Plane waves and Penrose limits

Blau 1

in GR studied since 1920s
in strings in 1990s

2 years ago

in IIB SUGRA maximally supersymmetric plane wave (with RR fields $F_5 = *F_5$)

(the other known solns: Minkowski, $AdS_5 \times S^5$) ↑ Penrose limit

string theory in this background is exactly solvable

BMN: AdS/CFT ⟶ plane wave / CFT duality

⇐ "advanced properties" of "special" plane waves – not will be covered

We shall cover "elementary properties" of "general" plane waves:

Plan

Plane waves → metrics
→ geodesics
→ curvature
→ isometries

Penrose limits (PL) → geodesic congruences
→ basic properties
→ examples

Omissions: global properties of plane waves

Conventions: $g_{\mu\nu}$ $(\overbrace{- + \ldots +}^{D})$ $D = d+2$

$[\nabla_\mu, \nabla_\nu] V^\lambda = R^\lambda{}_{\sigma\mu\nu} V^\sigma$ $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$

Motivation: plane waves in linearised GR $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $|h| \ll |\eta|$

⇒ wave equation

transverse polarisation ⇒ $ds^2 = -dt^2 + dz^2 + (\delta_{ij} + h_{ij}(t-z)) dy^i dy^j$

$U = z - t,\; V = (z+t)/2$

$\Rightarrow ds^2 = 2dU\,dV + (\delta_{ij} + h_{ij}(U))\, dy^i dy^j$

Plane wave in full GR

we assume ansatz $\boxed{ds^2 = 2dU\,dV + C_{ij}(U)\, dy^i dy^j}$ Rosen coordinates

C_{ij} is not unique for given ds^2

e.g. (1) $ds^2 = 2dU\,dV + d\vec{y}\,d\vec{y} \longleftrightarrow ds^2 = 2dU\,dV + U^2 d\vec{y}\,d\vec{y}$

$\vec{y} \to U\vec{y},\; V \to V + \ldots$

(2) $ds^2 = 2dU\,dV + e^{2U} d\vec{y}\,d\vec{y} \longleftrightarrow ds^2 = 2dU\,dV + \sinh^2 U\, d\vec{y}\,d\vec{y}$

⇒ these coords rather unconvenient, also usually have coord. singularities

Use $\frac{\partial}{\partial V}$ covariantly constant (parallel) 0 vector

$\Rightarrow$ More systematic approach: metrics with a parallel null vector

Z null, parallel $\Rightarrow$ we choose coords. so that $Z = \partial_V \Rightarrow g_{VV} = 0$, $\nabla_\mu Z_\nu = 0$ ($g_{\alpha\nu} Z^\nu = g_{\alpha V}$)

$\Leftrightarrow \underbrace{\nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 0}_{\text{Killing vector}} \; \& \; \underbrace{\nabla_\mu Z_\nu - \nabla_\nu Z_\mu = 0}_{\text{gradient}}$

$\Downarrow$ together with $\nabla_\alpha g_{\mu\nu} = 0$ i.e. $\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} = 0$, put $\alpha = V$

$\forall_{\mu,\nu} \; \partial_V g_{\mu\nu} = 0$ (since $Z_\mu = g_{\mu V} \Rightarrow$ satisfied if $\mu = V$ or $\nu = V$)

we split $X^\mu = (V, x^\alpha) \Rightarrow$

$\nabla_\alpha Z_\beta = \nabla_\beta Z_\alpha \Rightarrow Z_\alpha = \partial_\alpha U = g_{\alpha V}$

$$\Rightarrow ds^2 = g_{\mu\nu} dx^\mu dx^\nu \overset{g_{VV}=0}{=} 2 g_{\alpha V} dx^\alpha dV + g_{\alpha\beta} dx^\alpha dx^\beta =$$

$$= 2\, dU dV + g_{UU}(U, X^a) dU^2 + 2 g_{Ua}(U,X) dU dX^a + g_{ab}(U,X) dX^a dX^b$$

we put $X^\mu = (U, V, X^a)$

i.e. $$\boxed{ds^2 = 2dUdV + K(U,X)dU^2 + 2A_a(U,X)dUdX^a + g_{ab}(U,X)dX^a dX^b}$$

we specialise to

$\boxed{g_{ab} = \delta_{ab}}$ pp-waves plane-fronted & parallel rays (covariantly constant 0-vector)

Einstein eq. $\Rightarrow \partial^a F_{ab} = 0 \quad F_{ab} = \partial_a A_b - \partial_b A_a$

gauge transf. $V \to V + \lambda(x) \quad A_a \to A_a + \partial_a \lambda \Rightarrow$ one may put $A_a = 0$

Plane wave - special pp-wave

$$\boxed{ds^2 = 2dUdV + A_{ab}(U) X^a X^b dU^2 + d\vec{X}^2} \quad \text{Brinkman coordinates}$$

practically unique $ds^2 \xleftrightarrow[1-1]{\text{almost}} A_{ab}$

e.g. flat $\Leftrightarrow A_{ab}(U) = 0$

Geodesics

$$L = \tfrac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \dot{U}\dot{V} + \tfrac{1}{2} A_{ab} X^a X^b \dot{U}^2 + \tfrac{1}{2}\dot{\vec{X}}^2$$

$\Rightarrow p_V = \dot{U}$ constant $\quad p_V = 0 \to$ geodesics = straight lines boring

$p_V \neq 0 \to$ rescale U $\quad p_V = 1 \quad U = \tau$ (affine parameter) ("light-cone gauge")

$\Rightarrow$ harmonic oscillator Lagrangian

$$\ddot{X}^a(U) = A_{ab}(U) X^b(U) \qquad \omega^2_{ab} = -A_{ab}$$

$\ddot{V} = \ldots$ (boring)

Blue 2

<u>Curvature</u>

$$R_{UaUb} = -A_{ab} \quad \text{other elements vanish}$$

Ricci tensor $R_{UU} = \delta^{ab} R_{UaUb} = -\mathrm{Tr}\, A$

$g^{UU} = 0 \Rightarrow R = 0$

Vacuum Einstein equations $\mathrm{Tr}\, A = 0$

$D=4 \quad (d=2) \qquad$ e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$ds^2 = 2dUdV + (x^2-y^2)dV^2 + dx^2 + dy^2$$

$$A = \begin{pmatrix} f(U) & g(U) \\ g(U) & -f(U) \end{pmatrix}$$

$$ds^2 = 2dU\,dV + [f(U)(x^2-y^2) + 2g(U)xy]\,dU^2 + dx^2 + dy^2$$

2 arbitrary ~~equations~~ functions ~ 2 indep. polarizations of graviton

$R_{UU} \neq 0 \Rightarrow$

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \sim T_{\mu\nu}$$

$$\Downarrow$$

$R_{UU} \sim T_{UU} \neq 0 \qquad T_{\mu\nu} = 0$ otherwise (NAUB matter)

e.g. scalar field $\phi(U)$ or $F = dU \wedge (\ldots)$

Weyl tensor $C_{UaUb} = -(A_{ab} - \frac{1}{d}\delta_{ab}\mathrm{Tr}A)$

$= 0 \iff A_{ab}(U) = A(U)\,\delta_{ab}$

$d > 1 \Rightarrow$ plane wave is conformally flat $\iff A_{ab}(U) = A(U)\,\delta_{ab}$

