ is a holomorphic vector bundle $\tilde{E} \to \mathbb{M}\mathbb{T}$
$\tilde{E}(x, \lambda) = \{ s$ section ~~over E~~ of E over $Z(\lambda, \mu)$
s.t. $Ls = Ms = 0\}$

Theorem [Ward] (1) $\tilde{E} \to \mathbb{M}\mathbb{T}$ ~~diffy~~ determines (E, D_i, Φ)

(2) For generic choice of $\tilde{E}$, Chern class $0 \Rightarrow$ soln to Bogomolny perhaps with rational singularities

ad Proof: $\tilde{E}$ has bases over $\tilde{U}, \tilde{U}'$ with patching $P(\lambda, \mu) \in GL(n, \mathbb{C})$
Define $E_x = \{$ space of sections of $\tilde{E}|_{L(x^i)}$ ← $\mathbb{C}P^1 \subset \mathbb{M}\mathbb{T} \}$

on $L(x)$ $\quad P(\lambda, x^0 + \lambda x^1 + \lambda^2 x^2) = G(\lambda, x_i)^{-1} G'(\lambda', x_i)$

$$(lG) G^{-1} = (l G')(G'^{-1}) = -A_1 + \Phi + \lambda A_0 \text{ etc.}$$
similarly for m $\Rightarrow (A_i, \Phi)$ for Bog. eqns
also $\Rightarrow LG = 0$

MIKHAILOV : Introduction to symmetry methods & recursion operators

$1+1$ x, t $u(x;t)$

Evolutionary eqn (or system) $u_t = F(u, u_x, \dots, u^{(m)})$

$\mathrm{ord} F = \text{order of } F = m$

notation $u_0 = u,\ u_1 = u_x,\ u_2 = u_{xx}, \dots \Rightarrow \infty$ # of variables

... dynamical variables

$\mathcal{F}$ = diff. field of functions depending on finite number of variables $u_0, \dots, u_k, \dots$

$$\partial_x \rightarrow D \qquad D u_j = u_{j+1} \quad \Rightarrow D = u_1 \frac{\partial}{\partial u_0} + u_2 \frac{\partial}{\partial u_1} + \dots$$

$\Rightarrow$ in D always finite number of terms contribute (applied on $a \in \mathcal{F}$)

$$u_{0t} = F_0(u_0, \dots, u_m)$$

$$\partial_t \rightarrow D_t : \quad D_t u_{0t} \overset{!}{=} F_0(u_0, \dots, u_m)$$

$$\Rightarrow D_t = F_0 \frac{\partial}{\partial u_0} + ?$$

$$[\partial_x, \partial_t] = 0 \quad \Rightarrow \quad [D, D_t] \overset{!}{=} 0 .$$

assume $D_t = \sum_{j=0}^{\infty} F_j \frac{\partial}{\partial u_j}$

Remark: $X = \sum_i X_i \frac{\partial}{\partial u_i}$, $Y = \sum_i Y_i \frac{\partial}{\partial u_i} \Rightarrow [X, Y] = \sum_{i=0}^{\infty} (X(Y_i) - Y(X_i)) \frac{\partial}{\partial u_i}$.

$$\Rightarrow \quad F_m = D^m(F_0) \qquad\qquad D_t = \sum_{j=0}^{\infty} D^j(F_0) \frac{\partial}{\partial u_j} .$$

$\Rightarrow$ 2 compatible dynamical systems instead of 1 PDE

$$D u_j = u_{j+1} \qquad D_t u_j = F_j = D^j(F_0)$$

Symmetries of ODEs – see Sokolov's 1st lecture

— " — PDES $u_{0t} = F_0(u_0, \dots, u_m)$

(1) $\hat{u}_0 = u_0 + \tau\, G_0(u_0, \dots, u_m)$ $\quad \hat{u}_j = D^j(\hat{u}_0)$

$\Updownarrow$ $\Rightarrow \hat{u}_t - F(\hat{u}_0, \dots, \hat{u}_m) = \sigma(\tau^2)$

(2) $u_t = F(\dots)$ & $u_\tau = G(\dots)$ compatible

$\Updownarrow$

(3) $[D_t, D_\tau] = 0 \qquad [D, D_\tau] = 0 = [D, D_\tau]$

$\Updownarrow$ where $D_\tau = G_0 \frac{\partial}{\partial u_0} + D(G) \frac{\partial}{\partial u_1} + \dots$

(4) $G_*(F) - F_*(G) = 0$

where $\frac{\partial}{\partial \epsilon} F(u_0 + \epsilon v_0, u_1 + \epsilon v_1, \dots) \Big|_{\epsilon = 0} = F_*(V)$

Ex: KdV $u_t = u_3 + 6 u u_1$, i.e. $F_* = D^3 + 6uD + 6u_1$

$G_1 = u_1 \qquad G_5 = u_5 + 10 u_3 u + 20 u_2 u_3 + 30 u^2 u_1$

Galilean transf. $u_\tau = 1 + 6 t u_1$

Symmetry reductions

$D_t u = F(u, u_1, u_2, u_3)$, $\quad D_t u_1 = F_1(\dots u_4)$

assume symmetry $D_\tau u = G(u, \dots, u_5)$

$\Rightarrow D_t G = F_*(G) \qquad$ if $G = 0$ at $t = 0 \Rightarrow G = 0, \forall t$

$\Rightarrow G(u, \dots, u_5) = 0$ is invariant sub-manifold

$\Rightarrow u_5 = g(u, u_1, u_2, u_3, u_4)$, $u_6 = Dg(\dots)$ etc.

$\Rightarrow$ we arrive to system $D_t u = F, \dots, D_t u_4 = F_4$, remaining eqns. are satisfied because of $u_5 = g(u, \dots, u_4)$

we can solve this system and find a soln of original PDE

$A = \sum_k A_k D^k \qquad A^+ = \sum (-1)^k D^k A_k^T$

(A_k either ~~[illegible]~~ real function or real matrix function)

Remark: F-derivative for systems $(F_*)_{lm} = \sum \frac{\partial F^l}{\partial u_s^m} D^s$

index of variable

order of derivative

Ex: $A = uD^2 + u_1 D$ $A^+ = D^2 u - D u_1 - uD^2 + u_1 D$

Euler operator (variational derivative)

$$\frac{\delta h}{\delta u} = \sum_k (-1)^k D^k \left(\frac{\partial h}{\partial u_k} \right) = h_*^+(1)$$

(in matrix case the result is vector)

Ad first integrals

$S_t = S \overset{\shortparallel}{\wedge} S_{xx}$ $S \in \mathbb{R}^3$ Heisenberg eqn
$\times$ .. vector product

$\Rightarrow \; S S_t = (S \cdot S \wedge S_{xx}) = (S \wedge S) \cdot S_{xx} = 0 \Rightarrow (S^2)_t = 0$
in this case first integral, but usually (like KdV)
no first integrals

instead local conservation laws:

$\partial_t u = F(u, .., u^{(m)})$ $\partial_t \rho(u, .., u^{(k)}) = \partial_x \sigma(u, u_x, ..)$

$\Rightarrow \rho$ is conserved density

e.g. KdV $u_t = u_{xxx} + 6 u u_x = \partial_x (u_{xx} + 3u^2)$

$\Rightarrow \quad I = \int_0^L u \, dx$ where $\langle 0, L \rangle$ a period
$\Rightarrow$ ~~$F = 0$~~ $\partial_t I = 0$

Conservation laws should be linearly independent modulo total derivatives ($\rho_1 \sim \rho_2 \Longleftrightarrow \rho_1 - \rho_2 = Da, \; a \in \mathcal{F}$)

Order: $F \in \mathcal{F}$ $\operatorname{ord} F = \deg(F_*)$ (i.e. highest contributing derivative)

Order of conserv. laws:

Theorem: $\dfrac{\delta a}{\delta u} = 0, \; a \in \mathcal{F} \iff a \in \operatorname{Im} D + \mathbb{C}$
($\operatorname{Im} \mathbf{D} \equiv \{ f \in \mathcal{F} / \exists g \in \mathcal{F} : f = Dg \}$)

[Gelfand, Manin, Šubin]

$\Rightarrow \quad \operatorname{ord} \rho \equiv \deg \left(\left(\frac{\delta \rho}{\delta u} \right)_* \right)$

Ex: KdV $\operatorname{ord} u = 0$ $\operatorname{ord} u^2 = 0$ $\operatorname{ord} u_1^2 = 2$

KRUSKAL: Tutorial

Ablowitz-Ramani-Segur (ARS) Painlevé analysis for PDEs
Concept: (1) PDE (1+1)

(2) reduce to ODE by ~~[illegible]~~ symmetry reduction

~~[illegible]~~ check if it has Painlevé property
eventually after transformation → if not then PDE is not solvable using Gelfand-Levitan-Marčenko eqn (i.e. IST)

(3) if it has Painlevé property ⇒ keep working, i.e. look for IST

DEGASPERIS:

Ex: $Du = c_{00}^{(2)} u^2 + \tilde{F}(u)$ not integrable with $c_{00}^{(2)} \neq 0$ for any $\tilde{F}$

e.g. $u_t + u_{xxx} = u^2$ is not integrable

Higher orders

we assume $u = u^*$, real PDE, $F[-u] = -F[u]$ } simplifying assumptions

$E = \exp(i(k_0 x - \omega_0 t))$ $\omega_0 = \omega(k_0)$
$u = \sum_{\alpha=-\infty}^{+\infty} E^\alpha u^{(\alpha)}(\xi, t, t_2, \dots, t_m, \dots)$

$F(u) = \sum E^\alpha F^\alpha[u]$ $D^{(\alpha)} u^{(\alpha)} = F^{(\alpha)}$ $\alpha = 1, 3, 5, \dots$

$D^{(\alpha)} = \sum_m \varepsilon^m D_m^{(\alpha)}$ $u^{(\alpha)} = \sum \varepsilon^m \cdot u_m^{(\alpha)}$, $F^{(\alpha)} = \sum \varepsilon^m F_m^\alpha$

$\Rightarrow \sum_{m=1}^{n} D_{n-m}^{(\alpha)} u_m^{(\alpha)} = F_n^{(\alpha)}$

$D_0^{(\alpha)} = i[\omega(\alpha k_0) - \alpha \omega(k_0)] \Rightarrow D_0^{(\pm 1)} = 0$

$D_0^{(\alpha)} \neq 0 \quad \forall \alpha : |\alpha| \neq 1$

$u_n^{(1)} \equiv u_n \,,\; D_n^{(1)} \equiv D_n \,,\; F_n^{(1)} \equiv F_n$

$D_n = \partial_n - (i)^{n+1} \omega_n \partial_\xi^n \qquad n = 1,3, \sim$

$$\omega_n = \frac{1}{n!} \frac{d^n}{dk^n} \omega(k) \Big|_{k=k_0}$$

$D_1 u_n + D_2 u_{n-1} + \dots + D_n u_1 = F_{n+1}$

$\mathcal{P}_n \equiv \{$ nonlinear diff. polynomials deg n, gauge $=1$ $\}$

$\deg(\partial_\xi^\ell u_n) = \deg(\partial_\xi^\ell u_n^*) = n+\ell$

$F_{n+1} \in \mathcal{P}_{n+1} \qquad \mathcal{P}_2 = \emptyset,\ \dim(\mathcal{P}_3) = 1,\ \dim \mathcal{P}_4 = 4$

$D_1 u_1 = 0 \Rightarrow$ by recursion $D_1 u_n = 0$

$D_2 u_n + \dots + D_{n+1} u_1 = F_{n+2}$

$D_2 u_1 = F_3$ ... NLS eqn

$\partial_2 u_1 = i\omega_2 (u_{1\xi\xi} - 2c|u_1|^2 u_1) = K_2[u_1]$ assume $c \in \mathbb{R}$

next eqn. $D_2 u_2 + D_3 u_1 = F_4$ def. $M_2 = \partial_2 - \underbrace{K_2'[u]}_{\text{Frechet deriv.}} \cdot$

$M_2 u_2 + \underbrace{\boxed{D_3 u_1}}_{\text{secular term}} = \tilde{F}_4 \Rightarrow \partial_3 u_1 - \omega_3 (u_{1\xi\xi\xi} - 6c|u_1|^2 u_{1\xi}) \overset{!}{=} 0$

i.e. $\partial_3 u_1 = K_3[u_1]$ complex MKdV eqn.

by recursion conditions for freedom from secular terms

$$\partial_n u_1 = K_n[u_1]$$

Def: $\mathcal{P}_n(\ell) \subset \mathcal{P}_n$ subset of diff. polyn. which depend on u_m and u_m^* only for $1 \leq m \leq \ell$

$M_n \cdot \doteq \partial_n \cdot - K_n'[u_1] \cdot \qquad (M_n - D_n) u_m \in \mathcal{P}_{n+m}(m)$

Theorem: if $\partial_n u_1 = K_n[u], \partial_m u_1 = K_m[u] \Rightarrow [M_n, M_m] = 0$

Reduced multiplescale eqns.

$M_2 u_n + \dots + M_n u_2 = G_{n+2} \in \mathcal{P}_{n+2}$ ... without secular terms

Integrable PDEs

Linear PDEs : $G_n = 0 \quad M_n = D_n \quad M_\ell u_n = 0$

Nonlinear PDEs: e.g. $(\partial_t - \partial_x^3) u = 3(u^2 u_{xx} + 3 u u_x^2 + u^4 u_x)$

$(\partial_t - \partial_x^3) u = (c_2 u^2 + c_3 u^3)_x$

$(\partial_t - \partial_x^3) u = -\frac{1}{8} u_x^3 + c(\cosh u - 1) u_x$

$M_\ell u_n = f_\ell(n) \in \mathcal{P}_{n+\ell}(n-1)$

compatibility $\boxed{M_m f_\ell(n) = M_\ell f_m(n)}$

hierarchies

$\partial_2 u_1 = K_2, \quad \partial_3 u_1 = K_3, \dots$

$\partial_2 u_2 = K_2' u_2 + G(4) \quad \partial_3 u_3 = \dots$

$\partial_2 u_3 = \dots$

Asymptotic integrability test

Def: PDE is A_n-integrable if the following hierarchies of commuting flows exist

$\partial_\ell u_1 = K_\ell,\ \ell \geq 2$ etc. ... up to n ... (see above)

MIKHAILOV

Remark: Details, proofs: e.g. in book "What is integrability", 1991

Properties: $(\alpha\beta)_* = \alpha\beta_* + \beta\alpha_*$ $\quad\forall \alpha, \beta \in \mathcal{F}$

$$(D(\alpha))_* = D\circ\alpha_* = D(\alpha_*) + \alpha_* D$$

$$(D_t(\alpha))_* = D_t(\alpha_*) + \alpha_* F_{0*} \qquad (\Leftarrow D_t u_0 = F_0)$$

$$\left(\frac{\delta\alpha}{\delta u}\right)_* = \left[\left(\frac{\delta\alpha}{\delta u}\right)_*\right]^+$$

$$D_t \cdot \frac{\delta}{\delta u} - \frac{\delta}{\delta u} D_t + F_*^+ \frac{\delta}{\delta u} = 0$$

$D_t\rho = D\sigma$ $\quad \rho, \sigma \in \mathcal{F}$ local cons. law

$$\left(D_t \frac{\delta}{\delta u} - \frac{\delta}{\delta u} D_t + F_*^+ \frac{\delta}{\delta u}\right)\rho = 0$$

$$\frac{\delta}{\delta u} D_t \rho = \frac{\delta}{\delta u} D\sigma \equiv 0$$

$$\Rightarrow \boxed{D_t \frac{\delta\rho}{\delta u} + F_*^+ \frac{\delta\rho}{\delta u} = 0}$$ Lax equation (cosymmetry)

Def: Formal pseudodifferential operators: (FPD)

$$A = a_n D^n + a_{n-1} D^{n-1} + \dots + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \dots$$

where $a_n \neq 0$, $a_k \in \mathcal{F}$

Properties:

$$[D^k, D^m] \equiv 0 \qquad \forall k, m \in \mathbb{Z}$$

$$D^k \circ a \equiv aD^k + \binom{k}{2} D(a) D^{k-1} + \binom{k}{2} D^2(a) D^{k-2} + \dots$$

where $\binom{k}{p} = \left(\frac{k!}{p!(k-p)!}\right) = \frac{k(k-1)\dots(k-p+1)}{p!}$

FPD form a skew-symmetric field

$$A = a_n D^n + a_{n-1} D^{n-1} + \dots$$
$$B = b_{-n} D^{-n} + b_{-n-1} D^{-n-1} + \dots$$

$AB \equiv 1 \Rightarrow a_n D^n b_{-n}^{-n} + a_n D^n b_{-n-1} D^{-n-1} +$

$a_{n-1} D^{n-1} b_{-n} D^{-n} + \dots = 1$

$\Rightarrow a_n b_{-n} = 1 \qquad b_{-n} = \frac{1}{a_n}$

$n a_n D(b_{-n}) + a_n b_{-n-1} + a_{n-1} b_{-n} = 0$

$\Rightarrow b_{-n-1} = -\frac{a_{n-1}}{a_n} - \cancel{n} D\left(\frac{1}{a_n}\right)$ etc. $\Rightarrow$ algorithm for CAS

Moreover we can find n-th root of $A = a_n D^n + a_{n-1} D^{n-1} + \dots$

$$C = c_1 D + c_0 + c_{-1} D^{-1} + \dots$$

e.g. $A = D^2 + u \Rightarrow c_1^2 = 1$ $\quad$ i.e. $c_1 = \pm 1$

$2c_0 = 0 \qquad c_0 = 0$

$(C^2 = c_1^2 D^2 + c_1 D(c_1) D + (c_1 c_0 + c_0 c_1) D + \dots)$ $\quad 2c_{-1} = u \qquad c_{-1} = \frac{u}{2}$

$$C = D + \frac{u}{2} D^{-1} + \dots$$

Def: The residue of a formal series $A = \sum_{k\le m} a_k D^k$ with scalar coeffs $a_m \in \mathcal{F}$ is by definition the coefficient at D^{-1}
(or $\mathrm{Tr}\, a_{-1}$ in matrix case)
$\mathrm{res}\, A = a_{-1}$
The logarithmic residue of A of order n is defined as
$\mathrm{res}\log A = \frac{a_{n-1}}{a_n}$

$$D(\mathrm{res}\log A) = \mathrm{res}([D, A] A^{-1})$$

Theorem [Adler] $\quad \mathrm{res}[A, B] \in \mathrm{Im}\, D$
$\qquad = D(\sigma(A, B))$

$$\sigma(A, B) = \sum_{\substack{p+q+1\ge 0\\ p\le \deg A\\ q\le \deg B}} \binom{q}{p+q+1} \sum_{s=0}^{p+q} K_{pqs}, \quad K_{pqs} = (-1)^s D^s(a_q) D^{p+q-s}(b_p)$$

Theorem: $A = a_n D^n + \dots$, $\quad B = b_m D^m + \dots$

$$\Rightarrow \mathrm{res}\log ABA^{-1}B^{-1} = D(\dots)$$

Let G be a symmetry of order m $\quad D_\tau u = G(u_0, \to u_m)$

Then $D_t G = D_\tau F$, apply $*$ $\Rightarrow$ $D_t(G_*) + G_* F_* = D_\tau F_* + F_* G_*$

$\Rightarrow D_t(G_*) - [F_*, G_*] = D_\tau F_*$

$$G_* = G_m D^{m} + G_{m-1} D^{m-1} + \dots + G_0$$

If $F = F(u_0, \dots, u_3)$ $\qquad F_* = F_3 D^3 + F_2 D^2 + F_1 D + F_0$, $F_k = \frac{\partial F}{\partial u_k}$

Def: PDE is called a formal symmetry of order N if

$$\deg\left(D_t L - [F_*, L]\right) \leq \deg(F_*) + \deg(L) - N$$

Theorem: local symmetry of order N or two integrals of motion of order N_1, N_2, $N_1 + N_2 > N + m$
$\Rightarrow$ formal symmetry of order N.

If infinite hierarchy of symmetries $\Rightarrow \exists$ FPD L s.t. $\boxed{D_t(L) = [F_*, L]}$ $(*)$

Prop: L soln of $(*)$ $\Rightarrow$ L^k is also soln

$$\Downarrow$$

$$D_t(L^k) = [F_*, L^k]$$

$$D_t(\operatorname{res} L^k) \in \operatorname{Im} D \qquad D_t(\operatorname{res} \log L) \in \operatorname{Im} D$$

$\rho_{-1} = \operatorname{res} L^{-1}$, $\rho_0 = \operatorname{res} \log L$, $\rho_k = \operatorname{res} L^k$ define
"sequence of canonical conserv. laws"

if such densities are not conserved $\Rightarrow$ eqn is not integrable
i.e. doesn't have higher local conserv. laws

n-th order eqn $\quad D_t u_0 = F_0(u_0, \dots, u_n)$
First $n-1$ coeffs of F_{0*} satisfy equation for formal recursion op.

$$L^n = \frac{\partial F}{\partial u_n} D^n + \frac{\partial F}{\partial u_{n-1}} D^{n-1} + \dots + \frac{\partial F}{\partial u_2} D^2 + f_1 D + f_0 + f_{-1} D^{-1}$$

Therefore we know $n-1$ of L almost straight away

$$L = \left(\frac{\partial F}{\partial u_n}\right)^{1/n} D + \dots$$

$\rho_{-1} = \operatorname{res} L^{-1} = \left(\frac{\partial F}{\partial u_n}\right)^{-\frac{1}{n}}$, should be conserved density
$D_t \rho_{-1} \in \operatorname{Im} D$

Ex: $\quad u_t = u^k u_k = u^k \left(\frac{\partial^k}{\partial x^k}\right) u \qquad u^{k-2} u_k \in \operatorname{Im} D$
only for $k = 1, 2, 3$

Remark: C-integrability … integrable through analog of Cole-~~Hoff~~ Hopf

S-integrability … through IST

C-integrability $\Leftarrow$ higher order terms in the hierarchy are trivial (e.g. Burgers … ρ_k … total derivatives for $k \geq 1$)

---

ad local cons. laws $\qquad D_t \frac{\delta \rho}{\delta u} + F_*^+ \frac{\delta \rho}{\delta u} = 0$

$R = \left(\frac{\delta \rho}{\delta u}\right)_*$ $\qquad R_t + \underset{\substack{\uparrow \\ N+m}}{R} \underset{\substack{\nwarrow \\ N}}{F_*} + F_*^+ R = Q \qquad \deg Q = 2m$
ord $F = m$, $\deg R = N + m$

$\Rightarrow$ for many orders we can for $N \gg m$ forget Q $\Rightarrow$ recursion relations

$\Rightarrow$ even order eqns. cannot have higher order conserv. laws

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Conclusion:
Symmetries $\Rightarrow$ canonical densities
Conserv. laws $\Rightarrow$ further restrictions

NOVIKOV : skipped

TUTORIAL ON MIKHAILOV & SOKOLOV

Software for $\qquad$ http:// www.imop.csa.ru/diff

www.neva.ru/journal

KRUSKAL: Poly - Painleve' test

P_{II} $u'' = 2u^3 + zu + \alpha$ $u \sim \frac{\pm 1}{z - z_0} + \underbrace{\dots}_{\text{Taylor}}$

$u = -v^2$ $v''v + 2v'^2 = 2v^6 + 3v^2 + \alpha$

$\Rightarrow v \sim \frac{\pm 1}{\sqrt{z-z_0}}$ or $\frac{\pm i}{\sqrt{z-z_0}}$ branching point

without previous knowledge one finds $v \sim c(z-z_0)^{-\frac{1}{2}} + (\ )^{\frac{1}{2}} + (\ )^{3/2}$

$\Rightarrow$ should exist transf. to eqn with Painleve' prop.