BFHP solution

$d=8;\ D=10 \qquad A_{ab} = -\delta_{ab} \qquad ds^2 = 2dUdV - \vec{x}^2 dV^2 + d\vec{x}^2$

maximally SUSY solution of IIB SUGRA

typical matter: $T_{UU} \overset{!}{>} 0 \Rightarrow R_{UU} > 0 \iff \mathrm{Tr}\,A < 0$

$R_{UaUb} = -A_{ab}$, $R_{UU} = -\mathrm{Tr}\,A$ but no scalars can be constructed from them & their derivatives

$\Rightarrow$ all curvature invariants are zero $\Rightarrow$ it is impossible to write higher order correction to Einstein eq. $\Rightarrow$ should be exact solns. of string theory

Schmidt's argument: $g \to \lambda^2 g$

1) $\nabla_{\mu_1} \ldots \nabla_{\mu_k} R^{\alpha}{}_{\beta\gamma\delta}$ invariant under scaling of g, we need to raise some index (indices) to get curvature invariant $\Rightarrow$ cannot be inv. under $g \to \lambda^2 g$

2) If there is a coord. transf. which in fact just rescales g → at a fixed point x of this coord. transf. all curvature invariants are invariant & aren't invariant

homothety

⇒ must be zero at these fixed points x

3) For plane waves $\forall x$ $\exists$ homothety with fixed point x

e.g. $(U, x^a = 0, V = 0)$ $\quad (U, V, x) \to (U, \lambda^2 V, \lambda x)$

⇒ together with translational invariance of ds^2 we see that all curvature inv. are zero

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Note: usually intrinsic singularity ⟷ divergence in curvature invariant

But we have all curvature invariants $= 0$!

Geodesic deviation equation:

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu{}_{\nu\lambda\rho}\,\dot{x}^\nu \dot{x}^\lambda \,\delta x^\rho$$

in our case we have $U = \tau$ $\quad \ddot{x}^a = A_{ab} x^b$

we may choose δx^μ: $\delta U = 0$ $\Rightarrow$ $\frac{d^2}{dU^2}\delta x^a = A_{ab}\,\delta x^b$

⇒ for negative eigenvalues of A_{ab} gravitation is attractive → focusing of geodesics

positive — repulsive → defocusing —"—

if $A_{ab}(U)$ is singular at $U = U_0$ ⇒ tidal force becomes infinite → singularity

if there is singularity ⇒ every geodesics runs into it in finite time U_0

⇒ geodesically incomplete spacetime

transition to Rosen coords. $C_{ij} \leftrightarrow A_{ab}$

we introduce vielbein for $C_{ij} = E^a_i E^b_j \delta_{ab}$ $\quad y^i \to x^a = E^a_i y^i$

$\Rightarrow d\vec{x}^2 + (\;)dU^2 + \underbrace{(\dots)dU dx^a}$

eliminate these by shifting V

$\Rightarrow A_{ab} = \ddot{E}_{ai} E^i_b$, i.e. $\ddot{E}_{ai} = A_{ab} E^b_i$ again harmonic oscillator eqn.

2d L Indep. solns, pick d of them → $E \to C$

in fact geodesics in Rosen coords. are straight lines in Rosen coords but spirals in Brinkmann

If C_{ij} diagonal $\quad ds^2 = 2dUdV + C^2(U)d\vec{y}^2$

⇒ Brinkman coords. $ds^2 = 2dUdV + \frac{\ddot{c}}{c}(U)\vec{x}^2 dU^2 + d\vec{x}^2$ $\quad$ if $c(U)^2 = (aU+b)^2$ ⇒ flat metric

BC $ds^2 = 2dUdV + (x^2-y^2)dU^2 + dx^2 + dy^2$ **Blau 3**

$\Rightarrow$

RC $ds^2 = 2dUdV + \sinh^2 U\, dx^2 + \sin^2 U\, dy^2$

Isometries of plane waves

$Z = \partial_V$

in Rosen coords translations in X^i .. d symmetries

+ d more (in generic case)

} $(2d+1)$-dimensional Heisenberg algebra

in Brinkman coords.

Killing equations $\overset{?}{=} X$: $\quad L_X g_{\mu\nu} = 0 = \nabla_\mu X_\nu + \nabla_\nu X_\mu$

$2d$ Killing vectors: $\quad X(f_{Ia}) \qquad I = 1,..,2d$

f_{aI} .. $2d$ linearly independent solns of $\ddot{f}_a(U) = A_{ab}(U) f_b(U)$

$$X(f_I) = f_{Ia}\partial_a - \dot{f}_{Ia} x^a \partial_V$$

$+ \partial_V$

$[X(f_I), Z] = 0 \qquad [X(f_I), X(f_K)] = \underbrace{\left(\sum_a f_{Ia}\dot{f}_{Ka} - f_{Ka}\dot{f}_{Ia}\right)}_{} \underset{Z}{\underset{\parallel}{\partial_V}}$

Wronskian, indep. of U ($(\dot{\ }) = 0$ by inspection)

$\Rightarrow$ constant antisym. matrix W_{IK}, $\det W_{IK} \neq 0$

lin. indep. solns

$$[X(f_I), X(f_K)] = W_{IK} Z$$

we may put W_{IK} into canonical form $\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \\ & & & & \ddots \end{pmatrix}$

We split $\quad f_I \rightarrow \{p_a, q_a\}_{a=1..d}$ such that $p_{ab}(U_0) = \delta_{ab}$, $\dot{p}_{ab}(U_0) = 0$

$q_{ab}(U_0) = 0$, $\dot{q}_{ab}(U_0) = 0$

some fixed time U

$X(p_a) \equiv P_a,\ X(q_a) \equiv Q_a \Rightarrow \underbrace{[P_a, Q_b] = -\delta_{ab} Z,\ [Q_a, Q_b] = 0 = [P_a, P_b] \quad [Q_a, Z] = 0 = [P_a, Z]}_{\text{algebra of isometries in generic case}}$

More isometries

$A_{ab}(U) = A(u)\,\delta_{ab} \longrightarrow SO(d)$ invariance not very interesting

Heisenberg algebra acts transitively on $U = \text{const.}$ planes $\Rightarrow$ it would be interesting to have isometries involving $\frac{\partial}{\partial U}$

e.g. $A_{ab} = \text{constant} \Rightarrow X = \partial_U$ is a Killing vector

$$[X, Q_a] = P_a \qquad [X, P_a] = A_{ab} Q_b \qquad [X, Z] = 0$$

$\Rightarrow$ one may thing of X as something like $P^2 + \omega^2 Q^2$

$\Rightarrow$ Harmonic oscillator algebra

(any $K_\mu \to K_\mu \dot{X}^\mu$ conserved charge $\qquad P_V = -\frac{1}{2}(\underbrace{\dot{\vec{X}}^2 - A_{ab} X^a X^b}_{\text{Hamiltonian}}) + \varepsilon^{=0,\pm}$)

Z, Q_a, P_a, X acts transitively on spacetime .. homogeneous ~~symmetric space~~ space, actually even <u>symmetric</u> Cahen - Wallach space

locally symmetric $\Leftrightarrow \nabla_\lambda R_{\alpha\beta\gamma\delta} = 0 \underset{\text{for plane waves}}{\Longleftrightarrow} \partial_U A_{ab} = 0$

<u>? $\exists A_{ab}(U)$ is homogeneous ?</u>

YES: e.g. $A_{ab}(U) = \frac{B_{ab}}{U^2}$ invariant under $(U, V) \to (\lambda U, \lambda^{-1} V)$

$$X = U\partial_U - V\partial_V$$

$\exists$ two families of homogeneous plane waves

1) generalizes the Cahen - Wallach spaces

2) generalizes

<u>Penrose limits</u>

1976 "The limit of every spacetime is a plane wave." Penrose

silence (except 2 footnotes by Tseytlin)

2000 Güven

SUGRA solution $\xrightarrow[PL]{}$ new SUGRA solution

then

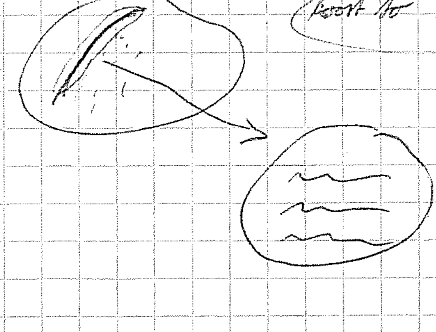
II B plane waves $A_{ab} = -\delta_{ab} \sim AdS_5 \times S_5$ but how? .. PL

<u>Picture</u>

$M, g_{\mu\nu}, \gamma$ null geodesic

everything outside the null geodesic gets pushed to ∞, the geodesic gets blown up to the whole space.

Boost to null geodesic & rescale the clocks. What you will see is a plane wave



γ adapted coord. system $\quad x^\mu \to (U, V, Y^i)$ $\quad$ (the $\exists$ of (U,V,Y^i) can be proved locally) $\quad$ $\boxed{\text{Class 4}}$

s.t. ∂_U is geodesic $\forall V, Y^i$

∂_U at $V = Y^i = 0$ is γ

& $ds^2 = 2dUdV + C(U,V,Y)\,dV^2 + C_i(U,V,Y)\,dVdY^i + C_{ij}(U,V,Y)\,dY^i dY^j$

($\Rightarrow$ ∂_U is null geodesic $\Leftarrow$ no terms containing dU)

$(U,V,Y) \to (\lambda^{-1}U, \lambda V, Y)$, $\lambda \to 0$ $\Rightarrow$ singular metric $\Rightarrow$ simultaneously blow up

– large volume limit $(U,V,Y) \to (\lambda U, \lambda V, \lambda Y)$ & $ds^2 \Rightarrow \lambda^{-2} ds^2$

$\Rightarrow$ combined $\quad (U,V,Y) \to (U, \lambda^2 V, \lambda Y)$ & $ds^2 \Rightarrow \lambda^{-2} ds^2$

$$\Rightarrow ds^2 \to \underbrace{2dU\,dV + C_{ij}(U,0,0)\,dY^i dY^j}_{\text{Penrose limit of the spacetime above .. plane wave in Rosen coords.}}$$

in the procedure above we shall consider only a patch .. adapted coord. system $\xrightarrow{PL}$

$\to$ patch of plane wave in Rosen coords. $\to$ plane wave in Brinkmann coords

$\underset{PL}{\sim}$ the whole null geodesic

Note: (2.40) in lecture notes is not correct (mess of 2 coord. systems)

General properties of Penrose limits

null geodesic γ : $\gamma(0)$

$\dot\gamma(0)$ $\quad$ PL doesn't depend on $|\dot\gamma(0)|$, only on the direction ($\sim$ rescaling affine parameter)

γ_1 & γ_2 related by isometry $\quad \gamma_1(0) \to \gamma_2(0)$, $\dot\gamma_1(0) \to \dot\gamma_2(0)$ $\Rightarrow$ $\underbrace{PL(\gamma_1) \,\&\, PL(\gamma_2) \text{ isometric}}_{\text{"covariance of PL"}}$

$\Rightarrow$ The more symmetries $\Rightarrow$ less distinct PLs

Plane wave : Heisenberg algebra of isometries

$\Rightarrow$ PL might have much more symmetries than the original metrics

What properties are preserved by PLs?

Geroch (1969) : Properties of families of spacetimes

$\Rightarrow$ $\quad \lambda \neq 0$ $\quad g_{\mu\nu}$ original metric $\Rightarrow (U,V,Y) \to (U, \lambda^2 V, \lambda Y)$, $g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}$

what happens as $\lambda \to 0$

E.g. 1) $g_{\mu\nu}$ Ricci flat $R_{\mu\nu}=0 \Rightarrow R_{\mu\nu}(g^{\lambda}_{\mu\nu})=0 \quad \forall\lambda>0 \Rightarrow R_{\mu\nu}(\bar{g}_{\mu\nu})=0$

PL metric

.. hereditary property of PLs

2) $g_{\mu\nu}$ conformally flat $C_{\mu\nu\lambda\varsigma}=0 \Rightarrow \bar{C}_{\mu\nu\lambda\varsigma}=0$ also $\bar{g}_{\mu\nu}$ is conformally flat

3) $\nabla_{\lambda} R_{\alpha\beta\gamma\delta}=0$ locally symmetric $\Rightarrow$ $\bar{g}_{\mu\nu}$ locally symmetric

$g_{\mu\nu}$ Einstein $\quad R_{\mu\nu}=\Lambda g_{\mu\nu} \quad \Rightarrow \quad$ PL: $R_{\mu\nu}(\bar{g})=0$

(large volume limit)

Typically topological properties are not hereditary.

More subtle hereditary properties:

isometries $\quad g_{\mu\nu} \quad n$ Killing vectors $\Rightarrow g^{\lambda}_{\mu\nu} \quad n$ Killing vectors (KV)

$\lambda\to 0$ they might become linearly dependent

Since (Geroch) proved: # of isometries can never decrease in the PL

we have $\qquad \#KV(\bar{g}_{\mu\nu}) \geq \max(n, 2d+1)$

Similarly for supersymmetries, at least half of them get preserved.

$g_{\mu\nu}$ homogeneous $\overset{?}{\Longrightarrow} \bar{g}_{\mu\nu}$ homogeneous .. not true (Kaigorodov's metrics)

if + & reductive $\quad \Rightarrow \bar{g}_{\mu\nu}$ homogeneous & reductive

PLs of maximally symmetric spacetimes

AdS, dS $\quad$ maximally symmetric $\to$ Einstein & conformally flat

$$\begin{array}{ccc} \text{Einstein} & & \text{conformally flat} \\ \downarrow \text{PL} & & \downarrow \text{PL} \\ \text{Ricci flat} & & \text{conf. flat} \end{array}$$

$$\underbrace{R_{\mu\nu}(\bar{g})=0}_{\text{Ricci flat}} = \underbrace{C_{\mu\nu\lambda\varsigma}(\bar{g})}_{\text{conf. flat}} \Rightarrow R_{\alpha\beta\gamma\delta}=0$$

$\Rightarrow$ PL is flat space for any γ

AdS × S

1) covariance $\Rightarrow$ at most 2 distinct PLs

$\rightsquigarrow \gamma$ has / doesn't have component in S

2) (locally) symmetric $\Longrightarrow A_{ab}=$ const.

flat space

E.g. on $AdS_5\times S^5 \qquad R_{AdS}=R_S$

Coords: cosmological coordinates on AdS, spherical on S^5 [Blatt 5]

$$-dt^2 + \sin^2 t \underbrace{d\tilde{\Omega}_4^2}_{\text{unit hyperboloid}} + d\theta^2 + \sin^2\theta\, d\Omega_4^2$$

$$U = (\theta - t)/\sqrt{2} \qquad V = (\theta + \tau)/\sqrt{2} \qquad \theta = (U+V)\frac{1}{\sqrt{2}} \qquad t = (V-U)\frac{1}{\sqrt{2}}$$

$$\Rightarrow ds^2 = 2dUdV + \sin^2((V-U)/\sqrt{2})\, d\tilde{\Omega}_4^2 + \sin^2((V+U)/\sqrt{2})\, d\Omega_4^2$$