PDEs: $\left\{ \begin{array}{l} u_t + 6uu_x + u_{xxx} = 0 \quad \text{KdV} \\ v_t = \left(v^{-\frac{1}{2}}\right)_{xxx}, \text{ subst. } w = v^{-\frac{1}{2}} \Rightarrow w_t = c\,w^3 w_{xxx} \end{array} \right.$

pass analog of naive Painleve' test for PDE

Dyme's eqn.

there $\exists$ nonlocal transf. between these eqns

Poly - Painleve' test :.. we allow branching such that the branch points are not dense in u-plane

.. we look at 2 or more singular points

(1) $u' = u^3 + u^2 \cdot z$ ~~(quadratic term can be absorbed by transf. of u)~~

(2) $u' = u^3 + u \cdot z$

(3) $u' = u^3 + 1 \cdot z$

balance of leading terms $\Rightarrow$ $u \sim c(z-z_0)^p$ leads to $(\ )^{p-1} = (\ )^{3p}$

$\Rightarrow p = -\frac{1}{2} \Rightarrow$ branching points $\Rightarrow$ not Painleve'

only (2) has a Poly-Painleve' property

$\frac{1}{2}(u^2)' = (u^2)^2 + u^2 z$ $u^2 = v$

$\Rightarrow \frac{1}{2}v' = v^2 + zv$ Riccati $\Rightarrow$ single-valued soln

$\Rightarrow$ u has globally 2 branches $\Rightarrow$ poly - Painleve' eqn

proof that others don't -- nontrivial

$u' = u^3 + z$

$z = Z + \delta\xi$ $|Z| >> 1$

$u = \gamma U$ $\xi = \mathcal{O}(1)$
$U = \mathcal{O}(1)$

$\Rightarrow \frac{\gamma}{\delta}\frac{\partial U}{\partial \xi} = \gamma^3 U^3 + Z + \delta\xi$

$\frac{dU}{d\xi} = \gamma^2 \delta U^3 + \frac{\delta}{\gamma} Z + \frac{\delta^2}{\gamma}\xi$, we assume $|\delta| << |Z|$

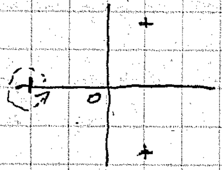
$\frac{\delta}{\gamma} Z = 1, \ \gamma^2\delta = 1$

$\Rightarrow \gamma = Z^{1/3}, \delta = Z^{-2/3}$

$\Rightarrow \frac{dU}{d\xi} = U^3 + 1 + \underset{\overset{\parallel}{\varepsilon} << 1}{Z^{-\frac{5}{3}}}\xi$

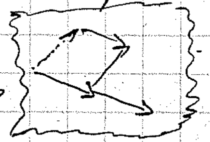
perturbation theory on ε 0th order: $u' = u^3 + 1$ $\int^u \frac{du}{u^3+1} = z + \text{const.}$

branch points in u-plane



roots of -1

$\Rightarrow$ in Z-plane



$\Rightarrow$ 3 periods but only 2 independent $\Rightarrow$ 2D lattice

$a \log(z-z_1) + b \log(z-z_2) + c \log(z-z_3) = t$

$\underbrace{\qquad}_{\text{branching is not dense}}$

to 0th order it is poly - Painleve', higher ~~d~~ orders $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$

$\xi(U)$ instead: $\frac{1}{\frac{d\xi}{dU}} = u^3 + 1 + \epsilon\xi \Rightarrow \frac{d\xi}{dU} = \frac{1}{u^3 + 1 + \varepsilon\xi} =$

$= \frac{1}{U^3+1}\left[1 - \frac{\varepsilon\xi}{u^3+1} + \frac{\varepsilon^2\xi^2}{(u^3+1)^2} + \dots\right]$

$\xi = \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \dots$

$\frac{d\xi_0}{dU} = \frac{1}{U^3+1} \Rightarrow \xi_0 = \int^u \frac{dU}{U^3+1}$, $\frac{d\xi_1}{dU} = \frac{-1}{(U^3+1)^2}\int^U \frac{dU}{U^3+1}$

$\Rightarrow$ to this order again parallelogram similar to the above

on the 2nd order 3 indep. logarithms $\Rightarrow$ dense branching $\Rightarrow$ eqn not poly - Painleve'

SANDERS : Number theory & the symmetry classification integrable systems

$\parallel$ evolution equation with ∞ infinitesimal symmetries

$n < \infty$ .. almost integrable systems of order n

Symmetry $S[u]$ $D_S K = D_K S$ where $D_K = \sum_{i=0}^{\infty} \frac{\partial K}{\partial u_i} D_x^i$

Frechet derivative of K

where $u_i = \frac{\partial u}{\partial x_i}$, $u = u(t,x)$ C^∞-functions of t, x

How to prove one has found all symmetries of all possible orders ?

We will assume eqns. polynomial in derivatives of u

Two tools : (1) Symbolic method to go from diff problems to algebra

(2) implicit function theorem for filtered Lie algebras

$\Rightarrow$ we need only low order computations.

Symbolic method $u_k \rightarrow \xi^k u$

to have commutativity

e.g. $u_1 u_2 \rightarrow \xi_1 \xi_2^2 u^2$ + average $\Rightarrow u_1 u_2 = \frac{1}{2}(\xi_1 \xi_2^2 + \xi_2 \xi_1^2) u^2$

Differentiation

$$D_x u_1 u_2 = u_1 u_3 + u_2^2 = \frac{1}{2}(\xi_1 \xi_2^3 + \xi_1^3 \xi_2 + 2 \xi_1^2 \xi_2^2) u^2 =$$
$$= (\xi_1 + \xi_2) \frac{1}{2} (\xi_1 \xi_2^2 + \xi_1^2 \xi_2) u^2$$

$$\Rightarrow D_x \equiv \sum_i \xi_i$$

Ex 1st symmetry of KdV $K_3 = K_3^0 + K_1^1 = u_3 + u u_1$

try $S_5 = S_5^0 + S_3^1 + \dots = u_5 + a_1 u u_3 + a_2 u_1 u_2 + \dots$

$$\Rightarrow D_x S_3^1 + u D_x S_5^0 + u_1 S_5^0 = D_x^5 K_1^1 + a_1 u D_x^3 K_3^0 + a_1 u_3 K_3^0 + a_2 u_1 D_x^2 K_3^0 + a_2 u_2 D_2 K_2^0$$

$$\Rightarrow (\xi_1 + \xi_2)^3 S_3^1 + (\xi_1^5 + \xi_2^5) \cancel{K}_1^1 = (\xi_1 + \xi_2)^5 K_1^1 + (\xi_1^3 + \xi_2^3) S_3^1$$

$$\Rightarrow S_3^1 = \frac{\cdots \xi \cdots}{\cdots \xi \cdots} K_1^1$$

$$G_n^1 (\xi_1, \xi_2) = (\xi_1 + \xi_2)^n - \xi_1^n - \xi_2^n$$

$$\Rightarrow S_3^1 = \frac{G_5^1 (\xi_1, \xi_2)}{G_3^1 (\xi_1, \xi_2)} K_1^1 \qquad (K_1^1 \text{ is linear in } \xi)$$

if S_3^1 is polynomial in ξ_1, ξ_2 $\Rightarrow$ real solution of symmetry eqns.

ξ_0 : $\xi_0 + \xi_1 + \xi_2 = 0$ $\Rightarrow G_n^1 = - \sum_{i=0}^{2} \xi_i^n$

$\Rightarrow G_n^1$ are invariants w.r.t S_3 (permutation of indices)

$$c_n = \sum_{i=0}^{k+1} \xi_i^n$$

$G_5^1 = c_2 c_3$ up to constant

$$\hat{S}_3^1 = c_2 \hat{K}_1^1 = \frac{5}{6} (\xi_1^3 + 2 \xi_1^2 \xi_2 + \xi_2^3) u^2$$

~~[illegible]~~ $[S_3^1, K_1^1] + [S_1^2, K_3^0] = 0$

$$\Rightarrow S_1^2 = \frac{5}{8} (\xi_1^2 + \dots) u^2$$

etc.

HIETARINTA: Hirota bilinear method

How to construct multisoliton solns of nonlinear PDE ?

1st step : transform eqn to bilinear form

Ex: KdV $u_{xxx} + 6 u u_x + u_t = 0$

def. w : $u = \partial_x^2 w$ $\Rightarrow \partial_x (w_{xxxx} + 3 w_{xx}^2 + w_{xt}) = 0$

$\Rightarrow w_{xxxx} + 3 w_{xx}^2 + w_{xt} = 0$ potential KdV

def. F $w = \alpha \log F$

$$F^2 \cdot (\underbrace{\text{something quadratic}}) + 3\alpha(2-\alpha)(\dots) F'^2 = 0$$

we choose $\alpha = 2$

$$\Rightarrow F_{xxxx} F - 4 F_{xxx} F_x + 3 F_{xx}^2 + F_{xt} F - F_x F_t = 0$$

i.e. $u = 2 \partial_x^2 \log F$

Bilinear form

Hirota's D-operator:

$\underline{D} :\equiv \quad D_x^m f \cdot g = (\partial_{x_1} - \partial_{x_2})^m f(x_1) g(x_2) \Big|_{x_2 = x_1 = x}$

$\Rightarrow \quad D_x f \cdot g = f_x g - f g_x$

$D_x D_t f g = f g_{xt} - f_x g_t - f_t g_x + f_{xt} g$

$\Rightarrow$ potential KdV $\Rightarrow (D_x^4 + D_x \cdot D_t) F \cdot F = 0$

Remark: $P(D) f \cdot g = P(-D) g \cdot f \qquad$ P .. polynomial (or any function)

$P(D)\, 1 \cdot f = P(-\partial) f, \quad P(D) f \cdot 1 = P(\partial) f$

$\uparrow$ ordin. derivative

$P(D) e^{px} \cdot e^{qx} = P(p - q)\, e^{(p+q)x}$ important for solitons (esp. $p = q \Rightarrow P(D) \cdot\cdot = 0$)

Class of eqns: ~~$P(D)$~~ $P(D_x, D_t, ..) F \cdot F = 0$

- KdV

- KP $\quad \partial_x (u_{xxx} + 6 u u_x - 4 u_t) = -3 u_{yy}$

  $\Rightarrow (D_x^4 - 4 D_x D_t + 3 D_y^2) F \cdot F = 0$

- Hirota - Satsuma shallow water eqn.

  $u_{xxt} + 3 u u_t - 3 u_x v_t - u_x = u_t, \quad v_x = -u$

  $\Rightarrow (D_x^3 D_t - D_x^2 - D_t D_x) F \cdot F = 0$

  its 2-dim. $\underset{(2+1)}{}$ generalization $\overset{\uparrow}{D_y}$ (otherwise the same)

- Sawada - Kotera eqn.

  $(D_x^6 + D_x D_t) F \cdot F = 0$

  and its 2+1-dim generalization

Class $\quad P(D_x, D_y \ldots) F \cdot F = 0$

$F = 1 \Rightarrow$ vacuum soln if $P(0, .., 0) = 0$

Multisoliton solns $\quad F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3$

finite perturb. expansion

1-soliton soln $\quad F = 1 + \varepsilon f_1$

$\Rightarrow P(D_x, \ldots)(1 \cdot 1 + \varepsilon 1 \cdot f_1 + \varepsilon f_1 \cdot 1 + \varepsilon^2 f_1 f_2) = 0$

order ε^0 vanishes

ε^1, P even $\Rightarrow P(\partial_x, \partial_y, \ldots) f_1 = 0 \overset{\text{ansatz}}{\Rightarrow} f_1 = e^{\eta}, \ \eta = px + qy + \omega t + \ldots + \text{const.}$

params $p, q, \ldots$ must satisfy dispersion relation $P(p, q, \ldots) = 0$

ε^2: $\quad P(\vec{D}) e^{\eta} e^{\eta} = e^{2\eta} P(\vec{p} - \vec{p}) = 0 \quad$ vanishes automatically

2-soliton soln (2-SS) $\qquad$ 2-SS from two 1-SS on the same vacuum

$F_1 = 1 + \varepsilon e^{\eta_1} \qquad F_2 = 1 + \varepsilon e^{\eta_2}$

$F = 1 + \varepsilon e^{\eta_1} + \varepsilon e^{\eta_2} + \varepsilon^2 A_{12} e^{\eta_1 + \eta_2}$

$\Rightarrow \quad F \cdot F \ldots$ 16 terms $\qquad P(D \ldots) F \cdot F = 0$

using the properties of 1-SS and $P(D) e^{\eta_1} e^{\eta_1} = 0$ lot of terms vanish, remaining terms:

$0 = P(D..)(A_{12} 1 \cdot e^{\eta_1 + \eta_2} + e^{\eta_1} \cdot e^{\eta_2} + e^{\eta_2} e^{\eta_1} + A_{12} e^{\eta_1 + \eta_2} \cdot 1)$

P is even $\Rightarrow \quad 2 A_{12} P(\vec{p}_1 + \vec{p}_2) + 2 P(\vec{p}_1 - \vec{p}_2) = 0$

$$A_{12} = -\frac{P(\vec{p}_1 - \vec{p}_2)}{P(\vec{p}_1 + \vec{p}_2)}$$

Exercise: KdV $\qquad A_{12} = \left(\frac{p_1 - p_2}{p_1 + p_2} \right)^2 \quad$ (using dispersion relations)

3-SS $\quad$ from various limits

$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12} e^{\eta_1 + \eta_2} + A_{13} e^{\eta_1 + \eta_3} + A_{23} e^{\eta_2 + \eta_3} + A_{12} A_{13} A_{23} e^{\eta_1 + \eta_2 + \eta_3}$

no freedom in coeffs.

$\Rightarrow$ ~~$A_{12}+A_{13}+A_{23}$~~ $\sum_{\sigma_i=\pm 1} P(\sigma_1\vec{p}_1+\sigma_2\vec{p}_2+\sigma_3\vec{p}_3)\,P(\sigma_1\vec{p}_1-\sigma_2\vec{p}_2)$

$$\cdot P(\sigma_2\vec{p}_2-\sigma_3\vec{p}_3)\,P(\sigma_3\vec{p}_3-\sigma_1\vec{p}_1)=0$$

if $\exists$ soln of this $\underbrace{\text{eqn.}}_{}$ $\overset{\text{+ disp. relation}}{\Rightarrow}$ 3-SS

if $\exists$ 3-SS then usually $\exists$ general soliton

$$F=\sum_{\substack{\mu_i=0,1\\ 1\le i\le N}}\exp\Big(\sum_{1\le i,j\le N}\varphi(i,j)\mu_i\mu_j+\sum_{i=1}^{N}\mu_i\eta_i\Big),\quad A_{ij}=e^{\varphi(i,j)}$$

$\boxed{\text{mKdV}}$ $\quad u=\partial_x w \quad \Rightarrow \partial_x(w_{xxx}+2w_x^3+w_t)=0$

$\Rightarrow$ potential mKdV $\quad w_{xxx}+2w_x^3+w_t=0$

$w=2\arctan(G/F) \qquad u=\frac{2D_xF\cdot G}{F^2+G^2}$ $\quad$ (can be guessed from 1-SS)

$$\Rightarrow 0=(F^2+G^2)\left[(D_x^3+D_t)G\cdot F\right]+3(D_xF\cdot G)\left[D_x^2(F\cdot F+G\cdot G)\right]$$

we split $\Rightarrow \quad (D_x^3+D_t)G\cdot F=0 \qquad D_x^2(F\cdot F+G\cdot G)=0$

other splitting possible $(\text{—''—}+2\lambda D_x)(G\cdot F)=0 \qquad (D_x^2+\lambda)\text{ —''— }=0$

where λ general function of x,t

$\boxed{\text{sine Gordon}}$ $\qquad \phi_{xx}-\phi_{tt}=\sin\phi$

$\phi=4\arctan(G/F) \Rightarrow \dots$ after splitting

$$(D_x^2-D_t^2-1)\,G\cdot F=0 \qquad (D_x^2-D_t^2)(F\cdot F-G\cdot G)=0$$

ZAKHAROV:

KP eqn

$$\frac{\partial u}{\partial t}+6uu_x+u_{xxx}=3\alpha\,\partial^{-1}u_{yy}$$

$\alpha=1 \quad KP_I \qquad \alpha=-1 \quad KP_{\underline{II}}$

Ham. form $\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\frac{\delta H}{\delta u} \qquad H=\frac{1}{2}\int[(u_x^2+3\alpha(\partial^{-1}u_y)^2)+u^3]\,dx\,dy$

$H=H_0+H_{int}$ $\qquad u(k,t)=\dots$ analog of Fourier

TUTORIAL ON DISCRETE SYMMETRIES $\quad\emptyset$ (omitted)

KODAMA: Near integrable systems

Lecture 1: ODEs, Normal forms

Let $u=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}\in\mathbb{C}^n \qquad u:\mathbb{R}\to\mathbb{C}^n:\ t\to u(t)$

$F:\Omega\subset\mathbb{C}^n\to\mathbb{C}^n \qquad u\to F(u)=\begin{pmatrix}F_1(u_1,\dots,u_n)\\ \vdots\\ F_n(u_1,\dots,u_n)\end{pmatrix}$

$\frac{du}{dt}=F(u)=X_F(u)$ $\quad$ i.e. $\frac{du_i}{dt}=F_i(u_1,\dots,u_n)$, $\quad X_F\equiv\sum_{i=1}^{n}F_i(u)\frac{\partial}{\partial u_i}$

Remark: The IVP (initial value problem) of $\frac{du(t)}{dt}=X_F(u(t)),\ u(0)=u\in\mathbb{C}^n$ can be written by $u(t)=\exp(tX_F)u=$

$$=(I+tX_F+\tfrac{1}{2}t^2X_F\circ X_F+\dots)u$$

Exercise: $\frac{du}{dt}=a=a\frac{\partial}{\partial u}u=X_F(u)$

$$\Rightarrow u(t)=\exp(tX_F)u=\Big(1+ta\frac{\partial}{\partial u}+\frac{1}{2}t^2a^2\frac{\partial^2}{\partial u^2}+\dots\Big)u$$

$$=u+ta$$

i.e. $\exp(ta\frac{\partial}{\partial u})u=u+ta$

Ex: $\frac{du}{dt}=au=X_F(u), \qquad X_F=au\frac{\partial}{\partial u}$

$u(t)=\exp(tX_F)u \qquad$ set $u=e^\theta\Rightarrow X_F=a\frac{\partial}{\partial\theta}\Rightarrow$

$$=\exp(ta\tfrac{\partial}{\partial\theta})e^\theta=e^{\theta+ta}=e^{ta}u$$

HW: $\frac{d}{dt}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}-u_2\\u_1\end{pmatrix}=X_F\begin{pmatrix}u_1\\u_2\end{pmatrix}$

Definition: Critical point of $\frac{du}{dt}=F(u)$

$u^0\in\mathbb{C}^n$ is a critical point of $\frac{du}{dt}=F(u)$ iff $F(u^0)=0 \quad (\forall i: F_i(u_1^0,\dots,u_N^0)=0)$

Question: Find the behaviour of the soln near a critical point

Assume: $F_i \in C^{\infty}(\Omega)$, Ω is a neighborhood of u^0 (we set $u^0 = 0$)

$$\Rightarrow F_i(u_1, \dots, u_n) = \sum_{j=1}^{n} a_{ij} \cdot u_j + \sum_{i,j=1}^{n} a_{ijk} u_j \cdot u_k + \dots$$
$$+ \sum \underbrace{u \dots u}_{m\text{-times}} + o(\|u\|^m)$$

i.e. $F_i(u) = F_i^{(1)}(u) + F_i^{(2)}(u) + \dots$ $\quad F_i^{(k)}(u) \in P_{\mathbb{C}}^{(k)}[u_1, \dots, u_n]$ homogenous polynomial of degree k

$$P_{\mathbb{C}}^{(k)}[u_1, \dots, u_n] = \operatorname{span}_{\mathbb{C}} \left\{ \prod_{i=1}^{n} u_i^{m_i} \,\middle|\, \begin{array}{l} \sum m_i = k \\ m_i \in \mathbb{Z}_{\geq 0} \end{array} \right\}$$

Remark:
$$\mathbb{C}[u_1, \dots, u_n] = \bigoplus_{k=0}^{\infty} P_{\mathbb{C}}^{(k)}[u_1, \dots, u_n]$$

(Example: $\dim P_{\mathbb{C}}^{(1)} = n$, $\dim P_{\mathbb{C}}^{(2)} = \frac{n(n+1)}{2}$, ...)