PL: $ds^2 = 2dUdV + \sin^2\frac{U}{\sqrt{2}}\, d\vec{y}_4^{\,2} + \sin^2\frac{U}{\sqrt{2}}\, d\vec{z}_4^{\,2}$

$\Rightarrow$ go from Rosen to Brinkmann coords:

PL of $AdS_5 \times S^5 = (A_{ab} = -\delta_{ab})$

$$AdS_4 \times S^7 \;\&\; AdS_7 \times S^4 \xrightarrow{PL} A_{ab} \qquad a,b = 1,\dots,9$$

$$(-1,-1,-1,-\tfrac{1}{4},-\tfrac{1}{4},\dots,-\tfrac{1}{4})$$

<u>Schwarzschild metric</u>

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 \underbrace{d\Omega_2^2}_{= d\theta^2 + \sin^2\theta d\varphi^2} \qquad f(r) = 1 - \frac{2m}{r}$$

$\varphi = $ const. defines a null geodesic $\forall \varphi$

$$L = -f(r)\dot{t}^2 - f(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 \qquad \Rightarrow E = \text{konst.}\;:\; \dot{t} = (f(r)/E)^{-1}$$

$$l = \text{konst.} \;\dots\; \dot{\theta} = l/r^2$$

$L = 0 \quad (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0)$

$$\Rightarrow \boxed{\dot{r}^2 = E^2 - 2V_{eff}(r)} \qquad V_{eff}(r) = f(r)\frac{l^2}{r^2} \qquad f(r) = 1 - \frac{2m}{r}$$

$$(\Leftarrow \ddot{r} = -V_{eff}') \quad \& \quad \boxed{\dot{t} = E/f \;\&\; \dot{\theta} = l/r^2}$$

Now use Hamilton-Jacobi formalism to find adapted coordinates

$$(r,t,\theta,\varphi) \longrightarrow (U, V, (\tilde{\theta},\tilde{\varphi}) = y_i)$$

$U = $ affine parameter $\quad r = r(U) \qquad d\phi = d\tilde{\phi} \quad$ i.e. $\phi = \tilde{\phi} + \phi_0$

$$d\theta = \dot{\theta}(U)dU + d\tilde{\theta}, \; dt = -E^{-1}dV + E^{-1}l d\tilde{\theta} + E/f\, dU$$

i.e. $dx^\mu = \dot{x}^\mu(U)dU + \dots$ general form of such transitions

$$(\Leftarrow g_{UU} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0)$$

PL of Schwarzschild metric (in Rosen coordinates)

$$ds^2 = 2dUdV + E^{-2}r(U)^2 d\tilde{\theta}^2 + r(U)^2 \sin^2(\tilde{\theta} + l\int r(U)^{-2})d\tilde{\phi}^2$$

$\Rightarrow$ go to Brinkmann coordinates

$$\Rightarrow A_{11} = \frac{(r(U)\,\dot{r}(U))^{\cdot\cdot}}{r(U)\,\dot{r}(U)} \qquad A_{22} = \frac{(r(U)\sin(\ ))^{\cdot\cdot}}{r(U)\sin(\ )}$$

$$\Rightarrow A_{11} = -\frac{3}{r}V' - V'' \qquad A_{22} = -\frac{1}{r}V' - \frac{L^2}{r^4}$$

Consistency check: Schwarzschild Ricci flat $\Rightarrow$ PL Ricci flat $\Rightarrow \mathrm{Tr}\,A = 0$?

yes $A_{11} = -A_{22}$

$$A_{11} = -A_{22} = \frac{3 m l^2}{r(U)^5}$$

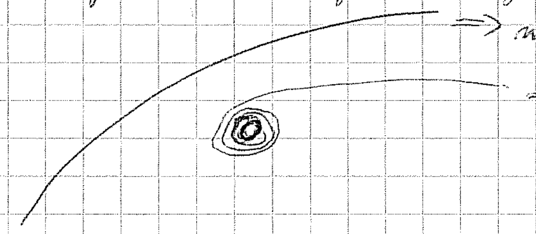
$$ds^2 = 2\,dU\,dV + (x^2 - y^2)\frac{3ml^2}{r(U)^5}dU^2 + dx^2 + dy^2$$

$\Rightarrow$ time dependent Ricci flat metric

Singularities in this metric $\Leftrightarrow A_{ab}(U_0)$ singular $\Leftrightarrow r(U) = 0$

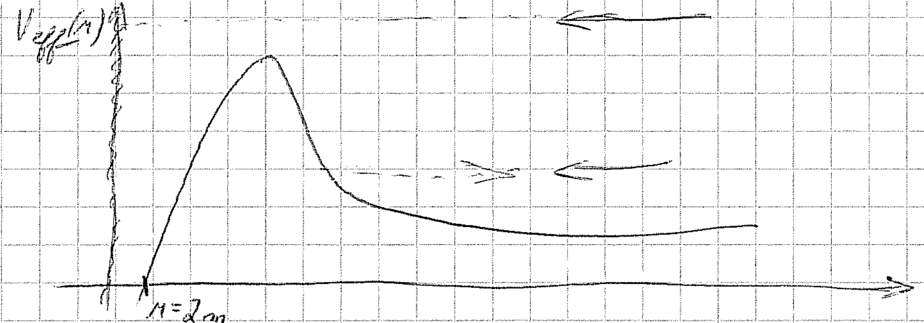
$\Rightarrow$ PL singular $\Leftrightarrow$ the original null geodesic runs into singularity

$\Rightarrow$



$\Rightarrow$ nonsingular PL

$\Rightarrow$ singular PL (singularity is reached in finite value of affine parameter)



$l^2 > 27\, m^2 E^2$ PL smooth
$l^2 < 27\, m^2 E^2$ PL singular

$r(U)$ near $r = 0$, $l \neq 0$: $\dot{r}^2 = E^2 - (1 - \frac{2m}{r})\frac{l^2}{r^2} \Rightarrow$ dominant term $\frac{1}{r^3}$ $\quad \dot{r}^2 \simeq \frac{\sqrt{2m}\,l}{r^{3/2}} \Rightarrow$

$\Rightarrow r(U)^5 \simeq \frac{25}{2}m^2U^2 \Rightarrow$ universal behaviour of PL of Schw. metric near singularity

$$ds^2 = 2\,dU\,dV + (x^2 - y^2)\frac{6}{25}\frac{1}{U^2}dU^2 + dx^2 + dy^2$$

- appears quite often (PL of RW, cosmological strings, ...) - typical near-singularity behaviour

Chern-Simons theory

Mariño 1

QFT in 3d ... 3-manifolds & knots ... Witten 1988

Topological strings

QFT → curvature invariants (Gromov-Witten), CY manifolds

't Hooft $\quad U(N)$ gauge theory ⟷ string theory

$\quad$ CS on M ⟷ X_M ⇒ connection between 2 kinds of topological invariants

Plan:

1. CS theory introduction
2. $1/N$ expansion in CS & 't Hooft relation to strings
3. CS as string theory
4. Applications to enumerative geometry