$\frac{du}{dt} = Au + F^2(u) + \dots$, $A \in \operatorname{Mat}(n, \mathbb{C})$ constant $\quad a_{ij} = \frac{\partial F_i}{\partial u_j}\Big|_{u=0}$

Assume: A semisimple (diagonalisable on $\mathbb{C}$)
i.e. eigenvectors span $\mathbb{C}^n$

Explicitly $(*)$ $\frac{du_i}{dt} = \sum_j a_{ij} u_j + (a_{i11} u_1^2 + a_{i12} u_1 u_2 + \dots + a_{i22} u_2^2 + \dots + a_{inn} u_n^2 + \dots)$

$$\Rightarrow \dim P_{\mathbb{C}}^{(2)} = \frac{n(n+1)}{2}$$

We want to transform this eqn $(*)$ into a simple, so-called normal form.

Def: Resonance (Poincaré type resonance)
$\lambda_1, \dots, \lambda_n$ eigenvalues of A (with multiplicities)

$(\lambda_1, \dots, \lambda_n)$ is resonant $\Longleftrightarrow$ $\exists \lambda_\ell$ s.t. $\lambda_\ell = \sum_{i=1}^{n} m_i \lambda_i$

where $\underset{\text{order of resonance}}{|\vec{m}|} = \sum_{i=1}^{n} m_i \geq 2$, $m_i \in \mathbb{Z}_{\geq 0}$.

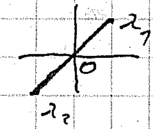
Ex: $n = 2$ $\quad n\lambda_1 = m\lambda_2$ is non-resonant if (n, m) are not divisible (mutually)
e.g. $2\lambda_1 = 3\lambda_2$, $10\lambda_1 = 6\lambda_2$ are non-resonant

$\lambda_1 = m\lambda_2$ $\quad m \geq 2$ is resonant with order m

Ex: $\lambda_1 + \lambda_2 = 0$ is resonant of order $2n+1$, $n \in \mathbb{N}$
$$\lambda_1 = (n+1)\lambda_1 + n\lambda_2$$

Remark: Poincaré Domain
$(\lambda_1, \dots, \lambda_n) \to$ closed convex hull of $\{\lambda_1, \dots, \lambda_n\}$
if $\underbrace{0}_{\text{the origin}}$ is outside $\Rightarrow$ finite number of resonances

Siegel Domain if CH contains the origin. see $\lambda_1 + \lambda_2 = 0$



Theorem 1: Poincaré
If there is no resonance then $\exists$ a $\overbrace{\text{formal change}}^{\text{not necessarily converges}}$ of coordinate
$$u = \phi(v) = v + \phi^{(2)}(v) + \dots = X_G(v), \quad \phi_i^{(k)} \in P_{\mathbb{C}}^{(k)}[v_1, \dots, v_n]$$
i component of vector
s.t. the transformed eqn for v is just $\frac{dv}{dt} = Av$ (i.e. linear)

Proof: later

Theorem 2 (Poincaré - Dulac): In general case $\exists$ a formal change of coords $u = \phi(v) = v + \phi^{(2)}(v) + \dots$ s.t. eqn for v is $\frac{dv}{dt} = Av + G^{(2)}(v) + G^{(3)}(v) + \dots = X_G(v)$
where $G_i^{(k)}(v) \in P_{\mathbb{C}}^{(k)}[v]$ is a symmetry of $\frac{dv}{dt} = Av$.

Def: $\frac{dv}{dt} = S(v)$ is a symmetry of $\frac{dv}{dt} = F(v)$ $\Longleftrightarrow$
(symmetry generator)

$$\frac{dS}{dt}(v) = X_F(S(v)) = \frac{dF}{d\tau} = X_S(F(v))$$

Proof of theorems:

Consider a transf. in the form $u = \phi(v) = \exp(X_\Phi)\, v =$
$= v + X_\Phi(v) + \frac{1}{2} X_\Phi(v) X_\Phi(v) + \dots$

$$\frac{du}{dt} = X_F(u) = X_F(\phi(v)) = \phi \circ X_F(v) = \sum_i \frac{dv_i}{dt}\frac{\partial \Phi}{\partial v_i} = X_G \circ \Phi(v)$$

$$X_G = \phi \circ X_F \circ \phi^{-1} = \exp X_\Phi \circ X_F \exp(X_\Phi)^{-1}$$

$$\Rightarrow X_G = X_F + [X_\Phi, X_F] + \frac{1}{2}[X_\Phi, [X_\Phi, X_F]] + \dots$$

$$X_F = X_{F^{(1)}} + X_{F^{(2)}} + \dots, \quad X_G = X_{G^{(1)}} + X_{G^{(2)}} + \dots$$

$X_\Phi = X_{\Phi^{(2)}} + \dots$ $\quad\Rightarrow X_{G^{(1)}} = X_{F^{(1)}} = (Ax)\cdot\nabla$
$\uparrow$ skipping linear transf.

$$X_{G^{(2)}} = X_{F^{(2)}} + [X_{\Phi^{(2)}}, X_{\Phi^{(2)}}]$$

$\Rightarrow$ Homological eqn $\quad [X_{F^{(1)}}, X_{\Phi^{(k)}}] = \underbrace{X_{\tilde{F}^{(k)}}}_{} - X_{G^{(k)}}$

given from previous steps

DISCUSSION SECTION: Painlevé test for PDE

BBM equation $\quad u_t + u_{xxt} + 6uu_x + \cancel{u_x} = 0$ shift $u \to u + \text{const} \Rightarrow$ term drops out
(not integrable)

vs.

KdV $\quad u_t + u_{xxx} + 6uu_x = 0$

(ARS conjecture: possible reductions all pass Painlevé test
- travelling wave $\quad u = w(x - ct)$
- scaling reduction $\quad u = (3t)^{-\frac{2}{3}} w\left(\frac{x}{(3t)^{\frac{1}{3}}}\right)$
- similarity reduction $\quad u \,\text{"}\!\sim\!\text{"}\, w(x - t^2)$)

for BBM no other reduction than travelling wave - this passes Painlevé test
one can easily find

$\Rightarrow$ how to distinguish between these PDEs ?

there really is one which doesn't pass Painlevé test

ODE .. expand around moveable singularity at $z = z_0$

PDE .. expand around singularity manifold $\quad \phi(x,t) = 0$

generically we assume $\phi_x \neq 0$ $\Rightarrow$ by implicit function theorem
we can choose $\phi \equiv x - \psi(t) = 0$

Weiss, Tabor, Carnevale $\quad$ WTC test

(1) Look for dominant balances

(2) Find the resonances in expansion (where appear arbitrary functions in expansion)

(3) Substitute expansion into the equation and check that resonance conditions are fulfilled

(1) $\quad u \sim a(x,t)\, \phi^{-\mu}$ $\qquad \underset{\sim}{\mu} > 0$
$u_x \sim -\mu \phi_x\, a(x,t)\, \phi^{-(\mu+1)}$ $\qquad$ for a singularity of a pole type
(for branching point can be $\mu < 0$)
$u_{xx} \sim \mu(\mu+1)(\phi_x)^2 a\, \phi^{-(\mu+2)}$
$u_{xxt} \sim -\mu(\mu+1)(\mu+2)(\phi_x)^2 \phi_t\, a\, \phi^{-(\mu+3)}$
$uu_x \sim -\mu a^2 \phi_x\, \phi^{-2\mu-1}$

BBM: balance of terms u_{xxt} & uu_x $\Rightarrow \mu + 3 = 2\mu + 1 \Rightarrow \mu = 2$

$$-24(\phi_x)^2\phi_t \sim 2\cdot 6\cdot \phi_x a \Rightarrow a = -2\phi_x\phi_t$$

$$u = -2\phi_x\phi_t\,\phi^{-2} + u_{-1}\phi^{-1} + \sum_{j=0}^{\infty} u_j \cdot \phi^j$$

KdV: $\quad u = -2(\phi_x)^2\phi^{-2} + 2\phi_{xx}\phi^{-1} + \tilde{u}$ $\quad \tilde{u} = \sum_{j=0}^{\infty} u_j \phi^j$
$= 2(\log\phi)_{xx} + \tilde{u}$

($\Rightarrow$ one can guess a substitution to bilinear form)

(2) $\quad u \sim a\phi^{-2}(1 + \varepsilon\phi^r)$ substitute into dominant terms and look to dominant terms in ε

$u_x \sim a\phi_x\left(-2\phi^{-3} + \varepsilon(n-2)\phi^{n-3}\right)$

Look for linear terms in ε from $u_{xxt} + 6uu_x$

$u_{xxt} \sim a\phi_x^2\phi_t\left(-2\overset{-2}{\cancel{4\phi}}\phi^{-3} + \varepsilon(n-2)(n-3)(n-4)\phi^{n-5}\right)$

$6uu_x \sim 6a^2\phi_x\left(-2\phi^{-3} + \varepsilon\left[(n-2)-2\right]\phi^{n-3}\right)\phi^{-2}$

Coeff. of $\varepsilon\phi^{n-5}$: $\phi_x\phi_t^2 a(n-2)(n-3)(n-4) + \phi_x 6a^2(n-4) \underset{!}{=} 0$

i.e. $0 = \phi_x a\left(\phi_x\phi_t(n-2)(n-3)\cancel{6} + 6a\right)(n-4)$ to have a resonance

$$0 = \phi_x^2\phi_t\ \underbrace{\left((n-2)(n-3)-12\right)}(n-4) = 0$$

$$(n+1)(n-4)(n-6) = 0$$

$\Rightarrow$ resonances $\underbrace{n=-1}_{\text{universal resonance}}$ $\quad n=4 \quad n=6$

the same both for KdV and BBM

$\Rightarrow$ ~~k~~ coef. of ϕ^2 and ϕ^4 should have arbitrary params

(3) one should plug these expansions in, check resonances

(another possibility: resonances appear earlier in potential version of given eqns.)

BBM: $u = -2v_x \quad v_t + v_{xxt} - 12v_x^2 = 0 \quad v \sim \phi_t\phi^{-1} + \sum_{j=0}^{\infty} v_j\phi^j$

resonances at $n=-1 \quad n=1 \quad n=6$

coeff. at $n=1$ fails the resonance check

$$\underbrace{\phi_t^2\phi_{xx} + \phi_x^2\phi_{tt} - 2\phi_x\phi_t\phi_{xt} = 0}$$

not valid for a general ϕ $\Rightarrow$ there is no soln of potential BBM in the form assumed $\Rightarrow$ there should ~~be~~ be a logarithm $\Rightarrow$ branching $\Rightarrow$ not Painlevé

potential KdV .. resonance conditions are satisfied

SANDERS:

SOKOLOV: Transformations and already solved classification problems

Def: Formal recursion operator $L = l_1D + l_0 + l_{-1}D^{-1} + \dots$ for $u_t = F[u_m,\dots,u)$ if $L_t - [F_*, L] = 0$ where $F_* = \sum_k \frac{\partial F}{\partial u_k}D^k$

We call $l_{-1} \equiv \operatorname{res} L$.

Remark: $D^n f = fD^n + nD(f)D^{n-1} + \frac{n(n-1)}{2}D^2(f)D^{n-2} + \dots$

In the following we assume $u_t = F(u_m,\dots,u_x,u)$, $m>1$

Theorem 1: Eqn $u_t = F$ with infinite hierarchy of higher symmetries $u_{\tau_i} = G_i(u_{m_i},\dots,u)$, $m_i \to \infty$ $\Longrightarrow$ eqn has a formal recursion operator

Theorem 2: Eqn $u_t = F$ with infinite hierarchy of conserved quantities $\rho_i(u_{m_i},\dots,u) \in \operatorname{Im} D$, $\frac{\partial^2\rho_i}{\partial u_{m_i}^2} \neq 0$, $m_i \to \infty$ $\Rightarrow \exists$ formal recursion operator

Theorem 3: $u_t = F$ related to linear eqn $v_t = v_n$ by a differential substitution $v = \varphi(\cancel{u_k},\dots,\cancel{u})$ $\Rightarrow$ ~~infinite~~ $\exists$ formal recursion operator. $\longleftarrow$ subst. from u to v $v = \varphi(u_k,\dots,u)$

$\Rightarrow \rho_i = \operatorname{res}(L^i) \quad i = -1,1,2,\dots, \quad \rho_0 = \frac{l_0}{l_1}$ are conserved densities for $u_t = F$ .... canonical densities

Ex: KdV $u_t = u_3 + 6uu_1$, $L = \left(D^2 + 4u + 2u_1D^{-1}\right)^{1/2}$

$$\rho_1 = 2u \quad \rho_2 = 2u_1 \quad \rho_3 = 2u_2 + u^2$$

Theorem 4: (1) Under assumptions of Th 2 all even canonical densities ρ_{2j} are trivial.

(2) Under assumptions of Th 3 all canonical densities are trivial.

a simple classification problem

$u_t = u_3 + f(u_1, u)$? all eqns having formal recursion operator

(1) ρ_1, equate coeffs of $D^3, D^2, \dots$ in $L_t - [F_*, L] = 0$

$\Rightarrow \rho_1 = \frac{1}{3}\frac{\partial f}{\partial u_1}$ should be conserved density

$\Rightarrow 0 \overset{!}{=} \frac{\delta}{\delta u}\left(\frac{\partial f}{\partial u_1}\right)$ ∀ systems with formal recursion operator

$\Rightarrow f(u_1, u) = \lambda u_1^3 + A(u) u_1^2 + B(u) u_1 + C(u)$

- where $0 = \lambda A'$ $\quad B''' + 8\lambda B' = 0$

$(B'C)' = 0 \quad AB' + 6\lambda C' = 0$

(2) $\rho_2 \Rightarrow \frac{\delta}{\delta u}\left(\frac{\partial f}{\partial u}\right) \overset{!}{=} 0 \Rightarrow$ only possible equations

KdV, MKdV, Calogero - Degasperis

Transformations (e.g. Cole - Hopf)
find group w.r.t which the original eqn. is invariant $\Rightarrow$ find group invariants and use the simplest one as a new variable $\Rightarrow$ new integrable equation

Ex: $u_t = u_2$, $u \to \tau u$

$\Rightarrow \hat{u} = \frac{u_x}{u} \Rightarrow$ Burgers $\hat{u}_t = \hat{u}_2 + 2\hat{u}\hat{u}_1$

If we admit $u_t = u_3 + f(u_2, u_1, u)$

$\Rightarrow$ we have two more integrable eqns. (up to "almost invertible" transformations (point transf., potentiation, differentiation))

Calogero - Degasperis 2, KN (… - Novikov)

Using Miura type eqns. we are left: MKdV, CD2.

Other known classifications $u_{xy} = F(u)$

$u_t = F(u_2, u_1, u, x, t)$

$u_t = u_5 + F(u_4, u_3, u_2, u_1, u)$

Systems $u_t = u_2 + F(u, v, u_1, v_2)$, ~~a~~ $v_t = -v_2 + G(u, v, u_1, v_1)$

KODAMA:

Summary $(n \to N)$

Theorem (Poincaré - Dulac): $\frac{du}{dt} = X_F(u) = A~~u~~ + F^{(2)}(u)$

where $F^k(u) \in \mathbb{P}_{\mathbb{C}}^{(k)}[u_1, \dots, u_N] = P_{\mathbb{C}}^{(k)}[u_1, \dots, u_N] \otimes_{\mathbb{C}} \mathbb{R}^N$

$\Rightarrow \exists$ a formal change of ~~variables~~ coordinates

$u = \phi(v) = v + \phi^{(2)}(v) + \dots$, $\phi^{(k)}(v) \in \mathbb{P}_{\mathbb{C}}^{(k)}[u_1, \dots, u_N]$

such that $\frac{dv}{dt} = X_G(v) = Av + G^{(2)}(v) + \dots$

where $G^{(k)}(v) \in \mathbb{P}_{\mathbb{C}}^{(k)}[u_1, \dots, u_N]$ is a symmetry of $\frac{dv}{dt} = Av$,

$[X_{Av}, X_{G^{(k)}}] = 0$.

Proof: see last lecture

$L_A \Phi^{(k)} = \left(\sum_{i=1}^N (Av)_i \frac{\partial}{\partial v_i} - A\right)\Phi^{(k)} = \tilde{F}^{(k)} - G^{(k)}$

non-resonant case $G^{(k)} = 0$

in general decomposition theorem $G^{(k)} = \pi(\tilde{F}^{(k)})$,

$\pi : \mathbb{P}_{\mathbb{C}}^{(k)} \to \operatorname{Ker} L_A \cap \mathbb{P}_{\mathbb{C}}^{(k)} \Rightarrow L_A G^{(k)} = 0$

$\Leftrightarrow [X_{Av}, X_{G^{(k)}}] = 0$

QED

Explicit form of $\operatorname{Ker} L_A \cap \mathbb{P}_{\mathbb{C}}^{(k)}$

assume $A = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$, $L_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(\sum \lambda_i v_i \frac{\partial}{\partial v_i}\right) - A$

Define $E_e^{(k)}(v) = e_e \prod_{i=1}^N v_i^{m_i}$ $\quad k = |m| = \sum_{i=1}^N m_i$

$$e_\ell = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow \ell + k \qquad A e_\ell = \lambda_\ell e_\ell$$

$$L_A \mathbb{E}_\ell^{(k)} = \left(\sum_{i=1}^{N} \lambda_i m_i\right) \mathbb{E}_\ell^{(k)} - \lambda_\ell \mathbb{E}_\ell^{(k)} = \left(\sum_i \lambda_i m_i - \lambda_\ell\right) \mathbb{E}_\ell^{(k)}$$

$$\Rightarrow \text{if } \lambda_\ell = \sum_i \lambda_i m_i \Rightarrow \mathbb{E}_\ell^{(k)} \in \ker L_A$$

$$\ker L_A \cap \mathbb{P}_{\mathbb{C}}^{(k)} = \operatorname{span}_{\ell}\, \mathbb{C} \left\{ \mathbb{E}_\ell^{(k)}(v) \,\middle|\, \lambda_\ell = \sum_i \lambda_i m_i \text{ for some } m_i \in \mathbb{Z}_{\geq 0} \right\}$$

Theorem (Poincaré - Dulac for convergence)

If $(\lambda_1, \dots, \lambda_N)$ belongs to the Poincaré domain and $F(v) \in C^\omega(\Omega)$. Then $u = \phi(v)$ converges in $D \subset \Omega$

$\left(\Rightarrow \frac{du}{dt} = F(u) \longleftrightarrow \frac{dv}{dt} = G(v) \qquad C^\omega\text{-equivalence}\right)$

Poincaré domain $\Rightarrow G$ .. polynomial

Ex 1: $\frac{d}{dt} u = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} u + F^{(2)}(u) + \dots \qquad \lambda_1 = 2\lambda_2$ resonance of order 2

$$\longrightarrow \frac{dv}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} v + \underbrace{G^{(2)}(v)}_{\text{series stops because of}}$$

$$\mathbb{E}_1^{(2)}(v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_2^2 \Rightarrow \ker L_A = \mathbb{C}\begin{pmatrix} v_2^2 \\ 0 \end{pmatrix} \Rightarrow G^{(2)} \sim \alpha \begin{pmatrix} v_2^2 \\ 0 \end{pmatrix}$$

Note:

$\ker L_A \cap P_{\mathbb{C}}^{(k)} = \{0\}$ for $k > 2$

Remark: $\frac{dv}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} v + \alpha \begin{pmatrix} v_2^2 \\ 0 \end{pmatrix} \xrightarrow{\hspace{2em}} \frac{dw}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} w$

only C^1-equivalence $\quad v_1 = w_1 + \alpha w_2^2 \ln w_2$

$v_2 = w_2$

Remark: If X_F is a hamiltonian vector field (then $X_F(v) = J \nabla_v \underset{\substack{\uparrow \\ \text{functions}}}{H_F}$)

$$\Rightarrow [X_G, X_F] = X_{\boxed{\{H_G, H_F\}}} \text{ Poisson bracket}$$

.. Birkhoff normal form (Hamilt. $\Rightarrow$ in Siegel domain)

$\Rightarrow$ convergence is very tricky

Ex 2: $\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + F^{(2)}(u) + \dots$

$\lambda_1 = i,\ \lambda_2 = -i,\ \lambda_1 + \lambda_2 = 0 \quad \rightarrow (\lambda_1, \lambda_2)$ has resonances of order $2m+1$, $m = 1, 2, \dots$

For $\lambda_1 = (m+1)\lambda_1 + m\lambda_2 \Rightarrow \mathbb{E}_1^{(2m+1)}(v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (v_1 v_2)^m v_1$

For $\lambda_2 = m\lambda_1 + (m+1)\lambda_2 \Rightarrow \mathbb{E}_2^{(2m+1)}(v) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (v_1 v_2)^m v_2$

Eqns in diag. form $\frac{d}{dt} v = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} v + F^{(2)}(v) + \dots$

Normal form: $\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} w + \sum_{m=1}^{\infty} (w_1 w_2)^m \left(\alpha_m \begin{pmatrix} w_1 \\ 0 \end{pmatrix} + \beta_m \begin{pmatrix} 0 \\ w_m \end{pmatrix}\right)$

$w_1 = \overline{w_2}$

In the real form: $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sum_{m=1}^{\infty} (x_1^2 + x_2^2)^m \left(a_m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b_m \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\right)$

$$\underbrace{a_m}_{\text{stretching}} x + \underbrace{b_m}_{\text{rotation}} A x$$

Remark: Symmetry algebra

$$X_0^m(x) = (x_1^2 + x_2^2)^m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ X_1^m(x) = (x_1^2 + x_2^2)^m \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Then $\quad X_1^0\left((x_1^2 + x_2^2)^n\right) = 0,\ X_0^0\left((x_1^2 + x_2^2)^n\right) = 2n (x_1^2 + x_2^2)^n$

$$[X_0^n, X_0^m] = 2(m-n) X_0^{m+m} \qquad [X_0^n, X_0^m] = 0$$

$$[X_0^n, X_1^m] = 2m X_1^{n+m}$$

i.e. the symmetry algebra is <u>not</u> commutative.

Comment on multiple-scaling method

$$\frac{du}{dt} = Au + \varepsilon F^{(2)}(u) + \varepsilon^2 F^{(3)}(u) + \dots$$
$$= Au + \varepsilon X^{(1)}(u) + \varepsilon^2 X^{(2)}(u) + \dots$$

Introduce $t_n = \varepsilon^n t \qquad n = 0, 1, 2, \dots$

$$\frac{d}{dt} = \sum_{n=0}^{\infty} \varepsilon^n \frac{\partial}{\partial t_n} \qquad \frac{\partial u}{\partial t_0} - Au = \sum_{n=1}^{\infty} \varepsilon^n \left(X^{(n)}(u) - \frac{\partial u}{\partial t_n} \right)$$

Now expand $\quad u = u_0 + \varepsilon u_1 + \dots$

$O(1)$ $\quad \frac{\partial u_0}{\partial t_0} = A u_0$ $\qquad O(\varepsilon)$: $\frac{\partial u_1}{\partial t_0} - A u_1 = X^{(1)}(u_0) - \frac{\partial u_0}{\partial t_1}$

Choose $\frac{\partial u_0}{\partial t_1} := S^{(1)}(u_0)$ where $S^{(1)}$ is a symmetry of $\frac{du_0}{dt_0} = Au_0$
(in order to get rid off the secular term)

$O(\varepsilon^2)$: $\quad \frac{\partial u_2}{\partial t_0} - A u_2 = X^{(2)}(u_0) - \frac{\partial u_0}{\partial t_2} + \dots$

Choose $\frac{\partial u_0}{\partial t_2} = S^{(2)}(u_0)$

$\Rightarrow \frac{du_0}{dt} = \frac{\partial u_0}{\partial t_0} + \varepsilon \frac{\partial u_0}{\partial t_1} + \varepsilon \frac{\partial u_0}{\partial t_2} + \dots = Au_0 + \varepsilon S^{(1)}(u_0) + \varepsilon^2 S^{(2)}(u_0) + \dots$

$\Rightarrow$ we have normal form

solvability of conds. on $S^{(i)}(u_0)$ lead to restrictions (see Degasperis' lectures)

EX: $\frac{d}{dt}u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u + \varepsilon(u_1^2+u_2^2)u + \varepsilon^2(u_1^2+u_2^2)^2 u + \varepsilon^3(u_1^2+u_2^2)^3 u$
$\Rightarrow$ conds on compatible

SOKOLOV : Tutorial

Remark: $\tilde t = f(t,x,u)$ $\quad \tilde x = g(t,x,u)$ $\quad \tilde u = h(t,x,u)$

How to compute $\tilde u_{\tilde t}$, $\tilde u_{\tilde x}$ etc.?

a simple calculation using diff. vector fields

$D_{\tilde t} = \alpha_1 D_t + \alpha_2 D_x$ $\qquad D_{\tilde x} = \beta_1 D_t + \beta_2 D_x$

and use $1 = D_{\tilde t}\tilde t = \alpha_1 D_t \tilde t + \alpha_2 D_x \tilde t$ put in def.

$0 = D_{\tilde t}\tilde x = \alpha_1 D_t \tilde x + \alpha_2 D_x \tilde x \Rightarrow \alpha_1, \alpha_2$ (functions)

similarly $\beta_1, \beta_2 \Rightarrow$ we know $D_{\tilde t}, D_{\tilde x}$

then $\tilde u_{\tilde t} = D_{\tilde t}\tilde u = \dots$ using defining relations of transf.

$\tilde u_{\tilde x} = D_{\tilde x}\tilde u = \dots = \text{function}(x,t,u,u_1)$

in this way we can transform eqns. given in ~-variables to other equiv. eqns in ordinary x, t, u-variables

Contact transformations

Miura $\quad u_t = u_{xxx} + 6uu_x \qquad v_t = v_{xxx} - 6v^2 v_x$

$u = \pm v_x + v^2$

$\Rightarrow$ solns of MKdV $\Rightarrow$ solns of KdV (immediately)
$\Leftarrow$ ... one must solve Riccati
i.e. not invertible

Legendre $\quad \tilde t = t \quad \tilde x = u_1 \quad \tilde u = u - xu_1$
is invertible ... an example of contact transf.

Def: Contact transf. $\tilde x = \varphi(x,u,u_1) \quad \tilde u = \psi(x,u,u_1)$
$\tilde u_i := \left(\frac{1}{D(\varphi)}D\right)^i(\psi)$

where $D(\varphi)\frac{\partial \psi}{\partial u_1} = D(\psi)\frac{\partial \varphi}{\partial u_1}$, i.e. $\frac{\partial \psi}{\partial u_1}\left(\sum_i u_i \frac{\partial \varphi}{\partial u_{i-1}} + \frac{\partial \varphi}{\partial x}\right) = \frac{\partial \varphi}{\partial u_1}\left(\sum_i u_i \frac{\partial \psi}{\partial u_{i-1}} + \frac{\partial \psi}{\partial x}\right)$

or even $\frac{\partial \psi}{\partial u_1}\left(u_1 \frac{\partial \varphi}{\partial u} + \frac{\partial \varphi}{\partial x}\right) = \frac{\partial \varphi}{\partial u_1}\left(u_1 \frac{\partial \psi}{\partial u} + \frac{\partial \psi}{\partial x}\right)$

$\Rightarrow$ one has $\left.\begin{aligned} \hat x &= \varphi(x,u,u_1) \\ \hat u &= \psi(x,u,u_1) \\ \hat u_1 &= \chi(x,u,u_1) \end{aligned}\right\} \Rightarrow$ we can invert this transformation locally (since $\hat u_1$ doesn't depend on u_2)

Bäcklund has proved that such cut is not possible for higher derivatives (u_2 or u_3 etc.)

Not developed for more dependent variables (and 2 indep. variables) since in this case only point transformations are invertible [Lie].

But $\exists$ strange transf. $\hat u = u$, $\hat v = v + \psi(u, u_x, \dots, u_k)$
which are invertible ... triangular transf.
... can be used to simplify certain eqns with high order derivatives.

HIETARINTA:
General observation

(1) No algorithmic procedure to bilinearise given eqn.

(2) No one knows how many dependent and independent variables use.

(3) Given bilinear eqn has several nonlinear versions

Gauge invariant $A(D_x, D_y \dots)\, e^{\theta} F \cdot e^{\theta} G = e^{2\theta} A(D_x, D_y \dots) F \cdot G$
$\theta = ax + by + \dots$

One can define the Hirota's bilinear form through the requirement of this gauge invariance.

Bilinear ⟶ Multi-linear
we request gauge invariance

e.g. $n=3$ $\quad T^{m} (T^{*})^{N-m}$ basis of trilinear operators
$T = \partial_x + j\partial_y + j^2 \partial_z \qquad j = e^{\frac{2\pi}{3}}$
$T^{*} = \partial_x + j^{-2}\partial_y + j\partial_z$ possibility of dispersionless situation (nontrivial polynomial in T, T^{*} ⇒ trivial polynomial in numbers)

KdV soliton content
$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}$
let we comove with soliton one ⇒ η_1 is small and soliton 2 is far away when $|\eta_2| \to \infty$

MKdV has two bilinearisations

MKdV & SG ⇒ class $B(D_{\vec{x}})\, G \cdot F = 0$
$A(D_{\vec{x}})\,(F \cdot F + \varepsilon G \cdot G) = 0$
A even B either even or odd
(mKdV $B = D_x^2 + D_t^2$, $A = D_x^2$, $\varepsilon = +1$, SG: $B = D_x^2 - D_t^2 - 1$, $A = D_x^2 - D_t^2$, $\varepsilon = -1$)
vacuum $F = 1$, $G = 0$ (we choose)
1SS $F = 1 + \alpha e^{\eta}$, $G = \beta e^{\eta}$
⇒ $\alpha A(\vec{p}) = 0$, $\beta B(\vec{p}) = 0$, $\alpha\beta B(0) = 0$ ⇒ 2 kinds of solitons
type a soliton $F = 1 + e^{\eta_A}$, $G = 0$ Dispersion relation $A(\vec{p}) = 0$
type b soliton $F = 1$ $G = e^{\eta_B}$ —"— $B(\vec{p}) = 0$

2-soliton solution: a+a $F = 1 + \varepsilon e^{\eta_1} + \varepsilon e^{\eta_2} + O(\varepsilon^2)$
$G = O(\varepsilon^2)$ with $A(\vec{p}_1) = A(\vec{p}_2) = 0$
we choose $G = 0$ ⇒ KdV type eqn for A
a+b $F = 1 + \varepsilon e^{\eta_1}$ $\quad 0 = A(\vec{p}_1) = B(\vec{p}_2)$
$G = e^{\eta_2} + L_{12} e^{\eta_1 + \eta_2}$
where $L_{12} = -\dfrac{B(\vec{p}_2 - \vec{p}_1)}{B(\vec{p}_2 + \vec{p}_2)}$

b+b $F = 1 - K_{12} e^{\eta_1 + \eta_2}$
$G = e^{\eta_1} + e^{\eta_2}$ $\quad K_{12} = \varepsilon \dfrac{A(\vec{p}_2 - \vec{p}_1)}{A(\vec{p}_2 + \vec{p}_1)}$
$B(\vec{p}_1) = B(\vec{p}_2) = 0$

NLS $iu_t + u_{xx} + 2\varepsilon |u|^2 u = 0$ $\quad u \dots \mathbb{C}$-function

$u = g/f$, g complex, f real
⇒ $(iD_t + D_x^2)\, g \cdot f = 0$ $\qquad D_x^2 f \cdot f = \varepsilon 2 |g|^2$

we choose $f = 1$ $g = 0$ vacuum … bright solitons

1SS: $g = e^{\eta}$ $\quad f = 1 + \alpha e^{\eta + \eta^{*}}$ $\quad \eta = px + \omega t \in \mathbb{C}$
where $i\omega + p^2 = 0$ $\qquad \alpha = \dfrac{\varepsilon}{(p + p^{*})^2}$

Generalization $B(D)\, G \cdot F = 0$
$A(D)\, F \cdot F = \varepsilon 2 |G|^2$
dispersion relation $B(\vec{p}) = B(-\vec{p}^{*}) = 0$, place $\alpha = \dfrac{\varepsilon}{A(\vec{p} + \vec{p}^{*})}$

Since we need 2nd order in ε for 1SS ⇒ 2SS not automatic (1SS is like usual 2SS, 2SS is like usual 4SS)

Dark solitons … another choice of vacuum $f = 1$, $g = \underbrace{e^{i\theta}}_{\text{pure phase factor}}$

Multicomponent equation Hirota-Satsuma
$u_t + u_{xxx} + 6uu_x - 6vv_x = 0$
$v_t - 2v_{xxx} - 6uv_x = 0$
⇒ bilinearisation resembles the one for NLS,
2 kinds of 1SS (KdV type, NLS type - but real)

Dromions   Davey–Stewartson eqn. $iu_t + (\partial_x^2 + \partial_y^2)u + uv = 0$

$$v_{xy} = 2(\partial_x^2 + \partial_y^2)|u|^2$$

$$u = G/F\,, \quad v = 2(\partial_x^2 + \partial_y^2)\log F\,.$$

1-SS : 2 types : plane waves solitons of the NLS type

+ "ghost" solitons ($u = 0$)

2-SS : e.g. 2 ghost solitons ⇒ suddenly appears and disappears peak in u ⇒ terminology ("ghosts")

1-dromion solution

Bilinearization of hierarchies :

shallow water wave eqn $v_{xxxt} + \alpha v_x v_{xt} + \beta v_t v_{xx} - v_{xx} - v_{xt} = 0$

integrable if $\alpha = \beta = 3$ or $\alpha = 4, \beta = 2$

$\Downarrow$ bilinear form $\Downarrow$ quadrilinear ($\sim f^4$) form

$\Downarrow$ introduce extra indep. variable (τ)
→ bilinear form

this happens usually for higher eqns in hierarchies

Bilinear Bäcklund transformations

KdV $(D_x^4 + D_x D_t)\, F\cdot F = 0$ $(*)$

$$F^2[(D_x^4 + D_x D_t)G\cdot G] - G^2[(D_x^4 + D_x D_t)]F\cdot F = 0 \quad (**)$$

we rewrite using identities

$$2D_x[(D_x^3 + D_t - \lambda D_x)\,G\cdot F]\cdot[FG] + 6D_x[(D_x^2 + \lambda)G\cdot F]$$
$$\cdot[D_x F\cdot G] = 0$$

⇒ splits into
$$\left.\begin{aligned}(D_x^3 + D_t - 3\lambda D_x)\,G\cdot F &= 0\\ (D_x^2 + \lambda)\,G\cdot F &= 0\end{aligned}\right\} (***)$$

⇒ if F satisfies $(D_x^4 + D_x D_t)\,F\cdot F = 0$ $(*)$.

and G satisfies $(**)$, i.e. $(***)$ ⇒ G satisfies $(*)$.

Still remains to prove consistency, i.e. $\exists\, G$? By calculation of commutator of Lax pair like expressions we get O.K. (But this is not valid for a general equation instead of KdV.)

HIETARINTA : Tutorial

http://users.utu.fi/hietarin/dromions

Exercises :

KdV $(D_x^4 + D_x D_y)\, F\cdot F = 0$ $A_{12} = -\dfrac{P(\vec{p}_1 - \vec{p}_2)}{P(\vec{p}_1 + \vec{p}_2)}$,

dispersion relation $p_i^4 + p_i\,\omega_i = 0$ ⇒ $P(\underset{(x,\,\omega(x))}{\vec{x}}) = x^4 + x\cdot\omega(x)$

$$\Rightarrow A_{12} = -\frac{(p_1 - p_2)^4 + (p_1 - p_2)(\omega_1 - \omega_2)}{(p_1 + p_2)^4 + (p_1 + p_2)(\omega_1 + \omega_2)} \underset{\omega_i = p_i^3 \text{ supposing } p_i \neq 0}{=}$$

$$= -\frac{(p_1 - p_2)[\;\cdots\;]}{(p_1 + p_2)[\;\cdots\;]} = -\frac{(p_1 - p_2)}{(p_1 + p_2)}\cdot\frac{(p_2 - p_1)}{(p_1 + p_2)}$$

$$= +\frac{(p_1 - p_2)^2}{(p_1 + p_2)^2}$$

mKdV et al. $B(D)\,G\cdot F = 0$

$A(D)\,(F\cdot F + \overset{\pm 1}{2}\,G\cdot G) = 0$

0 SS : $F = 1$, $G = 0$ ⇒ $A(0) = 0$

1 SS : $F = 1 + \varepsilon a e^{\eta_1}$ $G = \varepsilon b e^{\eta_2}$

$\eta_i = p_i x + q_i y + \omega_i t + \dots + \eta_c^0$

$\Rightarrow \quad B(D)\varepsilon b\, e^{\eta_1}(1+\varepsilon a e^{\eta_1}) = \varepsilon b\, B(p_1, q_1, \dots)\, e^{\eta_1} +$

$+ \varepsilon^2 ab\, B(0, 0, \dots) = 0$

$'' B(D)(e^{\eta_1}\cdot e^{\eta_1})$

$bB(\vec{p}) = 0 \qquad abB(\vec{0}) = 0$

$A(D)\,(1+\varepsilon a e^{\eta_1})(1+\varepsilon a e^{\eta_1}) + \tau \varepsilon^2 b^2 A(D) e^{\eta_1}\cdot e^{\eta_2} \overset{?}{=} 0$

$\cancel{A(0)} 1 + 2e^{\eta_1} A(\vec{p})\,\varepsilon a + \cancel{\varepsilon^2 a^2 A(0)} + \cancel{\tau \varepsilon^2 b^2 A(0)} = 0$ (see above %)

$\Rightarrow \quad a\cdot A(\vec{p}) = 0$

either $a = 0$ or $b = 0$ (otherwise either $\underbrace{2 * \text{disp. relation}}_{\text{unacceptable}}$ or vacuum)

1) $a = 0 \Rightarrow B(\vec{p}) = 0$ $\qquad b \equiv 1 \Rightarrow F = 1, \quad G = e^{\eta}$
2) $b = 0 \Rightarrow A(\vec{p}) = 0$ $\qquad F = 1 + e^{\eta_1}, \quad G = 0$

velocity $v = \frac{d\omega}{dp}$ should be $\frac{d^2\omega}{dp^2} \neq 0$ in order to have solitons with different velocities

a+a 2SS $\quad G = 0 \quad F = 1 + \varepsilon(e^{\eta_1} + e^{\eta_2}) + \varepsilon^2 A_{12} e^{\eta_1+\eta_2}$

a+b 2SS $\quad F = 1 + \varepsilon(e^{\eta_1} + 0) + o(\varepsilon^2), \quad G = \varepsilon(0 + e^{\eta_2}) + o(\varepsilon^2)$

assume $o(\varepsilon^2) = \varepsilon^2 K e^{\eta_1+\eta_2}$ & $\varepsilon^2 L e^{\eta_1+\eta_2}$

$$0 = B(D)\,(e^{\eta_2} + \varepsilon L e^{\eta_1+\eta_2})\cdot(1 + \varepsilon e^{\eta_1} + K e^{\eta_1+\eta_2}\varepsilon^2) =$$

$$= B(D)\big(e^{\eta_2}\cdot 1 + e^{\eta_2}\cdot e^{\eta_1} + K e^{\eta_2}\cdot e^{\eta_1+\eta_2} + L e^{\eta_1+\eta_2}\cdot 1 + L e^{\eta_1+\eta_2}\cdot e^{\eta_1} + LK e^{\eta_1+\eta_2}\cdot e^{\eta_1\eta_2}\big) =$$

we put $\varepsilon = 1$

$\Rightarrow K = 0$ the only term with this exp

DR of 2nd soliton

$$= \cancel{B(\vec{p}_2)\, e^{\eta_2}} + B(\vec{p}_2 - \vec{p}_1)\, e^{\eta_1+\eta_2} + K\, B(-\vec{p}_1)\, e^{\eta_1+2\eta_2}$$

$$+ LB(\vec{p}_1 + \vec{p}_2)\, e^{\eta_1+\eta_2} + \cancel{LB(\vec{p}_2)\, e^{2\eta_1+\eta_2}} + \cancel{LKB(0)\, e^{2\eta_1+\eta_2}}$$

$K = 0$

$$\Rightarrow \quad L = -\frac{B(\vec{p}_2 - \vec{p}_1)}{B(\vec{p}_1 + \vec{p}_2)} \wedge K = 0$$

put it into $A(D)\,(F\cdot F + \varepsilon G\cdot G) = 0$ and see when there is no contradiction. If there is, then try general ansatz $F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \dots$, $G = \dots$, $f_1 = e^{\eta_1}, g_1 = e^{\eta_2}$ and check whether the series terminates (it must in the integrable case).

b+b 2SS $\quad F = 1 + \varepsilon 0 + \varepsilon^2 K e^{\eta_1+\eta_2}$

$\qquad G = 0 + \varepsilon(e^{\eta_1} + e^{\eta_2}) + \varepsilon^2 L e^{\eta_1+\eta_2} \Rightarrow K, L$ + conditions

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SANDERS: Geometry → integrable systems

∃ two compatible Hamiltonian structures
one of Ham. structures symplectic (formally invertible)
⇒ ∃ recursion operator ⇒ conserved quantities etc.

KODAMA

KdV with higher order corrections

Ex: BBM $\quad u_t + u_x + 6uu_x - u_{xxt} = 0$

Linear dispersion relation

$u \to$ small $\quad u_t + u_x - u_{xxt} = 0 \qquad u = A\cos(kx - \omega t)$

$\Rightarrow \omega - k + k^2\omega = 0$

$\Rightarrow \omega = \frac{k}{1+k^2}$

Write $\quad (1 - \partial_x^2)(u_t + u_x) + \underbrace{6uu_x}_{\text{nonlinear}} + \underbrace{u_{xxx}}_{\text{dispersive}} = 0$

Long wave approximation $\lambda \to$ ~~long~~ large $\quad k = \frac{2\pi}{\lambda} \to$ small

$x' = \delta x \qquad \frac{\partial}{\partial x} \sim k \sim \delta \ll 1, \quad \omega \sim k \sim \delta$

$t' = \delta t \qquad \frac{\partial}{\partial t} \sim \omega \sim k \sim \delta \ll 1$

⇒ in new variables $\delta(1 - \delta^2\partial_x^2)(u_t + u_x) + \delta 6uu_x + \delta^3 u_{xxx} = 0$

$\delta \to 0 \qquad u_t + u_x + 6uu_x = 0$ ⇒ shockwave behaviour

(⇒ u_x large ⇒ not good approx.)