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1. Chern Simons theory

M, A gauge connection $G = U(N), SU(N)$ (we consider the trivial vector bundle)

$$S = \frac{k}{4\pi}\int_M \left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A\right) \qquad k \in \mathbb{Z} \text{ on quantisation, usually we take } M = S^3$$

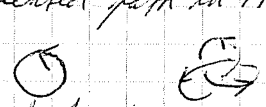
doesn't involve metric explicitly! (Compare YM $F\wedge {*F}$ ← involves metric $*$)

→ topological field theory

on quantisation anomalies → may appear a dependence on metric

Wilson loop operators

$$W_R^k(A) = \mathrm{Tr}_R\, U_k \qquad R \text{ irrep of } U(N) \qquad k \text{ oriented path in } M$$

e.g. 

unknot $\quad$ trefoil knot

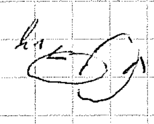
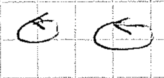
$$U_k = P\exp \oint_k A \quad \text{holonomy above the knot}$$

$\mathrm{Tr}\, U_k$ is gauge independent

Quantisation: generating functional in path-integral formalism etc.

partition function $$Z[M] = \int \mathcal{D}A\, e^{iS(A)} \qquad \text{topological invariant of } M$$

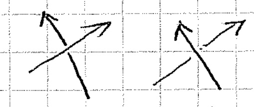
$$\langle W_{R_1}^{k_1}(A) \dots W_{R_L}^{k_L}(A) \rangle = \frac{1}{Z(M)} \int \mathcal{D}A \, e^{iS(A)} \prod_{i=1}^{L} W_{R_i}^{k_i}(A)$$

$\Rightarrow$ things like  and  etc.

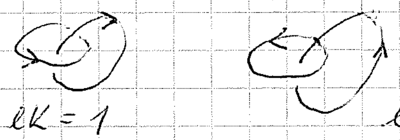
Hopf link — trivial link of 2 components

knot invariant: linking number for 2 loops

$$lK(K_1, K_2) = \frac{1}{2} \sum_{p \text{ crossings}} \epsilon(p)$$

ϵ: 

$+1$ $\quad -1$

$\Rightarrow$ 

$lK = 1 \qquad lK = -1$

Note:  $lK = 0 \Rightarrow$ in this sense topological invariant .. indep. of deformation

Eqns of motion: $\quad \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 0 \qquad F = 0 \qquad F(A) = 0$ flat connection

on S^3 only $A = 0$

M .. flat connections classified by $\pi_1(M) \to G$

e.g. $M = S^3/\mathbb{Z}_p = L(p,1)$ Lens space $\Rightarrow \mathbb{Z}_p \to U(N)$

we expand in path-integral around flat connection $A^{(c)}$

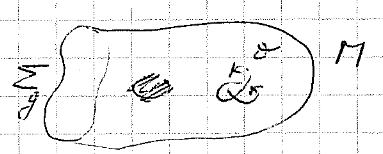
$$\Rightarrow \quad Z^{(c)}(M) = \underbrace{Z^{(c)}_{1\text{-loop}}(M)}\,\exp\Big(\sum_{l=1}^{\infty} S_l^{(c)} \cdot x^l\Big) \qquad x = \frac{2\pi i}{k+N}$$

if $Z^{(c)=\text{trivial}}_{\text{loop}}(M) \propto \frac{1}{\mathrm{vol}(G)}$ $\quad$ | $S_l \quad l = \#\text{vertices}/2$

$l = 1$ $\ominus$ $\qquad l = 2$ $\oplus$ ......

S_l = group factor $\times$ Feynman integral

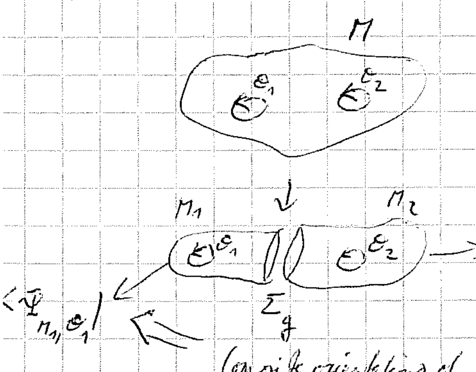
Canonical quantization

$M, \quad \partial M = \Sigma_g$ 

We associate to Σ_g Hilbert space

$$\Psi_{M,\mathcal{O}}(\mathcal{A}) = \int_{A|_\Sigma = \mathcal{A}} \mathcal{D}A \, e^{iS(A)} \mathcal{O} \in \mathcal{H}(\Sigma_g)$$

$$= \langle \mathcal{A} | \Psi_{M,\mathcal{O}} \rangle$$



$$Z(M, \mathcal{O}_1, \mathcal{O}_2) = \int \mathcal{D}A \, e^{iS(A)} \mathcal{O}_1 \mathcal{O}_2$$

$\to |\Psi_{M_2,\mathcal{O}_2}\rangle \in \mathcal{H}(\Sigma_g) \quad \Rightarrow \quad Z(M, \mathcal{O}_1, \mathcal{O}_2) = \langle \Psi_{M_1,\mathcal{O}_1} | \Psi_{M_2,\mathcal{O}_2} \rangle$

$\langle \Psi_{M_1,\mathcal{O}_1}|$

(opposite orientations of the boundaries) if the boundaries are identified in terms of $f:\Sigma_g\to\Sigma_g \Rightarrow U_f:\mathcal{H}(\Sigma_g)\to\mathcal{H}(\Sigma_g)$ $\boxed{\text{Marino 2}}$

$\Rightarrow \quad \langle \Psi_{M_1,\mathcal{O}_1}|U_f|\Psi_{M_2,\mathcal{O}_2}\rangle = Z(M,\mathcal{O}_1,\mathcal{O}_2)$

Witten result:

$\mathcal{H}(\Sigma_g)$ = space of conformal blocks of a 2D CFT (WZW) for group G and level k

e.g. $\mathcal{H}(S^2)$ = 1-dim Hilbert space

$\mathcal{H}(T^2)$ = states labeled by integrable representations of affine Lie algebra $\hat{G}$ at level k

R ... Young tableaux
or
Λ highest weight

E.g. $SU(2)$ $\boxed{\;\;\;\;\;}$ $j=0,1,2,\dots$ int. reps. at level $k \Rightarrow$ $j=0,1,\dots,k-1$

$\therefore \mathcal{H}(T^2)=\{|R\rangle\}$ we shall usually neglect the level k condition (assume k large enough)

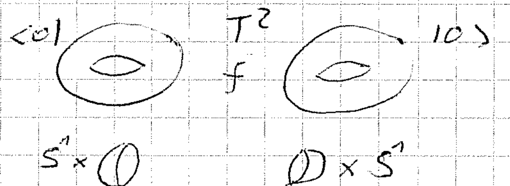
3-dim manifold with T^2 boundary ... solid 2-torus

$D\times S^1$

only 1 non-contractible cycle $\odot$ $\mathcal{O}=\mathrm{Tr}_R U_\odot$

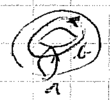
$\Rightarrow |R\rangle \to \Psi_{D\times S^1,\mathcal{O}}=\int \mathcal{D}A\, e^{iS(A)}\,\mathrm{Tr}_R U_\odot$