Small amplitude approx. $\quad u \sim \delta^2, \quad u = \delta^2 u'$

$\Rightarrow (1 - \delta^2\partial_x^2)(u_t + u_x) + \delta^2(6uu_x + u_{xxx}) = 0$

$\varepsilon = \delta^2 \qquad (1 - \varepsilon\partial_x^2)(u_t + u_x) + \varepsilon(6uu_x + u_{xxx}) = 0$

Remark: ∃ a solitary wave $\quad u(x,t) = \frac{2k^2}{1 - 4\varepsilon k^2}\operatorname{sech}^2\left[kx - \frac{k}{1 - 4\varepsilon k^2}t\right]$

$\varepsilon \to 0 \quad u(x,t) = 2k^2\operatorname{sech}^2[k(x - t)]$ (soln of $u_t + u_x = 0$)

ε-correction $u(x,t) = 2K^2 \operatorname{sech}^2[K(x-t) - 4\varepsilon K^3 t + O(\varepsilon^2)] + O(\varepsilon^2)$

modify the shape, frequency shift

Formally $u_t + u_x + \varepsilon(1-\varepsilon\partial_x^2)^{-1}(6uu_x + u_{xxx})$

$$= u_t + u_x + \sum_{n=0}^{N} \varepsilon^{n+1} F^{(n)}(u) + O(\varepsilon^{n+2}) = 0$$

$F^{(n)}(u) = \partial_x^{2n}(6uu_x + u_{xxx})$ differential polynomial

This expansion may be verified by imposing the initial condition

$|u(x,0)| \le M \exp(-\varepsilon^{1/2}|x|)$ as $|x| \to \infty$

$$\| u(x,0)\|^2_{H^\infty(\mathbb{R})} \equiv \sum_{n=0}^{\infty} \int_{\mathbb{R}} |\partial_x^n (u(x,0))|^2 dx < \infty$$

Definition : Weights of diff. polynomial

$W(u) = \text{weight}(u) = 2$ weight $(\frac{\partial}{\partial x}) = 1$

$\Rightarrow u(x,\cdot) = \delta^2 v(\delta x, \cdot) \Rightarrow F^{(n)}(u(x,\cdot)) = \delta^{2n+5} F^{(n)}(v(\xi,\cdot))$ δx

Def. $\mathcal{P}^{(m)}[u, u_x, \dots] = \left\{ \begin{array}{c}\text{the set of all differential polynomial} \\ \text{of weight } m\end{array}\right\}$

e.g. $m=2$ $\mathcal{P}_{\mathbb{R}}^{(2)}[u] = \mathbb{R}\cdot u$

$m=3$ $\mathcal{P}_{\mathbb{R}}^{(3)}[u] = \mathbb{R}\cdot u_x$

$\vdots$

$m=5$ $\mathcal{P}_{\mathbb{R}}^{(5)}[u] = \operatorname{span}_{\mathbb{R}}\{u_{xxx}, uu_x\}$

Consider : Since $u_t + u_x \Rightarrow u(x-t)$, $x' = x-t$, $t' = t$

linear part can be dropped

$t \to \varepsilon t$ .. rescaling, kills one ε in eqn

$$u_t + \sum_{n=0}^{N} \varepsilon^n F^{(n)}(u, \dots) + O(\varepsilon^{N+1}) = 0, \quad F^{(n)}(u,\dots) \in \mathcal{P}_{\mathbb{R}}^{(2n+5)}[u,\dots]$$

Question : Analyze the solution near a $\underset{\text{(N-soliton soln)}}{\underline{\text{particular}}}$ soln of KdV.

Normalize $F^{(0)}(u) = 6uu_x + u_{xxx}$

Method : 1. Try to find the corrections of integrals.

2. Find the normal form

1. Integral (conserved density) $I(u) = \int_{\mathbb{R}} \rho(\dots)\, dx$

Let $\{\rho_l^{(0)}(u,\dots) \mid l = 0,1,\dots\}$ be the set of all

Conserved densities for KdV $\rho_0^{(0)} = u$, $\rho_1^{(0)} = u^2$, $\rho_2^{(0)} = u_x^2 - 2u^3$

$\rho_l^{(0)} \in \mathcal{P}_{\mathbb{R}}^{2l+2} / \operatorname{Im} \partial_x$

$$\rho_l(u,\varepsilon) = \sum_{n=0}^{N} \varepsilon^n \rho_l^{(n)}(u) + O(\varepsilon^{N+1}),$$

$$\rho_l^{(n)} \in \mathcal{P}_{\mathbb{R}}^{(2(l+n)+2)}[u] / \operatorname{Im} \partial_x$$

$\Rightarrow \frac{\partial \rho}{\partial t} = O(\varepsilon^{N+1})$

$$\frac{\partial \rho_l}{\partial t} = L_F(\rho_l) = L_{F^{(0)}}(\rho_l^{(0)}) + \varepsilon L_{F^{(0)}}(\rho_l^{(1)}) + \varepsilon L_{F^{(1)}}(\rho_l^{(0)}) + \dots$$

$L_{F^{(0)}}(\rho_l^{(0)}) = 0$ ($\Leftarrow$ KdV) O.K.

$L_{F^{(0)}}(\rho_l^{(1)}) = -\underbrace{L_{F^{(1)}}(\rho_l^{(0)})}$

eqn for $\rho_l^{(1)}$ $\Leftarrow$ known

Since $W(\rho_l^{(k)}) = 2(l+k)+2$, $W(L_{F^{(0)}}(\rho_l^{(k)})) = 2(l+k)+5$

$$L_{F^{(0)}} : [\mathcal{P}^{(2n+2)}] \to [\mathcal{P}^{(2n+5)}] = \mathcal{P}^{(2n+5)} / \operatorname{Im} \partial_x$$

Lemma : $\dim(\operatorname{Ker} L_{F^{(0)}} \cap [\mathcal{P}_{\mathbb{R}}^{2l+2}]) = 1$

Solvability of $L_{F^{(0)}}$: $\operatorname{Dim}([\mathcal{P}_{\mathbb{R}}^{(2l+5)}]) = \operatorname{Dim}([\mathcal{P}_{\mathbb{R}}^{(2l+2)}]) - 1$

Proposition $\operatorname{Dim}([\mathcal{P}_{\mathbb{R}}^{(2l+5)}]) = \operatorname{Dim}([\mathcal{P}_{\mathbb{R}}^{(2l+2)}]) - 1$ is OK for $0 \le l \le 4$

$\operatorname{Dim}([\mathcal{P}_{\mathbb{R}}^{15}]) = 7 > \operatorname{Dim}([\cdot]^{(12)}) - 1 = 6$ $\exists_1$ constraint

$\dots \quad 17 \quad = 11 \quad \dots \quad (14) \qquad = 9$ $\exists_2$ constraints

$\rho_0^{(n)}$ ... ∃ for $n = 0,1,2,3,4$

$\rho_1^{(n)}$ ... ∃ for $n = 0,1,2,3$

$\rho_2^{(n)}$ ~ ∃ for $n = 0,1,2$

$\rho_3^{(n)}$ — " — $n=0$, for $n=1$ contradict

Theorem: (Normal form theorem)

$$u_t + F^{(0)}(u) + \varepsilon F^{(1)}(u) + \varepsilon^2 F^{(2)}(u) = \mathcal{O}(\varepsilon^3), \quad F^{(0)}(u) .. \text{KdV}$$

Then ∃ a formal change of coordinates $u = \phi(v) = v + \varepsilon \phi^{(1)}(v) + \dots$

such that $v_t + F^{(0)}(v) + \varepsilon G^{(1)}(v) + \varepsilon^2 G^{(2)}(v) = \mathcal{O}(\varepsilon^3)$

where

$$G^{(1)}(v) \text{ is } \text{KdV}_5 = v_{xxxxx} + 10 v v_{xxx} + 20 \cancel{x} v_{xx} + 30 v^2 v_x$$

$$G^{(2)}(v) = \text{KdV}_7 + \underbrace{R^{(2)}(v, \dots)}$$

in general non-integrable eqn, causes radiation etc (new soliton...)

(we can send two KdV solitons, then in interaction area radiation etc. due to this term $R^{(2)}(v,\dots)$)

HIETARINTA:

One should not have more equations than unknowns in bilinear form.

Search methods

The key property: ∃ of multisoliton solns

The class: Eqns written in Hirota bilinear form

1. KdV $P(D)F \cdot F$
2. MKdV / SG class $B(D) F \cdot F = 0$, $A(D)(F \cdot F + \varepsilon G \cdot G) = 0$
3. NLS
4. Benjamin O

dimension we consider free

$$\text{KdV}: 3SS: F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3} + A_{23} e^{\eta_2+\eta_3} + A_{123} e^{\eta_1+\eta_2+\eta_3}$$

$\eta_3 \to -\infty$ etc. $\Rightarrow A_{ij} = - \dfrac{P(\vec{p}_i - \vec{p}_j)}{P(\vec{p}_i + \vec{p}_j)}$ $\quad \eta_3 \to +\infty \Rightarrow A_{123} = A_{12} A_{13} A_{23}$

$P(D) F \cdot F = 0$ collect $e^{\eta_1+\eta_2+\eta_3}$ terms $\Rightarrow$ a terrible sum of polynomials must be zero

$$\sum_{\sigma_i \in \{-1,1\}} P(\sigma_1 p_1 - \sigma_2 p_2) P(\sigma_1 p_1 - \sigma_3 p_3) P(\sigma_1 p_2 - \sigma_3 p_3) \cdot P(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) = 0$$ on dispersion manifold

Let leading part of $P(p_i) = -(\text{rest of } P(p_i))$ e.g. $P(p) = p^4 + p\omega$ $\Rightarrow$ ~~let~~ $p_i^4 = -p_i \omega_i, \forall i$

put into eqn. above and so on, use REDUCE

$\Rightarrow$ in KdV case only 4 eqns allowing 3SS

MKdV / SG class $\Rightarrow$ 7 2-1-dim. eqns + reductions and some 1-1-dim eqns possessing 3SS & 4SS of all types (a+a, a+b, ...)

nonlinear forms from bilinear: for mKdV, SG use $F = e^{\frac{1}{2}(u+iv)}$, $G = e^{\frac{1}{2}(u-iv)}$

KODAMA: Tutorial

MIWA: Quantum integrable systems

REF: Miwa, Jimbo: Algebraic analysis of solvable lattice models

What is a problem of QMech? .. Diagonalisation of Hamiltonian

Toy example: harmonic oscillator

$H = -\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2\right)$ $\quad \overset{?}{f}, \lambda : Hf(x) = \lambda f(x),$
$f(x) = 0, |x| \to \infty$

Algebraic method $\quad H = QP + \frac{1}{2} \qquad Q = -\frac{1}{2}\left(\frac{d}{dx} - x\right)$

$$P = \frac{d}{dx} + x$$

$$[P, Q] = 1 \qquad [H, P] = -P \qquad [H, Q] = Q$$

P, Q generate Heisenberg algebra, P annihilation op., Q creation op.

Lemma: If $Hf = \lambda f$ then $H(Pf) = (\lambda - 1)Pf$ and $H(Qf) = (\lambda+1)Qf$

The vacuum vector $e^{-\frac{1}{2}x^2}$: $Pe^{-\frac{1}{2}x^2} = 0 \qquad He^{-\frac{1}{2}x^2} = \frac{1}{2}e^{-\frac{1}{2}x^2}$

The energy spectrum $\frac{1}{2} + m, \ m \in \mathbb{Z}_+^0, \quad f_m(x) = Q^m e^{-\frac{1}{2}x^2}$

What is a system of infinite degrees of freedom?

$\#(\text{variables}) = \infty$

Main example: the XXZ model

$$H = -\frac{1}{2}\sum_{m=-\infty}^{\infty}\left(\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y + \underset{\mathbb{R}}{\overset{}{\Delta}}\!\!\!\!\!\!\underset{\cap}{}\ \ \sigma_m^z\sigma_{m+1}^z\right)$$

acting on $\ldots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_m^x = \ldots \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \sigma^x \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \ldots \text{ etc.}$$

$\mathbb{C}^2 = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ $\qquad v_+$ spin up
$\qquad\qquad\qquad\qquad\quad v_y$ spin down for certain choice of direction

$$f = \sum c_{\ldots \varepsilon_m \varepsilon_{m+1} \ldots} \ \ldots \otimes v_{\varepsilon_m} \otimes v_{\varepsilon_{m+1}} \otimes \ldots$$

Main idea: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \cong$ bosonic Fock space
i.e. infinite set of harmonic oscillators

What is integrability in quantum systems with ∞ degrees of freedom?

N the size parameter $\qquad H^{(N)} = -\frac{1}{2}\sum_{m=1}^{N}(\ldots), \quad \sigma_{N+1} = \sigma_1$

There $\exists$ commuting family of operators .. "transfer matrices"

$T^{(N)}(\xi) \in \mathrm{End}\left(\bigotimes_{m=1}^{N}\mathbb{C}^2\right)$ satisfying

$[T^{(N)}(\xi_1), T^{(N)}(\xi_2)] = 0, \quad T^{(N)}(1) =$ the shift operator

$H^{(N)} = \frac{d}{d\xi}\log T^{(N)}(\xi)\Big|_{\xi=1}$ (up to constant)

$(\Rightarrow [T^{(N)}(\xi), H^{(N)}] = 0)$

Problem: infinite degrees of freedom $\Rightarrow$ may have infinitely degenerate eigenvalues

Vacuum state
one particle state $\quad \varepsilon(\xi), \tau(\xi)$ (energy + momentum define finite subspace)
m particle state $\quad \sum_{i=1}^{m}\varepsilon(\xi_i), \ \sum_{i=1}^{m}\tau(\xi_i)$

.. energy & momenta don't define a finite subspace

Integrability .. infinite degeneracy must be reduced to finite degeneracy
i.e. can see the particle

What is a symmetry?

X is a symmetry of $H \Leftrightarrow [H, X] = 0$

(1) abelian symmetry $\Leftrightarrow$ integrability (i.e. reduce the degeneracy to finite)

(2) non-abelian symmetry .. removes remaining finite degeneracy, may not commute with abelian symmetries

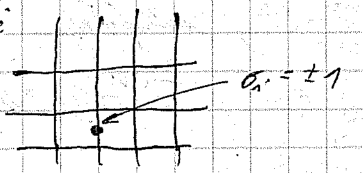
(3) dynamical symmetry .. $[H, X(\xi)] = \varepsilon(\xi)\cdot X(\xi)$
e.g. creation & annihilation operators

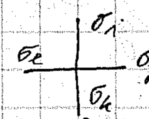
The six vortex model

XXZ .. a model of Q Statistical Mechanics of dimension 1

the 6V model   a model of CSM of dimension 2

square lattice



local interaction energy at a vertex


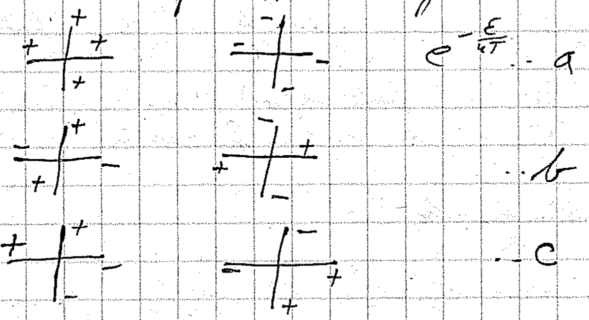
$\rightarrow \varepsilon(\sigma_i, \sigma_j, \sigma_k, \sigma_e)$

total energy $\varepsilon(\sigma) = \sum_{\text{vortex}} \varepsilon(\sigma_i, \sigma_j, \sigma_k, \sigma_e)$

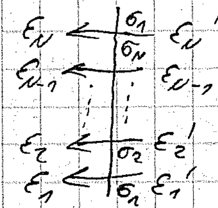
Boltzmann principle $e^{-\frac{\varepsilon}{kT}}$ relative probability of occurrence of the configuration

Configuration sum $\sum_{\sigma} e^{-\frac{\varepsilon}{kT}}$ sum over all configurations

the 6V model ⇐ six kinds of Boltzmann weight $e^{-\frac{\varepsilon(\sigma_i, \sigma_j, \sigma_k, \sigma_e)}{kT}} = R^{\sigma_i \sigma_j}_{\sigma_k \sigma_e}$



Transfer matrix $T^{(N)} \in \operatorname{End}\left(\bigotimes_{n=1}^{N} \mathbb{C}^2\right)$



$$T^{(N)}{}^{\varepsilon_1' \dots \varepsilon_N'}_{\varepsilon_1 \dots \varepsilon_N} = \sum_{\sigma_1, \dots, \sigma_N} \prod R^{\sigma_{n+1} \varepsilon_n'}_{\sigma_n \varepsilon_n}$$

ZAKHAROV: skipped

CLARKSON: Symmetries, Chazy eqn., Chazy hierarchy

Chazy (1909) $y''' = 2yy'' - 3(y')^2$

$$y''' = 2yy'' - 3(y')^2 + \frac{4}{36-k^2}(6y' - y^2)^2$$

Halphen system (1881)
$$\frac{d\omega_1}{dx} = \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3)$$
$$\frac{d\omega_2}{dx} = \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1)$$
$$\frac{d\omega_3}{dx} = \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2)$$

$\Rightarrow y(x) = -2[\omega_1(x) + \omega_2(x) + \omega_3(x)]$ solves Chazy

? Painlevé-type eqns of the form $y''' = F(x, y, y', y'') \Rightarrow$
lots of eqns (no complete classification up to now) including Chazy

Painlevé analysis $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\rho}$

maximum balance $\Rightarrow \rho = -1 \wedge (\overset{(a)}{a_0 = -6} \vee \overset{(b)}{a_0 = -3 + \frac{1}{2}k} \vee \overset{(c)}{a_0 = -3 - \frac{1}{2}k})$

$y(x) = \frac{a_0}{x - x_0} + p(x - x_0)^{r-1}$

⇒ resonances (a) $r = -1, -2, -3$ (b) $r = -1, 1, k$
(c) $r = -1, 1, -k$
⇒ (b), (c) leads to condition $k \in \mathbb{Z} - \{-6, 6\}$

ad (a) ? what does it mean: 3 negative resonances ?

$k = 1$ double resonance in (b), (c) ⇒ not Painlevé
$k = 0$ moveable logarithmic branch point ...

Finally Painlevé property $\Leftrightarrow k > 1 \wedge k \in \mathbb{Z}^+ \setminus \{6\}$

Exact solution: $y(x) = -\frac{6}{x - x_0} + \frac{A}{(x - x_0)^2}$ A, x_0 arbitrary constants

Chazy eqn $y''' = -2yy'' - 3(y')^2$

⇒ solution $y(x(s)) = 6\frac{d}{dx}\ln z_1(s)$, $x(s) = \frac{z_2(s)}{z_1(s)}$

where z_1, z_2 are indep. solns of $s(1-s)\frac{d^2z}{ds^2}+\left(\frac{1}{2}-\frac{7}{6}s\right)\frac{dz}{ds}-\frac{z}{144}=0$

inverse mapping $S(x)$ to $x(s)$ has a whole line of essential singularities

Halphen system (and consequently Chazy) can be reduced from self-dual Yang-Mills (with ∞-dim. Lie alg. of vector fields on S^2 as a fibre).

SHABAT:

SG: $S = \iint (q_x q_t + \cos 2q)\, dx\, dt \qquad \delta S = 0 \Rightarrow$

$q_{xt} + \sin 2q = 0$

Bäcklund transf. $\hat{q}_x - q_x = a(q+\hat{q})$ .. canonical transf.

$\Rightarrow \hat{q}_x \hat{q}_t - q_x q_t \in \mathrm{Im}\, D_x + \mathrm{Im}\, D_t$

Proof: $[q_x + a(q+\hat{q})]\hat{q}_t - q_x q_t = q_x(\hat{q}_t - q_t) + a(q+\hat{q})\hat{q}_t$

$\underset{\substack{\text{integr.}\\ \text{by parts}}}{\sim} -q(\hat{q}_t - q_t)_x + a(q+\hat{q})\hat{q}_t = -q[a(q+\hat{q})]_t + a(q+\hat{q})\hat{q}_t$

$= \ldots$ OK.

$a(q+\hat{q}) = \beta \sin(q+\hat{q})$ .. Bäcklund transf.

Generalization $\hat{q}_x - \epsilon \hat{q} = a(\epsilon q + \hat{q})$, $\epsilon^2 = 1$

i.e. two possible transf. $\hat{q}_x - q_x = \beta \sin(q+\hat{q}) \qquad \hat{q}_x + q_x = \gamma \sin(\hat{q} - q)$

$\hat{q}_x - q_x = \beta \sin(q+\hat{q}) \wedge \hat{q}_t + q_t = \gamma \sin(\hat{q} - q) \underset{\text{if } \beta\gamma = 1}{\Leftrightarrow} (\hat{q}_{xt} + \sin 2\hat{q} = 0 \Leftrightarrow q_{xt} + \sin 2q = 0)$

System $q_{j+2N} = -q_j \qquad (q_{j+1} - q_j)_x = \beta_j \sin(q_j + q_{j+1})$

$(\;\;\text{"}+\text{"}\;\;)_t = \ldots\ldots$

$N=1 \qquad (q_2 - q_1)_x = \ldots \qquad (-q_1 - q_2)_x - \ldots$

Proposition $[D_x, D_t] = 0$

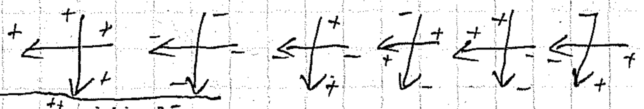
. . . .

MIWA: Tutorial

XXZ Hamiltonian $H = -\frac{1}{2}\left(\sum_{n=-\infty}^{\infty} \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z\right)$

XXZ model ⟷ 6V model

$$\begin{tikzcd}
 & sl_2 \arrow[dl, "\text{Today also affinisation}"'] \arrow[dr, "q\text{-deformation}"] & \\
U(\hat{sl}_2) \arrow[dr] & & U_q(sl_2) \arrow[dl] \\
 & U_q(\hat{sl}_2) \; \text{quantum affine alg.} &
\end{tikzcd}$$

6V model 

$$R = \begin{array}{c|cccc} & ++ & \cdot & \cdot & -- \\ \hline ++ & a & & & \\ +- & & b & c & \\ -+ & & c & b & \\ -- & & & & a \end{array}$$

$R \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ the first tensor comp. ↓, the second tensor comp ←

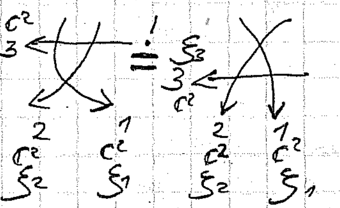
transfer matrix $T^{(N)} \in \mathrm{End}\left(\bigotimes_{n=1}^{N} \mathbb{C}^2\right)$

$\Rightarrow \sum_\sigma e^{-\frac{\varepsilon}{kT}} = \mathrm{trace}_{\bigotimes_{n=1}^{N}\mathbb{C}^2} \left(T^{(N)}\right)^N$

re-parametrize $a, b, c \to x, q, \xi$

$a = \frac{1}{x}, \quad b = \frac{1}{x}\frac{(1-\xi^2)q}{1-q^2\xi^2}, \quad c = \frac{1}{x}\frac{(1-q^2)\xi}{1-q^2\xi^2}$

then in correspondence XXZ ⟷ 6V we put $\Delta = \frac{q+q^{-1}}{2}$

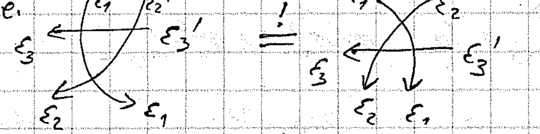
the Yang-Baxter equation 

$R \otimes \mathbb{1} = R_{12},\; R_{13},\; R_{23} = \mathbb{1} \otimes R$

$R_{12}(\xi_1/\xi_2)$ $R_{13}(\xi_1/\xi_3)$ $R_{23}(\xi_2/\xi_3)$
(inhomogeneous lattice)

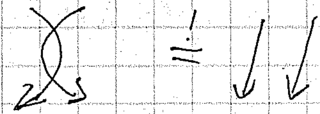
$$R_{12}(\xi_1/\xi_2)\cdot R_{13}(\xi_1/\xi_3) R_{23}(\xi_2/\xi_3) = R_{23}(\xi_2/\xi_3)$$
$$\cdot R_{13}(\xi_1/\xi_3)\cdot R_{12}(\xi_1/\xi_2) \qquad \boxed{\boxed{YBE}}$$

i.e.



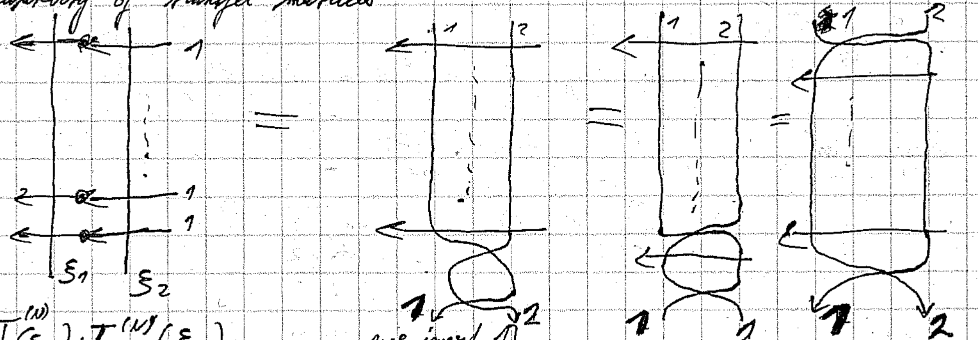
$$(\ )^{\varepsilon_1'\varepsilon_2'\varepsilon_3'}_{\varepsilon_1\varepsilon_2\varepsilon_3} = (\ )^{\varepsilon_1'\varepsilon_2'\varepsilon_3'}_{\varepsilon_1\varepsilon_2\varepsilon_3}$$

Lemma:



, i.e $R_{12}(\xi_1/\xi_2) R_{21}(\xi_2/\xi_1)$
$= \mathbb{1}$ (normalization by choice of π)
unitarity relation

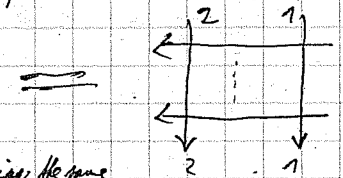
Commutativity of transfer matrices



$T^{(N)}(\xi_1)\cdot T^{(N)}(\xi_2)$
for a fixed vertical lines

we insert $\mathbb{1}$

use YBE



doing the same
from the other side

$$\Rightarrow T^{(N)}(\xi_1)\, T^{(N)}(\xi_2) = T^{(N)}(\xi_2) T^{(N)}(\xi_1)$$

$T^{(N)}$ commute $\Rightarrow$ commuting family of operators

Remark : $\Delta = 1$, suppose $N < +\infty$
"$[H, sl_2] = 0$"
i.e. $\forall\, x \in sl_2$: $[H, \sum_n \mathbb{1} \otimes \dots \otimes \underset{\overset{\uparrow}{n}}{x} \otimes \dots \otimes \mathbb{1}] = 0$
non-abelian symmetry
$H\ell = \lambda \ell \Rightarrow H(x\ell) = \lambda(x\ell)$

Affinization of $\overset{\text{enveloping algebra}}{U(sl_2)}$ $U(\hat{sl_2})$
Motivation: in the limit $N \to \infty$ we may have more degeneracy, i.e. more symmetry

$\Delta \neq 1 \Rightarrow$ deformation of symmetry algebra $U_q(sl_2)$

Finally $N \to \infty \wedge \Delta \neq 1 \Rightarrow$ quantum affine algebra $U_q(\hat{sl_2})$

$U(sl_2)$: generators $e\left(=\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)$ $f\left(=\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)$ & $h\left(=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$
$[e,f] = h$ $[h,e] = 2e$ $[h,f] = -2f$

$\hat{sl_2} = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$
$[X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes T^{m+n} + (m\delta_{m,-n} \operatorname{tr}(XY))C$

other definition
$U(\hat{sl_2})$ $e_1 = e \otimes 1$ $f_1 = f \otimes 1$ $h_1 = h \otimes 1$
$e_0 = f \otimes t^{-1}$ $f_0 = e \otimes t$ $h_0 = c - h$

$[e_i, f_j] = \delta_{ij} h_j$; $[h_i, f_j] = -a_{ij} f_j$
$[h_i, e_j] = a_{ij} e_j$, $(a_{ij}) = \begin{pmatrix}2&-2\\-2&2\end{pmatrix}$ (and more)

$U_q(sl_2)$ $e, f, q^h = k$ generators

$$[e,f] = \frac{q^h - q^{-h}}{q - q^{-1}} \xrightarrow[q \to 1]{} h$$

$[h,e] = 2e \longleftrightarrow q^h e q^{-h} = q^2 e$
$[h,f] = -2f \longleftrightarrow q^h f q^{-h} = q^{-2} f$

$U_q(\hat{sl_2})$ e_i, f_i, $t_i = q^{h_i}$

$[e_i, f_j] = \dfrac{t_i - t_i^{-1}}{q - q^{-1}}\, \delta_{ij}$, $t_i e_j t_i^{-1} = q^{a_{ij}} e_j$,
$t_i f_j t_i^{-1} = q^{-a_{ij}} f_j$ (and more)

FLASCHKA : Representation theory & Integrability

Heuristic approach to semi-classical quantization

Basic example $SU(2) = \left\{\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix} \middle| |a|^2 + |b|^2 = 1\right\}$
or $U(2) = \{2\times2 \mid g^{-1} = g^*\}$

$su(2)$ $\quad X=\begin{pmatrix} i\xi & -\eta+i\zeta \\ \eta+i\zeta & -i\xi \end{pmatrix}$

$\sigma_1=\frac{1}{2}\begin{pmatrix}0&i\\i&0\end{pmatrix}$, $\sigma_2=\frac{1}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ $\quad \sigma_3=\frac{1}{2}\begin{pmatrix}i&0\\0&-i\end{pmatrix}$

$X\in su(2) \longleftrightarrow \tilde{X}=(\xi,\eta,\zeta)\in \mathbb{R}^3$

$[\sigma_1,\sigma_2]=\sigma_3$ etc., $(\Rightarrow [X,Y]^{\sim}=\tilde{X}\times\tilde{Y})$, $(X,Y)=-\frac{1}{2}\mathrm{Tr}\,XY$

$(\Rightarrow (X,X)=\xi^2+\eta^2+\zeta^2)$

Orbits $\quad \mathcal{O}_\mu=\left\{g^{-1}\begin{pmatrix}i\mu&0\\0&-i\mu\end{pmatrix}g \,/\, g\in SU(2)\right\}$

or $\quad \mathcal{O}_\mu=\left\{g^{-1}\begin{pmatrix}i\mu&0\\0&0\end{pmatrix}g \,/\, g\in U(2)\right\}$

We want to identify $\mathcal{O}_\mu$ with Riemann sphere or $\mathbb{C}P^1$

If $\{\lambda\cdot(v_1,v_2)\,/\,\lambda\in\mathbb{C}\}$ is line in $\mathbb{C}^2$, give it coordinate

$z=\frac{v_1}{v_2}$ if $v_2\neq 0$ $\qquad w=\frac{1}{z}=\frac{v_2}{v_1}$ if $v_1\neq 0$

~~To the~~ To this line associate matrix $X=i\mu\begin{pmatrix}v_1\\v_2\end{pmatrix}(\bar{v}_1,\bar{v}_2)\in U(2)$

Remark: the area form on the Riemann sphere is scaled.

Representations

Let $\mathcal{H}_N=\{f(\lambda z_1,\lambda z_2)=\lambda^N f(z_1,z_2),\ \forall\lambda\in\mathbb{C}\}$

If $F\in\mathcal{H}_N$: when $z_2\neq 0$ set $F_0(z)=F(z,1)$

$z_1\neq 0$ set ~~[illegible]~~ $F_\infty(w)=$ ~~[illegible]~~ $F(1,w)$

When $z\neq 0, w\neq 0$ $\quad F_\infty(w)=F(\frac{1}{z},1)=\frac{1}{z^N}F(z,1)$ ~~[illegible]~~

~~[illegible]~~ $=\frac{1}{z^N}F_0(z)$

F_0, F_∞ are local representatives of a section of a linebundle over $\mathbb{C}P^1$.

Likewise for $U(3)$ $\quad \left\{g\begin{pmatrix}i\mu&&\\&0&\\&&0\end{pmatrix}g^{-1}\right\}\simeq\mathbb{C}P^2_\mu$ etc.

Homogeneous polynomials give representation.

Example: $N=0$ $\quad \mathcal{H}_N=\mathbb{C}$, g acts as 1

$N=2$ Basis z_1^2, z_1z_2, z_2^2

$U(2)$ $\quad \begin{pmatrix}a&b\\c&d\end{pmatrix}: f(z_1,z_2)\to f(az_1+bz_2, cz_1+dz_2)$

$=(cz_1+dz_2)^m f_0\left(\frac{az_1+bz_2}{cz_1+dz_2}\right) \underset{z=\frac{z_1}{z_2}}{\overset{\uparrow}{=}} (cz+d)^m f_0\left(\frac{az+b}{cz+d}\right)$

Specialise to $\begin{pmatrix}t_1&0\\0&t_2\end{pmatrix}$: $\quad z^2\to\left(\frac{t_1z+0}{0+t_2}\right)^2 t_2^2=t_1^2z^2$

$z\to t_1 z$

$1\to 1$

On $\begin{pmatrix}t_1&0\\0&t_2\end{pmatrix}$:

	weights
$z_1^2\to t_1^2z_1^2$	(2,0)
$z_1z_2\to t_1t_2\,z_1z_2$	(1,1)
$z_2^2\to t_2^2\,z_2^2$	(0,2)

In general $U(n)$ $\quad \begin{pmatrix}t_1&&\\&t_2&\\&&\ddots\end{pmatrix}v=t_1^{a_1}t_2^{a_2}\cdots t_n^{a_n}v$

then $a=(a_1,a_2,\ldots,a_n)$ is called a weight, v is a weight vector.

Fundamental theorem: Every irrep has a basis of weight vectors. There is a highest weight ~~[illegible]~~ λ and it characterises the representation.

Example: Now take product of orbits, $\mathbb{C}P^1_N$, $\mathbb{C}P^1_\mu$

e.g. $\mathbb{C}P^1\times\mathbb{C}P^1$, $N=1$. ~~sections~~ $\mathcal{H}_N=\mathrm{span}\{z_1,z_2\}$ or $\{\alpha z+\beta\}$ if $z\neq\infty$

$\{u\}$ $\quad \{v\}$

Sections $\quad 1=s_1(u)$ $\qquad 1=s_1(v)$

$u=s_2(u)$ $\qquad z=s_2(v)$

$(u,v)\in\mathbb{C}P^1\times\mathbb{C}P^1$ $\quad$ "product" of the bundles, basis $s_1(u)\cdot s_1(v)$, $s_1(u)s_2(v)$, $s_2(u)s_1(v)$, $s_2(u)s_2(v)$

This is just the tensor product of the two $\mathbb{C}^2$s $\quad \mathbb{C}^2\otimes\mathbb{C}^2$.

Better basis $\quad s_1(u)s_1(v)$, $s_1(u)s_2(v)+s_2(u)s_1(v)$, $s_2(u)s_2(v)$, $s_1(u)s_2(v)-s_2(u)s_1(v)$

Weights $\quad \underbrace{(0,2)\quad(1,1)\quad(0,2)}_{\mathcal{H}_2}\qquad \underbrace{(1,1)}$

$\underset{\text{group action}}{g\circ t}=(\det g)\cdot t$

i.e. rep on $\mathcal{H}_1\otimes$ rep on $\mathcal{H}_1$ = rep on $\mathcal{H}_2\oplus$ det. repr. on 1-dim vector space

Polygons (A. Klyachkin, M. Kapovich, J. Millson)

~~Take~~ Back to su(2) $\mathcal{N} = \sigma_{r_1} \times \dots \times \sigma_{r_m}$ = n-tuple of $(X_1, \dots, X_n)$ vectors in $\mathbb{R}^3$

$\underline{r} = (r_1, \dots, r_n)$

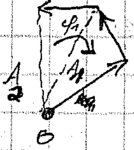
$$\tilde{M}_{\underline{r}} \equiv \{X_1 + \dots + X_n = 0\} \quad \text{polygon}$$

$$M_{\underline{r}} \equiv \{X_1 + \dots + X_n = 0\} / SU(2)$$

where the action of SU(2) on $(X_1, \dots, X_n)$ is $(g^{-1}X_1 g, \dots, g^{-1}X_n g)$

Thus: $M_{\underline{r}}$ = space of closed n-gons in $\mathbb{R}^3$ with vertex 1 fixed at origin, up to rotation.

Theorem: $M_{\underline{r}}$ is a symplectic manifold. The lengths $\|A_i\|$ of diagonals from the origin to vertices are action variables. The dihedral angles are the conjugate angles.
If $r_1, \dots, r_n \in \mathbb{Z}_{\geq 0}$ then the number of tori with integral actions equals the number of times the 1-dim representation occurs in the tensor product $\mathcal{H}_{r_1} \otimes \dots \otimes \mathcal{H}_{r_n}$.



(Not exact formulation !)

Discussion: CLARKSON: IST

P.A.CLARKSON @ ukc. ac.uk - ask for slides (transparencies)

M. Ablowitz P. Clarkson: "Solitons, Nonlinear Evolution Eqns. & Inverse Scattering" CUP 1991

OLVER: Multihamiltonian structures & integrability

Equations with more than one Hamiltonian structure $\rightsquigarrow$ Completely integrable

Hamilton's equations $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$ $\quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$

$H(p, q)$ Hamiltonian

$$u = \begin{pmatrix} q \\ p \end{pmatrix} \qquad \frac{du}{dt} = J \nabla H \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$J^T = -J$$

Poisson bracket
(Poisson structures) $\frac{dF}{dt} = \{F, H\} = \sum_i \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} =$

$$= \nabla F \cdot J \cdot \nabla H$$

$\{\ ,\ \}$ bilinear, skew-symmetric, Jacobi identity, Leibniz rule

ad Jacobi $\sum_l J_{il} \frac{\partial J_{jk}}{\partial u_l} + \text{cyclic}(i, j, k) = 0$

Rigid body eqns: $\frac{du_1}{dt} = \frac{I_2 - I_3}{I_2 I_3} u_2 u_3$ etc.

$\boxed{\text{dim} = 3 \text{ odd !}}$ $\quad H = \frac{u_1^2}{I_1} + \frac{u_2^2}{I_2} + \frac{u_3^2}{I_3}$ $\quad \frac{d\vec{u}}{dt} = \vec{u} \wedge \nabla H$

$\Rightarrow$ we have $\{F, H\} = u \cdot (\nabla F \times \nabla H)$

Lie-Poisson structure $\quad \mathfrak{g}$ - Lie alg. e.g. so(3)

$\mathfrak{g}^*$ - dual of $\mathfrak{g}$, $u \in \mathfrak{g}^*$: $\quad \{F, H\}(u) = \langle u, [\underset{\mathfrak{g}^{**} = \mathfrak{g}}{\nabla F, \nabla H}] \rangle$

is a Poisson bracket (Kirilov, Kostant, .. but originally from Lie)

rank J = dim M i.e. $\exists J^{-1}$ $\Rightarrow \Omega = d\vec{u} \wedge J^{-1} d\vec{u}$ symplectic form, Jacobi $\Rightarrow d\Omega = 0$
Darboux $\Rightarrow \exists$ locally p_i, q_i : $dp_i \wedge dq_i = \Omega$

In general Darboux: If rank J = const. $= 2n \leq$ dim M $\Rightarrow$
$\exists$ coords $\underbrace{p^i, q^i}_{2n}$, z^k such that

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dz^k}{dt} = 0$$

$C(z_k)$ distinguished functions (Casimirs) $\quad \{H, C\} = 0, \forall H$

e.g. for rigid body $C = u_1^2 + u_2^2 + u_3^2$

Schouten bracket

Multi-vector: section of $\Lambda^k TM$, i.e $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \dots$

$[\underset{k\text{-vector}}{\alpha}, \underset{l\text{-vector}}{\beta}]$ $k+l-1$-vector

$[v, w]$ Lie bracket for v, w 1-vectors (vector fields)

$[v, f] = v(f)$ v vector field, f function

we assume $[\,,\,]$ bilinear, $[\beta, \alpha] = (-1)^{kl}[\alpha, \beta]$

$(-1)^{kl}[\underset{k}{\alpha}, [\underset{l}{\beta}, \underset{m}{\gamma}]] +$ " $+$ " $= 0$ super-Jacobi

$$[\alpha, \beta\wedge\gamma] = [\alpha, \beta]\wedge\gamma + (-1)^{(k+1)l}\beta\wedge[\cancel{\beta}\alpha, \gamma]$$

Poisson bracket $\Longleftrightarrow$ Θ bivector

$$\{F, H\} = \langle dF \wedge dH, \Theta\rangle \text{ such that } [\Theta, \Theta] = 0$$

explicitly $\Theta = \sum_{i,j} J_{ij}(u) \frac{\partial}{\partial u_i} \wedge \frac{\partial}{\partial u_j}$

Poisson complex:
$$\begin{tikzcd} \Lambda_0 T \arrow[r, "{[\theta,\cdot]}"] & \Lambda_1 T \arrow[r, "{[\theta,\cdot]}"] \arrow[d, equal] & \Lambda_2 T \arrow[r, "{[\theta,\cdot]}"] & \Lambda_3 T \arrow[r] & {} \\ & TM & & & \end{tikzcd}$$

- exact $\mathrm{Im}[\theta, \cdot] = \mathrm{Ker}[\theta, \cdot]$
in symplectic case (then
in fact de Rham complex)

$H \longrightarrow [\theta, H] = \hat{V}_H$

$V \longrightarrow [\theta, V] = 0 \Rightarrow V$ locally Hamiltonian

in general exact with
exception of Z's
(for fixed rank J, otherwise unknown)

PDEs

Arnold - ideal fluid eqn., KdV $\frac{\partial u}{\partial t} = D_x \frac{\delta H}{\delta u}$ $H = \int\left(-\frac{1}{2}u_x^2 + \frac{1}{6}u^3\right)dx$, $J = D_x$

J skew-adjoint ← variational derivative

$$\{F, H\} = \int\left(\frac{\delta F}{\delta u} D_x \frac{\delta H}{\delta u}\right)dx$$

Jacobi - rather complicated in general, in this case follows since D_x doesn't depend on u

$$\frac{\partial u}{\partial t} = \left(D_x^3 + \frac{2}{3}u D_x + \frac{1}{3}u_x\right)\frac{\delta \tilde{H}}{\delta u} \qquad \tilde{H} = \int \frac{1}{2}u^2 dx$$

other Hamiltonian, new J

ZAKHAROV: skipped

CLARKSON: Chazy eqn.

Simplifying diff. eqns. in MAPLE - package diffgrob2

Theorem (Lie): an ODE of order n with a n-dimensional Lie symmetry algebra can be integrated by quadrature provided that the symmetry algebra is solvable.

Finally solution of Chazy eqn in terms of sols of $\overbrace{\text{stationary Schröd. eqn.}}^{\text{Lamé eqn.}}$ with Weierstrass function as a potential and consequently in terms of hypergeometric functions.