$\langle R'|R\rangle=\delta_{RR'}$ possible normalisation

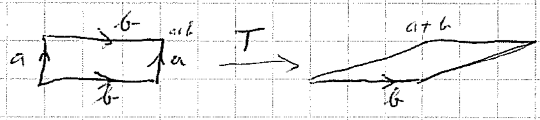


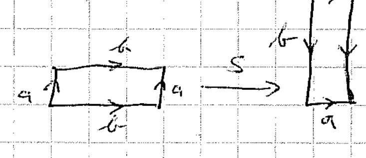
gluing $\to S^1\times \bigcirc = S^1\times S^2 \Rightarrow Z(S^1\times S^2)=\langle 0|0\rangle=1$

Ex: $f:T^2\to T^2$ $\quad SL(2,\mathbb{Z})$ $\quad T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $\quad S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$



$\begin{pmatrix}a\\b\end{pmatrix}$ $\quad T\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a+b\\b\end{pmatrix}$ $\quad S\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-b\\a\end{pmatrix}$





$\Rightarrow T,S:\mathcal{H}(T^2)\to\mathcal{H}(T^2)$

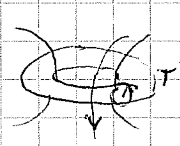
$T|R\rangle = e^{2\pi i(h_R-c/24)}|R\rangle$ $\qquad h_R=\frac{c_R}{2(k+N)}$ $\quad c_R$: quadratic Casimir of R

$c=\frac{k(N^2-1)}{k+N}$ central charge (for $SU(N)$)

$S|R\rangle=\sum_{R'} S_{R'R}|R'\rangle$ $\qquad S_{R'R}=\frac{1}{(k+N)^{N/2}}\sum_{w\in W}\exp\left(-\frac{2\pi i}{k+N}(\Lambda_R+\rho)\, w(\Lambda_{R'}+\rho)\right)\epsilon(w)$

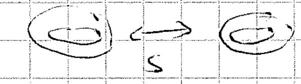
ρ .. Weyl vector $\rho = \sum \lambda_i$ $\qquad$ $\Lambda_R, \Lambda_{R'}$ highest weights

S^3 ... $\mathbb{R}^3 \cup \{\infty\}$ 1-point compactification



$\Rightarrow$ S^3 as union of 2 tori and their common boundary

... a contractible loop on $\partial T^2 = S^1 \times D \leftrightarrow$ non-contractible loop on $\partial \tilde{T}^2$



$$Z(S^3) = S_{00} = \langle 0|S|0\rangle$$

$$= \frac{1}{(k+N)^{N/2}} \sum_{w \in W} \epsilon(w)\, e^{-\frac{2\pi i}{k+N} \rho\, w(\rho)}$$

$\Rightarrow$ non-perturbative result (i.e. obtained without perturbation expansion but may be obtained that way)

$$\langle 0|S|R\rangle = Z(S^3, \mathrm{Tr}_R U) = S_{0R}$$

$$\langle W_R^{\bigcirc} \rangle = \frac{S_{0R}}{S_{00}} \quad \text{quantum dimension of } R$$

similarly $\quad \langle W_R^{lk=-1} \rangle = \frac{S_{RR'}}{S_{00}} \qquad \langle W_R^{lk=1} \rangle = \frac{S^{-1}_{RR'}}{S_{00}} = \frac{S_{\bar{R}R'}}{S_{00}}$

since

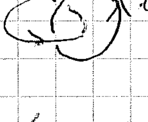
$$\mathrm{Tr}_R U_{R^{-1}} = \mathrm{Tr}_R U_R^{-1} = \mathrm{Tr}_{\bar{R}} U_R$$

Deformation of knots $\Rightarrow$ $\langle .. \rangle$ invariant

R = fundamental $\quad \langle \mathrm{Tr}_R U_R \rangle_{S^3} = \frac{S_{0R}}{S_{00}} = \frac{\lambda^{\frac12} - \lambda^{-\frac12}}{q^{\frac12} - q^{-\frac12}} \qquad q = e^{\frac{2\pi i}{k+N}}, \ \lambda = q^N$

factorizes ($\bigcirc \xleftrightarrow{\text{distance} \to \infty} \bigcirc$) also a more precise argument exists

$$\langle \mathrm{Tr}_R U_{k_1} \mathrm{Tr}_R U_{k_2} \rangle = \langle \mathrm{Tr}_R U_{k_1} \rangle \langle \mathrm{Tr}_R U_{k_2} \rangle = \left(\frac{\lambda^{\frac12} - \lambda^{-\frac12}}{q^{\frac12} - q^{-\frac12}} \right)^2$$



$lk = -1$

$$\langle \mathrm{Tr}_R U_{k_1} \mathrm{Tr}_R U_{k_2} \rangle = \left(\frac{\lambda^{\frac12} - \lambda^{-\frac12}}{q^{\frac12} - q^{-\frac12}} \right)^2 + \lambda - 1 \longrightarrow \text{distinguishes these knots}$$

$\langle .. \rangle$ are always rational functions of $q^{\pm 1/2}, \lambda^{\pm 1/2}$

$$k \longrightarrow \frac{\langle W_R^k \rangle}{\langle W_R^{\bigcirc} \rangle} = \text{polynomial in } q^{\pm\frac12}, \lambda^{\pm\frac12}$$

if $G = SU(2)$, R = fundam. ... Jones polynomial $P_R(q)$, $\lambda = q^2$

$G = SU(N)$, R = fundam. HOMFLY polynomial

$1/N$ expansion

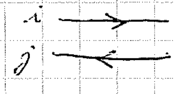
$$Z(M) = Z_{1\text{-loop}} \exp\left\{ \sum_{\ell=1}^{\infty} S_\ell\, x^\ell \right\}$$

$\ell + 1$ loops, 2ℓ vertices

Group factors $\quad l=1 \quad \ominus \quad N(N^2-1)$

$\qquad\qquad\quad l=2 \quad \ominus \quad N^2(N^2-1)$

We shall reorganize the perturbative expansion

$A_{ij} \Leftarrow U(N) = N \wedge \bar{N}$ $\Rightarrow$  diagram

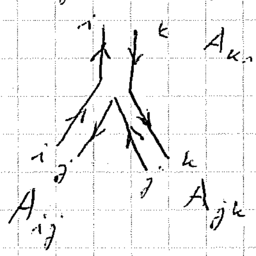
fundam. antifund. repr.

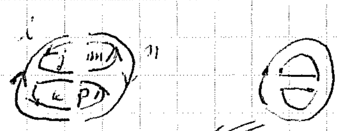
kinetic term

$x = \frac{2\pi i}{k+N}$ $\qquad \frac{1}{x}\,\mathrm{Tr}(A \wedge dA) \longrightarrow \frac{1}{x} A_{ij} \wedge dA_{ji} = \frac{1}{x} A_{ij}\, d A_{kl}\, \delta_{il}\, \delta_{jk}$

$\Rightarrow \langle A_{ij}\, A_{kl} \rangle \propto x\, \delta_{il}\, \delta_{jk}$ $\qquad \langle A_{ij} A_{kl} \rangle = \overline{A_{ij} A_{kl}}$

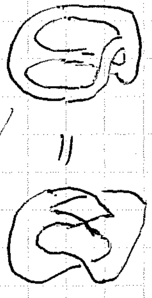
$\frac{1}{x}\,\mathrm{Tr}(A \wedge A \wedge A) = \frac{1}{x} \sum_{ijk} A_{ij} \wedge A_{jk} \wedge A_{ki}$



$\ominus \longrightarrow$ 

$(x^0 N^3 \sim)$ $\qquad \langle \sum_{ijkmnp} A_{ij}\, A_{jk}\, A_{ki}\, A_{mn}\, A_{np}\, A_{pm} \rangle$

$= \sum_{ijkmnp} \delta_{in}\, \delta_{jn}\, \delta_{pn}\, \delta_{kp}\, \delta_{ik} = N^3$

$(x^0 N \sim)$  $\qquad \langle \sum_{ijkmnp} A_{ij}\, A_{jk}\, A_{ki}\, A_{mn}\, A_{np}\, A_{pm} \rangle = \sum \delta_{in}\, \delta_{jm}\, \delta_{ip}\, \delta_{kn}\, \delta_{kn}\, \delta_{ip} = N$

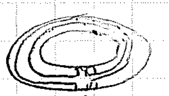
$\Rightarrow$ double lines ... topologically dif. diagrams $\rightarrow$ different N

hole $\rightarrow N$.

every propagator $\rightarrow x$.

every vertex $\rightarrow \frac{1}{x}$.