Tutorial: OLVER

Jacobi identity

$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\delta H}{\delta u}$ e.g. $\frac{\partial u}{\partial t} = \left(D_x^3 + \frac{2}{3}u D_x + \frac{1}{3}u_x\right)u$

$H = \int \frac{1}{2}u^2 dx$

$$\{F, H\} = \int \frac{\delta F}{\delta u}\mathcal{D}\frac{\delta H}{\delta u}\, dx$$

Functional - formal $\int P[u]dx$ $P \sim \tilde{P} \Longleftrightarrow P = \tilde{P} + D_x B$

$\frac{\delta P}{\delta u} = \sum_{k=0}^{\infty}(-D_x)^k \frac{\partial P}{\partial u_k}$ $u_k = D_x^k u$ sum finite but upper bound depends on concrete P

$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$ Jacobi

Functional Multivector – dual to differential forms

Evolutionary vector fields $v_Q = Q[u]\frac{\partial}{\partial u}$ $\qquad \frac{\partial u}{\partial t} = Q[u]$

prolongation $\mathrm{pr}\, v_Q = \sum_k D_x^k Q \frac{\partial}{\partial u_k} \Rightarrow \frac{\partial P[u]}{\partial t} = \mathrm{pr}\, v_Q(P)$

$$= \sum_k D_x^k Q \frac{\partial P}{\partial u_k}$$

Conservation law $0 = \frac{\partial}{\partial t}\int P\,dx = \int \frac{\partial P}{\partial t}dx = \int \mathrm{pr}\,v_Q(P)dx$

$$\Leftrightarrow \mathrm{pr}\,v_Q(P) = D_x Z$$

A general uni-vector will be of the form $\int\sum_k (R_k^{[u]}\theta_k)dx$ $\qquad \theta_k \sim \frac{\partial}{\partial u_k}$, $\theta_k = D_x^k\theta$

$$D_x(R_j\theta_j) = (D_x R_j)\theta_j + R_j\theta_{j+1}$$

$$\Rightarrow \int\sum_k R_k\theta_k\, dx = \int \sum_k R_k D^k\theta\, dx = \int\Big(\underbrace{\sum_k(-D^k)R_k}_{Q}\Big)\theta\, dx$$

$\Rightarrow$ ∀ uni-vector has canonical form $\int Q\cdot\theta\, dx$

e.g. $\int(u\theta + u_x\theta_x)dx = \int(u-u_{xx})\theta\, dx$

Bivector $\int(\sum A_{jk}\theta_j\wedge\theta_k)\,dx$ $\qquad D_x(A\theta_j\wedge\theta_k) = (D_xA)\theta_j\wedge\theta_k + A\theta_{j+1}\wedge\theta_k + A\theta_j\wedge\theta_{k+1}$

$$\theta_j\wedge\theta_k = -\theta_k\wedge\theta_j$$

$\Theta = \int[\theta\wedge\mathcal{D}(\theta)]\,dx$ $\qquad \left(\{\mathcal{F},H\} = \langle\Theta, \frac{\delta\mathcal{F}}{\delta u},\frac{\delta H}{\delta u}\rangle = \int \frac{\delta \mathcal{F}}{\delta u}\mathcal{D}\frac{\delta H}{\delta u}dx\right)$

KdV: $\mathcal{D} = D_x$ $\qquad \Theta = \int(\Theta\wedge\Theta_x)\,dx$

$\mathcal{D} = D_x^3+\dots$ $\qquad \Theta = \int\theta\wedge(D_x^3+\frac{2}{3}uD_x+\frac{1}{3}u_x)\theta =$

$$= \int(\theta\wedge\theta_{xxx}+\tfrac{2}{3}u\,\theta\wedge\theta_x)\,dx + ~~\int \text{total derivative}~~$$

in general $\langle A; L, M\rangle = \int\sum A_{jk}\det\begin{pmatrix}D_x^jL & D_x^jM\\ D_x^kL & D_x^kM\end{pmatrix}dx$

(before $\langle\theta_j; L\rangle = D_x^jL$)

trivector $[\theta,\theta]=0$? (then Jacobi ⇔ $\langle[\theta,\theta]; \frac{\delta a}{\delta u},\frac{\delta b}{\delta u},\frac{\delta c}{\delta u}\rangle = 0$)

To compute $[\theta,\theta]$ define the formal evolutionary vector field

$$\mathrm{pr}\,v_{\mathcal{D}\theta} = \sum_k D^k(\mathcal{D}\theta)\frac{\partial}{\partial u_k}$$

$\mathrm{pr}\,v_{\mathcal{D}\theta}[\mathcal{D}]$ acts on coefficients of $\mathcal{D}$, not on $\mathcal{D}_x$

e.g. $\mathrm{pr}\,v_{\theta_{xxx}+\frac{2}{3}\theta_x+\frac{1}{2}u_x}(D_x^3+\frac{2}{3}u_xD_x+\frac{1}{3}u_x) = \frac{2}{3}\underbrace{(\theta_{xxx}+\frac{2}{3}u\theta_x+\frac{1}{3}u_x)}_{D_x}$

$$+ \frac{1}{3}D_x(\dots)$$

$$[\theta,\theta] = \mathrm{pr}\,v_{\mathcal{D}\theta}(\Theta) = \int\theta\wedge\mathrm{pr}_{\mathcal{D}\theta}(\mathcal{D})\wedge\theta\, dx$$

Ex: $\Theta = \int(\theta\wedge\theta_{xxx}+\frac{2}{3}u\,\theta\wedge\theta_x)\,dx$

$$\mathrm{pr}\,v_{\mathcal{D}\theta}(\Theta) = \int\Big[\underbrace{\mathrm{pr}\,v_{\mathcal{D}\theta}(\theta\wedge\theta_{xxx})}_{=0} + \tfrac{2}{3}\mathrm{pr}\,v_{\mathcal{D}\theta}(u\,\theta\wedge\theta_x)\Big]dx$$

pr $v_{\mathcal{D}\theta}$ on θ gives automatically 0

$$= \tfrac{2}{3}\int(\mathcal{D}\theta\wedge\theta\wedge\theta_x)\,dx = \tfrac{2}{3}\int\Big[(\theta_{xxx}+\tfrac{2}{3}u\theta_x+\tfrac{1}{3}u_x\theta)$$

$$\wedge\theta\wedge\theta_x\Big]dx = \tfrac{2}{3}\int\theta_{xxx}\wedge\theta\wedge\theta_x\, dx =$$

$$= -\tfrac{2}{3}\int\theta_{xx}\wedge D_x(\theta\wedge\theta_x)\,dx = -\tfrac{2}{3}\int\underbrace{\theta_{xx}\wedge\theta_x\wedge\theta_x}_{\overset{\shortparallel}{0}}dx$$

$$-\tfrac{2}{3}\int\underbrace{\theta_{xx}\wedge\theta\wedge\theta_{xx}}_{\overset{\shortparallel}{0}} = \underline{0}\,\|$$

$\Rightarrow$ Jacobi identity for 2nd Hamiltonian structure for KdV

WINTERNITZ

References incl. Friš, Mandrosov, .., Uhlíř, P.W.
Phys. Lett 16, 354 (1965)

$$H = -\tfrac{1}{2}\Delta + V(x_1,\dots,x_n),\ H\psi = E\psi$$

Superintegrable systems: $H, X_1,\dots,X_{n-1}, Y_1,\dots,Y_{n-1}$ – algebraically independent

$$[H,X_i]=0 \quad [X_i,X_k]=0 \quad [H,Y_i]=0$$

Separable $\psi = \psi_1(z_1)\dots\psi_n(z_n)$ $\qquad z_i = z_i(x_1,\dots,x_n)$ ‖ we also usually require $[Y_i,Y_j]=0$

Multiseparable : $\exists$ more than one system of coords. in which is system separable

Exactly solvable : is possible to calculate spectrum of H ~~explicit~~ algebraically $\Leftrightarrow$
flag of subspaces $S_0 \subset S_1 \subset S_2 \subset .. \subset S_k \subset S_{k+1} \subset ..$
$HS_j \subseteq S_j$

Lie point symmetries   generalised symmetries (Lie-Bäcklund)
may form an ∞-dim. Lie alg.

superintegrable : ∞ dimensional non-Abelian algebra of generalised symmetries

Recursion operator : applied to symmetries gives another symmetry

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MIWA : Quantum integrable systems

$$H = -\frac{1}{2}\sum_{n=\infty}^{\infty}\left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta\sigma_n^z \sigma_{n+1}^z\right) \qquad XXZ \text{ Hamiltonian}$$

$T(\xi)$ transfer matrix $\quad [T(\xi_1), T(\xi_2)] = 0, \; H = \frac{d}{d\xi}\log T(\xi)\Big|_{\xi=1}$

$$T^{(N)}(\xi) = tr(R_{01} R_{02} \,..\, R_{0N})$$

YBE for R $\quad R_{12}(\xi_1/\xi_2)R_{13}(\xi_1/\xi_3)R_{23}(\xi_2/\xi_3) = R_{23}(..)R_{13}(..)R_{12}(..)$

$R = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}$ $\quad a, b, c$ functions of κ, q, ξ (see yesterday's lecture)

What is the R-matrix ?
R - matrix is the intertwiner
algebra A , 2 representation M_1, M_2 $\quad M_1 \xrightarrow{X} M_2$
X is intertwiner iff

$$\begin{tikzcd} M_1 \arrow[r, "X"] \arrow[d, "a"'] & M_2 \arrow[d, "a"] \\ M_1 \arrow[r, "X"'] & M_2 \end{tikzcd}$$
$\forall a \in A$

$\mathbb{C}^2 \quad U(sl_2) \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$U(\hat{sl}_2) \quad \hat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$

in any repr. of $U(sl_2)$
if we put $t = 1, c = 0 \Rightarrow$ we have repr. of sl_2, also we can in this way construct 2-dim. repr. of $U(\hat{sl}_2)$

automorphism $\quad \rho_z : U(\hat{sl}_2) \to U(\hat{sl}_2)$
$z = \xi^2 \qquad t \to t/z$

under ρ_z we have $\quad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_1 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$
$e_0 = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, f_0 = \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}, h_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$U_q(sl_2) \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad q^h = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$

$U_q(\hat{sl}_2) \quad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad t_1 = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$
$e_0 = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \quad f_0 = \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}, \; t_0 = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$

Action on $\mathbb{C}^2 \otimes \mathbb{C}^2$

$sl_2 \ni X \qquad \Delta(x) = x \otimes \mathbb{1} + \mathbb{1} \otimes X \qquad$ coproduct

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^3 \oplus \mathbb{C}^1 \ni (v_+ \otimes v_- - v_- \otimes v_+)$$
$$\cup \; v_+ \otimes v_+ \underset{e}{\overset{f}{\rightleftarrows}} (v_+ \otimes v_- + v_- \otimes v_+) \underset{e}{\overset{f}{\rightleftarrows}} v_- \otimes v_-$$

intertwiner $\quad Hom_A(M_1, M_2)$

$$R \in Hom_{U(sl_2)}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^2) = Hom_{sl_2}(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

$R(v_+ \otimes v_+) \overset{!}{=} v_+ \otimes v_+ \quad$ normalization

$R(v_+ \otimes v_- - v_- \otimes v_+) \overset{!}{=} (-c)(v_+ \otimes v_- - v_- \otimes v_+)$

$$R = \begin{pmatrix} 1 & & & \\ & \frac{1-\varepsilon}{2} & \frac{1+\varepsilon}{2} & \\ & \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} & \\ & & & 1 \end{pmatrix}$$

Coproduct on $U_q(sl_2) \qquad \Delta(e) = e \otimes \mathbb{1} + q^h \otimes e$

$\Delta(f) = f \otimes q^{-h} + \mathbb{1} \otimes f$

$\Delta(q^h) = q^h \otimes q^h$

One may check properties of bialgebra

How the tensor product decomposes?

$$\begin{tikzcd} \mathbb{C}^3 : v_+\otimes v_+ , \arrow[r, bend left, "f"] & q^{-1} v_-\otimes v_+ + v_+\otimes v_- , \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & v_-\otimes v_- \arrow[l, bend left, "e"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C}^1 : v_+\otimes v_- - q\, v_-\otimes v_+ \arrow[r, "e"] \arrow[dr, "f"'] & 0 \\ & 0 \end{tikzcd}$$

$$\mathrm{Hom}_{U_q(sl_2)}\Big(\underset{\underset{\sum x_{(1)}\otimes x_{(2)}}{\shortparallel}}{\overset{\mathbb{C}^2\otimes\mathbb{C}^2}{\Delta(x)}},\ \underset{\underset{\sum x_{(2)}\otimes x_{(1)}}{\shortparallel}}{\overset{\mathbb{C}^2\otimes\mathbb{C}^2}{\Delta'(x)}}\Big)$$

interwinor

$$R = \frac{1}{q+q^{-1}}\begin{pmatrix} q+q^{-1} & 0 & 0 & 0 \\ 0 & 1-c, & q^{-1}+cq & 0 \\ 0 & q+cq^{-1}, & 1-c & 0 \\ 0 & 0 & 0 & q+q^{-1} \end{pmatrix}$$

satisfies YBE if $c = -q^2$ $\qquad R = \begin{pmatrix} 1 & & & \\ & q & 1-q^2 & \\ & & q & \\ & & & 1 \end{pmatrix}$

$U(\hat{sl}_2)$ $\qquad \Delta(x) = x\otimes 1 + 1\otimes x$

$$\mathrm{Hom}_{U(\hat{sl}_2)}\big((\mathbb{C}^2)_{z_1}, (\mathbb{C}^2)_{z_2}\big)$$

$$\begin{tikzcd} 2v_-\otimes v_- & v_-\otimes v_+ + v_+\otimes v_- \arrow[l, "\Delta(f_1)"'] & v_+\otimes v_+ \arrow[l, "\Delta(f_1)=f\otimes 1"'] \arrow[dl, "\Delta(f_0)=f\otimes t^{-1}"] \\ & z_1 v_-\otimes v_+ + z_2 v_+\otimes v_- & \end{tikzcd}$$

$\hat{sl}_2\otimes\mathbb{C}[t,t^{-1}]$

If $z_1 \neq z_2 \Rightarrow$ we have irrep on $\mathbb{C}^2\otimes\mathbb{C}^2 \Rightarrow$ interwinor $= 1$

$U_q(\hat{sl}_2)$ $\qquad (\mathbb{C}^2)_{z_1}, (\mathbb{C}^2)_{z_1}$

$$\Delta(e_i) = e_i\otimes 1 + t_i\otimes e_i$$
$$\Delta(f_i) = f_i\otimes t_i^{-1} + 1\otimes f_i$$

$$R \in \mathrm{Hom}_{U_q(\hat{sl}_2)}\Big(\underset{\Delta}{(\mathbb{C}^2)_{z_1}\otimes(\mathbb{C}^2)_{z_2}},\ \underset{\Delta'}{(\mathbb{C}^2)_{z_1}\otimes(\mathbb{C}^2)_{z_2}}\Big)$$

Interwinor $\exists_1$ $\qquad R(z_1/z_2) = \dfrac{1}{1-q^2 z_1/z_2}\,(\ \cdots\ )$

$$(\cdots) = \begin{pmatrix} z_1/z_2 (=\xi^2) & & & \\ & (1-z_1/z_2)q & (1-q^2) & \\ & (1-q^2)\, z_1/z_2 & (1-z_1/z_2)q & \\ & & & z_1/z_2 \end{pmatrix}$$

$\Rightarrow R(z_1/z_2)$ ~~satisfy~~ satisfies YBE $\qquad R_{12}(z_1)R_{13}(z_1\cdot z_2)R_{23}(z_2) = \cdots$

putting $z_1 = z_2 = 0 \Rightarrow R$ given before satisfies YBE (constant)

Proof: $(\mathbb{C}^2)_{z_1}\otimes(\mathbb{C}^2)_{z_2}\otimes(\mathbb{C}^2)_{z_3}$

$$\begin{tikzcd} & \sum \underset{1}{x_{(1)}}\otimes\underset{2}{x_{(2)}}\otimes\underset{3}{x_{(3)}} \arrow[dl, "R_{12}"'] \arrow[dr, "R_{23}"] & \\ 2\ 1\ 3 \arrow[d, "R_{13}"'] & & 1\ 3\ 2 \arrow[d, "R_{13}"] \\ \begin{matrix} 2\ 3\ 1 \\ \cancel{3\ 1\ 2} \end{matrix} \arrow[dr, "R_{23}"'] & & 3\ 1\ 2 \arrow[dl, "R_{12}"] \\ & 3\ 2\ 1 & \end{tikzcd}$$

using the uniquity of R we proved YBE

since

$$\begin{tikzcd} \underset{1}{(\mathbb{C}^2)_{z_1}}\otimes\underset{2}{(\mathbb{C}^2)_{z_2}} \arrow[r, "R"] \arrow[d] & \underset{2}{(\mathbb{C}^2)_{z_1}}\otimes\underset{1}{(\mathbb{C}^2)_{z_2}} \arrow[d] \\ \cdots \arrow[r, "R"] & \cdots \end{tikzcd}$$

from def. of R we have:

Non-abelian symmetry $[T(z), \Delta^{(\infty)}(U_q(\hat{sl}_2))] = 0$ $\qquad T$ product of ∞ # of Rs; $\Delta^{(\infty)}$ infinite # of coproducts

Particle structure of H_{XXZ}

$\ldots\otimes\mathbb{C}^2\otimes\mathbb{C}^2\ldots$ decomposed into ~~sts~~ irreps of $U_q(\hat{sl}_2)$

$$\Leftrightarrow \underset{\text{vacuum}}{\mathbb{C}} \oplus \underset{\text{1-particle states}}{\int dz\,(\mathbb{C}^2)_z} \oplus \int dz_1 dz_2\,(\mathbb{C}^2)_{z_1}\otimes(\mathbb{C}^2)_{z_2}\Big/ \left\{\begin{matrix} R(z_1/z_2) \\ \cdot P \end{matrix}\right.$$

to avoid overcounting

$\oplus \ldots$

WINTERNITZ : . —

OLVER: Multihamiltonian structures

$$\frac{\partial u}{\partial t} = \mathcal{D}\,\frac{\delta H}{\delta u} \qquad H[u] = \int H[u]\,du \quad \text{Hamiltonian}$$

$$\mathcal{D}^* = -\mathcal{D} \qquad \Theta = \int (\theta \wedge \mathcal{D}\theta)\,dx \quad \text{Poisson bivector}$$

$$[\Theta, \Theta] = 0 \longrightarrow \text{Jacobi identity}$$ (see tutorial or book)

Examples: ① Any constant coefficient skew-adjoint operator, e.g. D_x

② $D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x$

③ Dubrovin - Novikov $\mathcal{D}^{\alpha\beta} = g^{\alpha\beta}(u) D_x + \sum_\gamma h^{\alpha\beta}_{\gamma}(u)\, u^\gamma_x$

in "systems of hydrodynamic type"

Hamiltonian $\Longleftrightarrow$ $g_{\alpha\beta}\, du^\alpha du^\beta$ is a flat Riemannian metric ($g_{\alpha\beta} = (g^{\alpha\beta})^{-1}$),

$$h^{\alpha\beta}_\gamma = \cancel{\partial g^{\alpha\beta}} - g^{\alpha\delta}\Gamma^\beta_{\delta\gamma}$$

Complex $\quad 0 \longrightarrow \Lambda_0 \xrightarrow{[\Theta, \;]} \Lambda_1 \xrightarrow{[\Theta, \;]} \Lambda_2 \longrightarrow \cdots$

functionals → Ham. vector fields $\qquad$ complex is exact e.g. ∀ all ~~hydrodynamic~~ hydrodynamics type op. with $k=0$

$$[\Theta, H] \longrightarrow V_{\mathcal{D}\frac{\delta H}{\delta u}}$$

Remark: $K = \frac{\partial u}{\partial t}$ is Hamiltonian $\underset{\Leftarrow \text{ if complex is exact}}{\Longrightarrow}$ $[\Theta, V_K] = 0$, i.e.

$$V_K(\mathcal{D}) = D_K \mathcal{D} + \mathcal{D} D_K^*$$

A system is called bihamiltonian iff it can be written in Hamiltonian form in 2 different ways

$$\frac{\partial u}{\partial t} = \mathcal{D}_1 \frac{\delta H_1}{\delta u} = \mathcal{D}_2 \frac{\delta H_0}{\delta u}$$ and satisfying

compatibility condition: $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_1 + \mathcal{D}_2$ are all Hamiltonian.

i.e. we have $\quad [\Theta_1, \Theta_1] = 0 = [\Theta_2, \Theta_2] = 0 = [\Theta_1, \Theta_2]$

$\Rightarrow \forall a, b \in \mathbb{R}$: $\; a\mathcal{D}_1 + b\mathcal{D}_2$ is Hamiltonian

Theorem (Magri): $\qquad$ e.g. exactness of Θ_1-complex at Λ_1

A biHamiltonian system (under some assumptions) is completely integrable in the sense that there is an infinite collection of conservation laws $H_0, H_1, H_2, \ldots$ such that $\{H_i, H_j\} = 0$ in either Poisson structure.

$$\begin{tikzcd} H_0 \arrow[d, "\mathcal{D}_1"'] \arrow[dr, "\mathcal{D}_2"] & H_1 \arrow[d, "\mathcal{D}_1"] \arrow[dr, "\mathcal{D}_2"] & H_2 \arrow[d, "\mathcal{D}_1"] & \cdots \\ V_0 & V_1 & V_2 = V_{K_2} & \end{tikzcd}$$

Higher order flows $\quad \frac{\partial u}{\partial t} = K_2 = \mathcal{D}_1 \frac{\delta H_2}{\delta u} = \mathcal{D}_2 \frac{\delta H_1}{\delta u}$

Flows are mutually commuting and $\mathcal{D}_2 \mathcal{D}_1^{-1} = \mathcal{R}$ hereditary recursion operator

Proof: $\quad V_n = [\Theta_1, H_n] = [\Theta_2, H_{n-1}] \overset{?}{\Longrightarrow} V_{n+1} = [\Theta_2, H_n] \overset{?}{=} [\Theta_1, H_{n+1}]$

Θ_1-complex exact $\Rightarrow$ $\Updownarrow\; 0 \overset{?}{=} [\Theta_1, V_{n+1}] = [\Theta_1, [\Theta_2, H_n]] =$

$$= -[\Theta_2, [\Theta_1, H_n]] - [H_1, \underbrace{[\Theta_1, \Theta_2]}_{\overset{\shortparallel}{0}}] \Leftarrow \text{compatibility}$$

$$= \pm[\Theta_2, [\Theta_2, H_{n-1}]] = 0$$

↖ induction hypothesis

$i \neq j \quad \{H_i, H_j\}_1 = V_{H_i}(H_j) =$

$= \{H_{i-1}, H_j\}_2 \Rightarrow$ by iteration $\;\ldots = -\{H_j, H_{i-1}\}_2 = -\{H_{j+1}, H_{i-1}\}_1$

$= \{H_{i-1}, H_{j+1}\}_1 = \ldots = \{H_k, H_l\}_{1 \text{ or } 2} = 0 \Rightarrow$ are in involution

QED

Hamiltonian perturbation theory

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\delta H}{\delta u}$$ $\qquad$ e.g. water waves

$$u = v + \varepsilon\varphi(v) + \varepsilon^2\varphi(v) + \ldots$$

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots \qquad \mathcal{D} \to \mathcal{D}_0 + \varepsilon\mathcal{D}_1 + \varepsilon^2\mathcal{D}_2 + \ldots$$

$$\Rightarrow \frac{\partial v}{\partial t} = (\mathcal{D}_0 + \varepsilon \mathcal{D}_1)\left(\frac{\delta H_0}{\delta u} + \varepsilon \frac{\delta H_1}{\delta u}\right) = \mathcal{D}_0 \frac{\delta H_0}{\delta u} + \varepsilon \left(\mathcal{D}_1 \frac{\delta H_0}{\delta u} + \mathcal{D}_0 \frac{\delta H_1}{\delta u}\right)$$
$$+ \varepsilon^2 (\dots)$$

$\Rightarrow V_t = V_x + \varepsilon (V_{xxx} + V V_x)$, since we have truncated the eqn $\Rightarrow$ we should not assume that the result is Hamiltonian system

but if $\mathcal{D}_1 \frac{\delta H_0}{\delta u} = \lambda \mathcal{D}_0 \frac{\delta H_1}{\delta u}$ we have Hamiltonian, even biHamiltonian system.

another remark: Tri Hamiltonian structures

$\mathcal{D}_0 = D_x \qquad \mathcal{D}_1 = D_x^3 \qquad \mathcal{D}_2 = u D_x + D_x u$

mutually compatible, Hamiltonian

$\mathcal{D}_0 \pm \mathcal{D}_1 = D_x \pm D_x^3 \qquad \mathcal{D}_1 + \mathcal{D}_2 = \dots\dots$

$\Rightarrow$ from KdV Hamiltonian we can get to Camassa-Holm eqns. ($\subset$ with nonlinear dispersion)

($\Rightarrow$ solitons = peakons $\frac{?}{!}$, nonlocal higher conservation laws)

similar approach can be applied to mKdV etc.