$\Big\} \Rightarrow$ diagram $\sim x^{E-V} N^h \underset{\wedge}{=} x^{2g-2+h} N^h$

non-planar, can be drawn on torus  similarly other diagrams $\Rightarrow 2g-2 = E-V-h$

e.g.  $\quad 3-2-3 \Rightarrow g=0$

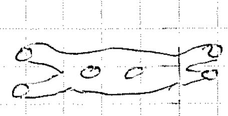
$\qquad 3-2-1 \Rightarrow g=1$

$$F = \log Z(M) = \log Z_{1\text{-loop}} + \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h}\, x^{2g-2+h} N^h$$

symmetry factors $\int_M dx \int_M dy\, \phi(x-y)$

$t = xN \Rightarrow F = \log Z_{1\text{-loop}} + \sum \sum F\, x^{2g-2}\, t^h \qquad t$ .. t'Hooft parameter

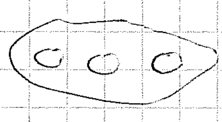
Λ: in string theory

 $\rightarrow N^h\, g_s^{2g-2+h} \Rightarrow$ there might be a correspondence of string and field theory

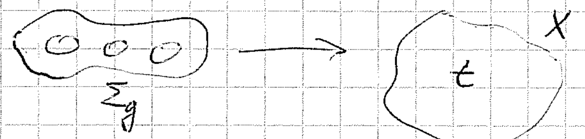
– open string interpretation (1-loops of open strings)

$$x = t/N$$

$$\sum_{g=0}^{\infty} x^{2g-2} \underbrace{\left(\sum_{h=1}^{\infty} F_{g,h} t^h \right)}_{F_g(t)} = \sum_{g=0}^{\infty} x^{2g-2} F_g(t) = \sum_{g=0}^{\infty} N^{-(2g-2)} F_g(t)$$



String theory



t somehow characterizes the geometry of X

$$= N^2 \underset{\sim}{F_0}(t) + F_1(t) + F_2(t) N^{-2} + \dots$$

the leading contribution from genus $= 0$ surface in $N \to \infty$ limit

but we must sum over all possible holes!

$F_0(t) =$  $t^3 +$  $t^4 + \dots =$  (t)

$F_1(t) =$  $t +$  $t^2 + \dots =$  (t)

$\sim$ closed string theory

Computation of $F_{g,h}$ for Chern-Simons theory

$$Z(S^3) = \frac{1}{(k+N)^{N/2}} \prod_{\alpha > 0} 2 \sin\left(\frac{\pi (\alpha \cdot \rho)}{k+N} \right) \qquad \alpha = e_k - e_l \quad k < l$$

$$= \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left(2 \sin \frac{\pi j}{k+N} \right)^{N-j} \qquad \rho = \frac{1}{2} \sum (N - 2k + 1) e_k$$

$$F(S^3) = \log Z = -\frac{N}{2} \log(N+k) + \sum_{j=1}^{N-1} \log \left[2 \sin \frac{\pi j}{k+N} \right] = F^{1\text{-loop}} + F^{\text{vacuum bubbles}}$$

$$F^{1\text{-loop}} = \log \frac{(2\pi g_s)^{\frac{1}{2} N^2}}{\mathrm{vol}(U(N))} \qquad F^{\text{vacuum bubbles}} = \sum_{j=1}^{N-1} (N-j) \sum_{n=1}^{\infty} \log \left(1 - \frac{j^2 g_s^2}{4\pi^2 n^2} \right)$$

previously $F = \log Z_{1\text{-loop}} + \log Z^{\text{vacuum bubbles}}$

where $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right)$, $g_s \equiv x$

$$\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} \, g_s^{2g-2+h} N^h$$

$\to \dots \Rightarrow$

$$g = 0 \qquad F_{0,h} = \frac{\overbrace{B_{h-2}}^{\text{Bernoulli number}}}{(h-2)\, h!}, \quad h \geq 4 \qquad \text{etc.}$$

$$F_0(t) = \sum_{h=4}^{\infty} F_{0,h} t^h$$

Toy:

$U(1)$ CS $\qquad S = \frac{k}{4\pi}\int_M A\wedge dA$

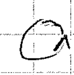
$$\langle e^{\oint_{k_1}A}\, e^{\oint_{k_2}A}\rangle = e^{\left(\langle \oint_{k_1}A\oint_{k_2}A\rangle + \langle\oint_{k_2}A\oint_{k_2}A\rangle + \langle\oint_{k_1}A\oint_{k_2}A\rangle\right)}$$

$$\downarrow$$

$$\frac{1}{4\pi}\oint_{k_1}dx^\mu\oint_{k_2}dy^\nu\,\varepsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3} = \phi(k_1)$$

~~Gausscross~~ self-crossing number

Note: one shall in computing linking number not consider crossings on the same rope, otherwise we have

e.g.  0 vs.  1 ? $\to$ not topological invariant.

$\Rightarrow$ when integrating in $\langle\ \ \rangle$ we shall put instead of the same knot twice we put two copies of the knot - number by integers .. # of mutual twists

$\Rightarrow$ Chern-Simons theory is not exactly a theory of knots but a theory of „framed knots"

On S^3 $\exists$ standard, canonical framing $\to$ on S^3 O.K., otherwise $\langle W_R^{k}(A)\rangle(p_k)$

change of framing $p\to p+n \Rightarrow$ a phase factor appears in front of $\langle\ldots\rangle$

Summarization:

CS theory $\qquad S = \frac{k}{4\pi}\int_M A\wedge dA + \frac{2}{3}A\wedge A\wedge A$

canonical quantization $\to$ explicit $\mathcal{H}(\Sigma_g)$, $Z(S^3)$ to all orders

General idea $\qquad$ gauge theory $U(N)$ $\underset{1/N \text{ expansion}}{\longleftrightarrow}$ string theory

$$F = \sum F_{g,h}\, x^{2g-2}t^h \quad\Longrightarrow F_g(t) = \sum_{h=1}^{\infty}F_{g,h}t^h$$

x coupling constant $\xrightarrow{2\pi/k+N}$ g_{YM}^2

$t = Nx$

Connection with string theory

1) CS is TFT $\longleftrightarrow$ should be: topological string

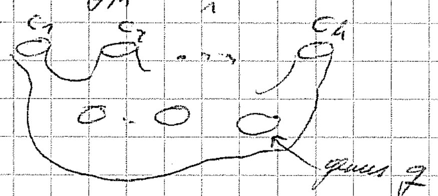
1st we find open string interpretation for $F_{g,h}$ (Witten hep-th/9207094)

target T^*M

Closed topological string (A-model) twisted susy theory with top. gravity

$$f: \Sigma_g \underset{\text{holomorphic}}{\longrightarrow} X \quad CY\ 3\text{-fold}$$