TUTORIAL : cancelled

MIWA: XXZ Hamiltonian $H^{(N)}_{XXZ} = \dots$, $[T^{(N)}(\xi), H^{(N)}_{XXZ}] = 0$

$$T^{(N)}(\xi) = \operatorname{trace}_{(\mathbb{C}^2)_0} \prod_{i=1}^{N} R_{0i}(\xi) \quad \text{i.e. operator on } \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{N\text{-times}}$$

R matrix intertwiner

$$\Rightarrow [T^{(N)}(\xi_1), T^{(N)}(\xi_2)] = 0$$

$$N \to \infty \quad [T(\xi), \Delta^{(\infty)}(U_q(\hat{sl}_2))] = 0$$

$$T(z) = \dots R_{0n}(\xi) R_{0n+1}(\xi) \dots$$

$$R_{0'-N}(\xi) \dots R_{0'N}(\xi) \Big(\sum \underbrace{x_{(1)} \otimes \dots \otimes x_{(2N+2)}}_{(\mathbb{C}^2)_0 \otimes (\mathbb{C}^2)_N \otimes \dots \otimes (\mathbb{C}^2)_{-N}} \Big)$$

2n+2 terms

$$= \Big(\sum x_{(2n+2)} \otimes x_{(1)} \otimes \dots \otimes x_{(2n+1)} \Big) R_{0'-N}(\xi) \dots R_{0'N}(\xi)$$

formally $N \to \infty \Rightarrow [T(\xi), \Delta^{(\infty)}(U_q(\hat{sl}_2))] = 0$

Representations of $U_q(\hat{sl}_2)$ on finite tensor product of $\mathbb{C}^2$ is (for $q \neq 1$) irreducible, on the contrary for ∞-tensor product the representation is highly reducible

We will consider $\Delta < -1 \Leftrightarrow -1 < q < 0$, $\Delta \to -\infty \Rightarrow q \to 0_-$

$\Delta = \frac{q + q^{-1}}{2}$, $H \sim \sum_n \sigma_n^z \sigma_{n+1}^z$

vacuum $\dots v_+ \otimes v_- \otimes v_+ \otimes v_- \otimes v_+ \dots$
or $\dots v_- \otimes v_+ \otimes v_- \otimes v_+ \otimes v_- \dots$
$\dots$ energy $-\infty$

} ... spontaneous symmetry breaking

$q = 0$ Kashiwara's theory of crystals

$U_q(sl_2)$

$$v_+ \otimes v_+ \xrightarrow{f_1} q^{-1} v_- \otimes v_+ + v_+ \otimes v_-$$

$$v_+ \otimes v_+ \xrightarrow{\tilde{f}_1} v_- \otimes v_+ \xrightarrow{\tilde{f}_1} v_- \otimes v_-$$

$$q v_- \otimes v_+ - v_+ \otimes v_- \xrightarrow{f_1} 0, \quad v_+ \otimes v_- \xrightarrow{\tilde{f}_1} 0$$

i.e. for $q = 0$:

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^3 \oplus \mathbb{C}$$

$$v_+ \otimes v_+ \xrightarrow{\tilde{f}_1} v_- \otimes v_+ \xrightarrow{\tilde{f}_1} v_- \otimes v_-$$

$$v_+ \otimes v_- \in \mathbb{C}$$

$U_q(\hat{sl}_2)$:

$$\begin{tikzcd}
 & v_+ \otimes v_+ \arrow[dl, "\tilde{f}_1"'] & \\
v_- \otimes v_+ \arrow[dr, "\tilde{f}_1"'] & & v_+ \otimes v_- \arrow[ul, "\tilde{f}_0"'] \\
 & v_- \otimes v_- \arrow[ur, "\tilde{f}_0"'] &
\end{tikzcd}$$

$\cdots + - + - + - \cdots$

$U_q(sl_2)_1$ acts on —"— as singlet repr.

$U_q(sl_2)_0$ .. $(-+)(-+)(-+)\ldots$ is singlet of the repr. $U_q(sl_2)_0$

$\Rightarrow$ we have removed degeneracy using $U_q(sl_2)$

One particle $\cdots(+-)+\}(+-)(+-)\ldots$ $\tilde{f}_1 \in U_q(sl_2)_1$

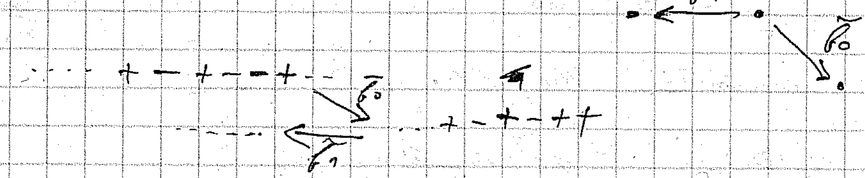
$\cdots(+-)\}-(+-)(+-)\ldots$ $\tilde{f}_0 \in U_q(sl_2)_0$

$\ldots(-+)\}+(-+)(-+)(-+)\ldots$

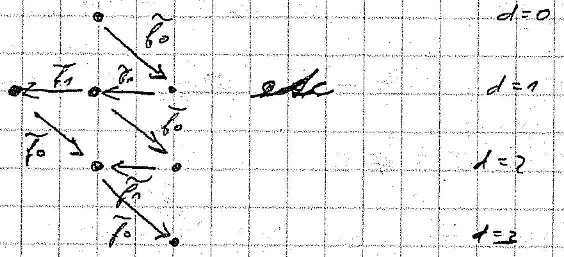
$\Rightarrow$ domain wall can move along the chain

We split the lattice into left and right half

$$\cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \,|\, \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots$$



$\Rightarrow$ we have



$\Rightarrow$ Character of the bosonic Fock space (i.e. of ∞# of harmonic oscillators)

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{d=0}^{\infty} n(d)\, q^d$$

$i = 0,1$

$$\cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2_{\text{vac}} \underset{U_q(sl_2)}{\overset{\uparrow}{\simeq}} L_0 \oplus L_1$$ level 1 highest weight representation

d .. degree operator

The vacuum vector $|vac\rangle = T(\xi)^2 |vac\rangle$ $\langle vac|vac\rangle = 1$

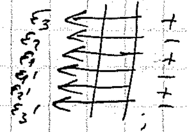
$$= T(\xi)^{2N}|vac\rangle = \lim_{N\to\infty} T(\xi)^{2N}|vac\rangle =$$

$$= \lim_{N\to\infty} T(\xi)^{2N}|\text{vector}\rangle \quad \forall \text{vector}$$

the result for ∞-product doesn't under certain conds. depend on original vector

$\Rightarrow |vac\rangle = \sum \text{coeff.}\, v_{\varepsilon_1} \otimes v_{\varepsilon_1'} \otimes \ldots$

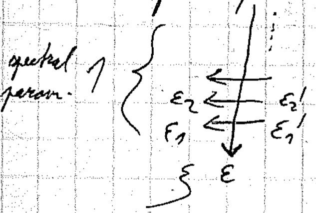
we fix the boundary e.g.



The whole space $(L_0 \oplus L_1) \otimes (L_0 \oplus L_1) \cong \mathrm{End}_{\mathbb{C}}(L_0 \oplus L_1)$

vacuum $(-q)^d$

The half transfer matrix



we choose boundary conds $\in L_0 \xrightarrow{\Phi(\xi)} L_1 \otimes (\mathbb{C}^2)_\xi$

$$\Phi(\xi) v = \sum_{\varepsilon} \Phi_\varepsilon(\xi)\, v \otimes v_\varepsilon$$

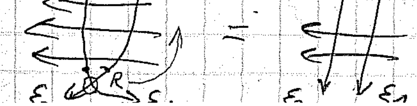
again the intertwiner

$|vac\rangle = (-q)^d$

properties (1) $\xi^d\, \Phi_\varepsilon(\xi)\, \xi^{-d} = \Phi_\varepsilon(\xi\xi)$

(2) $\sum R(\xi_1/\xi_2)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_1' \varepsilon_2'}\, \Phi_{\varepsilon_1'}(\xi_1)\, \Phi_{(\varepsilon_2')}(\xi_2)$

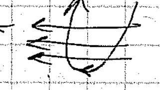
$= \Phi_{\varepsilon_2}(\xi_2)\, \Phi_{\varepsilon_1}(\xi_1)$



the full transfer matrix $f \in \mathrm{End}_{\mathbb{C}}(L_0 \oplus L_1)$

$$T(\xi) f = \sum \Phi_{\varepsilon}(\xi) \circ f \circ \Phi_{-\varepsilon}(\xi)$$

$$T(\xi)\,|vac\rangle_0 = \sum \Phi_{-\varepsilon}(\xi) \circ (-g)^{d} \circ \Phi_{\varepsilon}(\xi) = (-g)^{d} \sum \Phi_{-\varepsilon}(-g^{-1}\xi) \cdot \Phi_{\varepsilon}(\xi)$$

$= id$ ($\Leftarrow$  + unitarity)

$$= |vac\rangle_1 \Rightarrow T^2(\xi)\,|vac\rangle_{0,1} = |vac\rangle_{0,1}$$

$\Rightarrow$ eigenvectors of transfer matrix (similarly 1-particle states, 2-particle states) from representation theory

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FLASCHKA: Representation theory

V-representation of $U(n)$.

Lie alg. $u(n)$ of $U(n)$ $= \{$skew hermitean$\} \cong \underbrace{\sqrt{-1}\,u(n)}_{\text{hermitean}}$

Ex: 1) $U(2)$ on $V = \mathbb{C}$

$g \cdot 1 = (\det g)^2 \cdot 1$ weight $(2,2)$

2) on $\mathcal{H}_2 = \cdots$ weights $(0,2)$ $(1,1)$ $(2,0)$ etc.

Theorem: If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$ are integers, and $k \in \mathbb{Z}$, then $(\lambda_1 + k, \lambda_2 + k, \ldots, \lambda_n + k)$ is the highest weight of an irrep. of $U(n)$. Every irrep has such a highest weight and there is a basis consisting of weight vectors.

Fact: The representation $V^{\underline{\lambda}}$, $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ can be built from the orbit $\left\{ g \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} g^{-1} \,\middle|\, g \in U(n) \right\}$ in $\sqrt{-1}\,u(n)$ (Hermitean matrices).

(Borel-Weyl Theorem, geometric quantization).

P.B. $\{f, g\}(X) = \sqrt{-1}\,\mathrm{Tr}\, X [\nabla f(X), \nabla g(X)]$

$$\dot{X} = \sqrt{-1}\,[\nabla f(X), X]$$

$\sigma_{\underline{\lambda}^{(1)}} \times \ldots \times \sigma_{\underline{\lambda}^{(m)}}$ $f(X_1, \ldots, X_m)$ $\{f, g\}(\vec{X}) = \sum_{i=1}^{m} \{f, g\}_i(\vec{X})$

(like $\mathbb{R}^{2n} = \mathbb{R}^2 + \ldots + \mathbb{R}^2$, $\{\ \} = \sum_i \frac{\partial}{\partial}\frac{\partial}{\partial} - \frac{\partial}{\partial}\frac{\partial}{\partial}$)

Sketch of quantization:

F function $\rightarrow$ $\hat{F}$ operator
1st rule 1 $\rightarrow$ multiplication by 1
2nd rule $\{F, G\}$ $\rightarrow$ $i\,[\hat{F}, \hat{G}]$
3rd rule $\overline{F}$ $\rightarrow$ $\hat{F}^{*}$ + "minimality"

$\hat{F} = -i X_F$ satisfies 2nd rule not 1st

Modify $\hat{F} = F - i X_F$ satisfies 1st, not 2nd

Yet another modification $\hat{F} = F - i X_F - \sum p_j \frac{\partial F}{\partial p_j}$

satisfies 1, 2, 3 (How may be guessed $F = H \Rightarrow F - \sum p_j \frac{\partial F}{\partial p_j} = -L$ Lagran.)

Now $\hat{q} = q + i \frac{\partial}{\partial p}$ $\hat{p} = -i \frac{\partial}{\partial q}$

but these act on $\psi(q, p)$!

$\Rightarrow$ Need (somehow) eliminate one variable, e.g. restrict to functions constant in p (i.e. independent of p).

To do this on $\sigma_{\underline{\lambda}}$ we generalize $\hat{F} = F - i X_F - \sum p_j \frac{\partial F}{\partial p_j}$ and "throw out half the variables"

$$\sum p_j \frac{\partial F}{\partial p_j} = \underbrace{\left(\sum p_j\, dq_j\right)}_{\alpha}\left(\sum \frac{\partial F}{\partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial F}{\partial q_k}\frac{\partial}{\partial p_k}\right)$$

$$d\alpha = \sum dp_j \wedge dq_j$$

On a coordinate neighborhood U_j on $\mathcal{O}_\lambda$, try $\hat{H}_{U_j} = F - iX_F + \alpha_j\,(X_F)$

where $d\alpha_j = \omega|_{U_j}$ ($\leftarrow$ sympl. form).

α is determined up to df, may not $\exists$ globally.

Need to patch these together $\Rightarrow$ restriction on symplectic form ω: $\iint_{\text{sphere}} \omega \in 2\pi\mathbb{Z}$, $(\Leftarrow \lambda_i \in \mathbb{Z})$

$\hat{F}$ acts on sections of line bundle (instead of functions) of which $\{\alpha_j\}$ are connection, ω is curvature.

"Half" the variables

1) Complex coordinates on $\underline{\mathcal{O}_\lambda}$: $z_1, \dots, z_k, \bar{z}_1, \dots, \bar{z}_k$

Look only at sections depending on z, i.e. $\frac{\partial}{\bar{z}_k} s = 0$

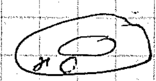
The space of (surviving) holomorphic sections is a finite-dim. vector space.

The group $U(n)$ acts on this: get a representation

Other method

2) If you have an integrable system on $\underline{\mathcal{O}_\lambda}$, actions J_k, angles θ_k

$\omega = \sum dJ_k \wedge d\theta_k$. Try find sections covariantly constant on tori (i.e. "indep. of θ_k").



Parallel transport around γ_1: pick up factor

$$e^{i\oint_{\gamma_1} a}\; e^{\cdot}\quad , \text{e.g. } \theta_2 = \dots = \theta_m = \text{const.} \Rightarrow e^{2\pi i J_1}$$

etc.

Unless all $e^{2\pi i J_k} = 1$, the section is "multivalued".

Def: Bohr-Sommerfeld torus if $J_k \in \mathbb{Z}$.

Rep $V^{(2,0,0)}$ of $U(3)$ basis $\underset{(2,0,0)}{z_1^2}$ $z_1 z_2$ z_2^2 $z_1 z_3$ $z_2 z_3$ z_3^2

Restrict to $U(2) \Rightarrow \mathcal{H}_2(U(2)) + \mathcal{H}_1(U(2)) + \mathcal{H}_0(U(2))$

$z_1^2 \; z_1 z_2 \; z_2^2 \qquad z_1 \; z_2 \; (\cdot z_3) \qquad 1 \; (z_3^2)$

Another restriction to $U(1) \Rightarrow$ 6 1-dim representation spaces, they give basis of original $U(3)$

Theorem (Wey): ....

$\Rightarrow$ Gelfand-Tsetlin patterns index a weight vector basis of V^3

$$\begin{matrix} \lambda_1 & \lambda_2 & \dots & \lambda_m \\ & \mu_1 & \dots & \mu_{m-1} \\ & & \nu_1 \dots \nu_{m-1} & \end{matrix}$$

e.g. for $U(3)$ $\begin{matrix}2&0&0\\&2\;0&\\&2&\end{matrix}$, $\begin{matrix}2&0&0\\&2\;0&\\&1&\end{matrix}$, $\begin{matrix}2&0&0\\&2\;0&\\&0&\end{matrix}$, $\begin{matrix}2&0&0\\&1\;0&\\&1&\end{matrix}$, $\begin{matrix}2&0&0\\&1\;0&\\&0&\end{matrix}$, $\begin{matrix}2&0&0\\&0\;0&\\&0&\end{matrix}$.

Guillemin-Sternberg: matrix analog, take a matrix X with eigenvalues $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^n$, $X \in \underline{\mathcal{O}_\lambda}$

$$\left(\begin{array}{c|c} U(n-1) & \\ \hline & \end{array}\right)$$

look for ~~eigenvectors~~ eigenvalues of $(n-1)\times(n-1)$ submatrix

$\rightarrow \lambda_1 \le \mu_1 \le \lambda_2 \le \mu_3 \le \dots \le \mu_{n-1} \le \lambda_n$

etc.

Count: # of these eigenvalues is $\frac{1}{2} \dim_{\mathbb{R}} \underline{\mathcal{O}_\lambda}$ $(= \dim_{\mathbb{C}} \underline{\mathcal{O}_\lambda})$

Proposition: Let μ, ν be any two

Then $\{\mu, \nu\} = 0$ and generate 2π-periodic flows.

Proof: Use L hermitean matrix $\Rightarrow \{L v = \overset{\text{single eigenvalue}}{\lambda} v,\ \|v\| = 1 \Rightarrow \nabla\lambda(L) = |v\rangle\langle v|$

= proj. onto v

$$\{\varphi_\mu, \varphi_\nu\} = \sqrt{-1}\,\mathrm{Tr}\, X \Big[\underset{\nabla\mu}{\begin{pmatrix} P_\mu & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\nabla\nu}{\begin{pmatrix} P_\nu & 0 \\ 0 & 0 \end{pmatrix}} \Big] =$$

$$= \sqrt{-1}\,\mathrm{Tr}\Big[X, \begin{pmatrix} P_\mu & 0 \\ 0 & 0 \end{pmatrix}\Big] \begin{pmatrix} P_\nu & 0 \\ 0 & 0 \end{pmatrix}$$

Minor of X commutes with P_μ

$$= \sqrt{-1}\,\mathrm{Tr} \begin{pmatrix} & * \\ * & \end{pmatrix} \begin{pmatrix} P_\nu & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$\dot{X} = \Big[\begin{pmatrix} P_\mu & 0 \\ 0 & 0 \end{pmatrix}, X \Big] = \left(\begin{array}{c|c} 0 & \\ \hline & \end{array} \right)$ $\Rightarrow$ minor μ never changes $\Rightarrow$

P_μ is constant $\Rightarrow$ $X(t) = e^{i t P_\mu} X(0) e^{-i t P_\mu}$

.. has period 2π

Conclusion: ~~integral values~~ Bohr-Sommerfeld tori index the weight basis of the representation.

Ad action variables
$$\begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 & \\ & & \nu_1 & & \end{matrix}$$
e.g.
$$\begin{matrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{matrix} \Rightarrow S^1: \begin{pmatrix} 1 & e^{i\psi} & 0 \\ e^{-i\psi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
corresp. torus: circle

$$\begin{matrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{matrix} \Rightarrow \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ \bar{a} & \bar{b} & 1 \end{pmatrix}, \quad |a|^2 + |b|^2 = 1 \quad . \; S^3,$$
i.e. not torus!

Discussion

Remark: integrability of LODE (Kruskal)

$y'' + a(x) y' + b(x) y = 0$

$y_1(x), y_2(x)$ 2 particular solns

general soln. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$y'(\;) = c_1 y_1'(\;) + c_2 y_2'(\;)$

$\Rightarrow c_1 = \ldots, \; c_2 = \ldots$ 2 indep. "first integrals"