Open topological string $\quad f: \Sigma_{g,h} \longrightarrow X \qquad \partial\Sigma_{g,h} = \bigcup_i C_i$



boundary cond. $f(\partial\Sigma_{g,h}) \overset{!}{=} \mathcal{L} \subset X$

fixed Lagrangian submanifold of X of dim 3

$(CY\ 3\text{-fold} \Rightarrow J : J|_{\mathcal{L}} = 0)$

$i \overset{N}{\circ}$ origin $\sim\sim\sim \overset{\bar{N}}{\circ} j$ end $\quad$ Chan-Paton factors

"quarks" on the ends of oriented string (i.e. transf. in N resp. $\bar{N}$ repr. of $U(N)$

$\Rightarrow$ after quantization $\Rightarrow \quad |0\rangle \to |i,\bar{j}\rangle$ adjoint $(N\times\bar{N})$ of $U(N)$

usually $\quad \alpha^{\mu}_{-1}|0\rangle \Longleftrightarrow A^{\mu} \quad U(1)$ gauge field

now $\quad \alpha^{\mu}_{-1}|i,\bar{j}\rangle \Longleftrightarrow A^{\mu}_{i\bar{j}} \quad U(N)$ — " —

We put Chan-Paton factors on Top. op. string $\Rightarrow \underbrace{N \text{ D-branes wrapping } \mathcal{L}}$

("topological D-branes")

in path-integral $\quad e^{-L} \longrightarrow e^{-L} \prod_{i=1}^{h = \#\text{ of holes}} \operatorname{Tr} P\exp \oint_{C_i} f^{*}(A)$

$\Rightarrow$ boundary of open strings $\sim$ Wilson loops

$T^*M \qquad M$ real 3-manifold $\quad$ local coords $q_a, p_a \quad \Rightarrow J = \sum_{a=1}^{3} dp_a \wedge dq_a$

= Kähler structure for certain complex structure

is Ricci flat $\Rightarrow$ noncompact CY

$T^*S^3 \quad$ deformed conifold

$$\sum_{\mu=1}^{4} y_{\mu}^2 = a \in \mathbb{R}, \qquad y_{\mu} = x_{\mu} + i v_{\mu} \Rightarrow \sum_{\mu=1}^{4} x_{\mu}^2 - v_{\mu}^2 = a,\ \sum_{\mu=1}^{4} x_{\mu} v_{\mu} = 0$$

$\Rightarrow a = (\text{radius})^2$ of a sphere

$a \to 0 \Rightarrow$ singular CY manifold - conifold

M - Lagrangian submanifold of T^*M

String theory $\underset{X}{} \overset{?}{\longrightarrow} S(\phi)$ spacetime dynamics (action) ... string field theory

Φ = tachyon $T(p)|p\rangle + A_{\mu}(p)\,\alpha^{\mu}_{-1}|p\rangle + \dots$ = functional of the string modes

for $\underset{\text{open}}{\wedge}$ bosonic string theory $\Rightarrow Q_{BRST} = \dots\ ), \quad S(\phi) = \frac{1}{g_s}\int\left(\frac{1}{2}\Phi \star Q_{BRST}\Phi + \frac{1}{3}\Phi\star\Phi\star\Phi\right)$

$\star$ makes 2 string functional into another one

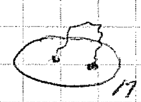
Marino 5

Q

$T_{\mu\nu} = \{Q, b_{\mu\nu}\}$ ... Q-exact $\quad \left(T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} \Rightarrow Z = \int D\phi\, e^{S(\phi)} \Rightarrow \right.$

needed to prove the topological invariance $\quad \left. \Rightarrow \frac{\delta Z}{\delta g_{\mu\nu}} = \int D\phi\, e^{S(\phi)} \{Q, b_{\mu\nu}\} = 0 \right)$

On T^*M: on M and open strings



Parameter $\ast t$. in its limit $\to \infty$ strings extended into T^*M become ∞-massive $\Rightarrow$ decouple

$\Rightarrow$ using topol. invariance we reduce to point particles

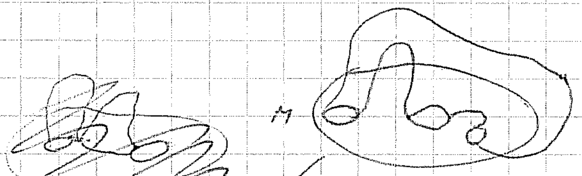
$$\Rightarrow \Phi = A_\mu^{ij}(x)\, dx^\mu \quad \text{on } M$$

$$\cancel{\times} \longrightarrow \wedge$$

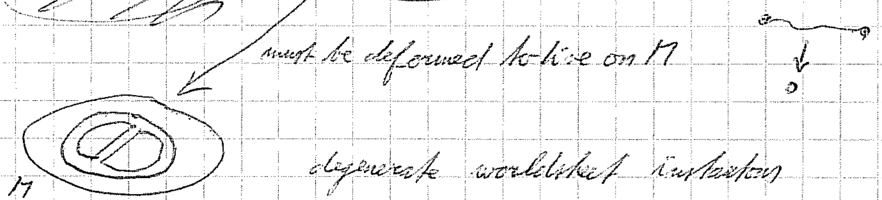
$$\int \longrightarrow \int_M$$

$\Rightarrow \boxed{S(\text{open string topological strings on } T^*M) = S_{CS}(A) \text{ with } g_s = \frac{2\pi}{k+N}}$

topological strings ~ instanton



must be deformed to live on M

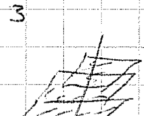


degenerate worldsheet instanton

Closed string interpretation

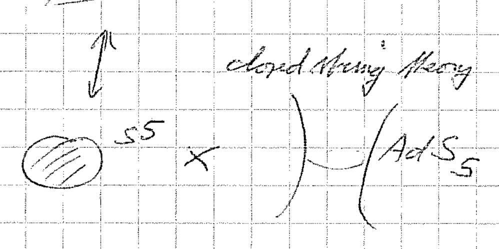
AdS / CFT correspondence

II B on $\mathbb{R}^{1,9}$ $\quad$ N D3-branes on $\mathbb{R}^{1,3}$ $\quad$ because of D-branes $\sim$ open string theory $\sim$ SYM$_4$



$\updownarrow$

II B on $AdS_5 \times S^5$ with no D-branes $\quad$ closed string theory



We have

open topological strings on T^*S^3 $\quad$ N D-branes

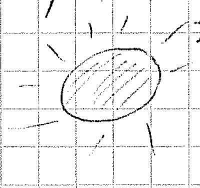
$=$ deformed conifold

$(\leftrightarrow$ CS on $S^3)$ $\quad \updownarrow ?$ $\quad \downarrow ?$

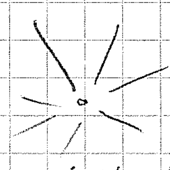
closed top. string on X with no D-branes $\quad$ X resolved conifold

Conifold transition

T^*S^3 $\xleftarrow{\text{deformation}}$ $\xrightarrow{\text{blow up}}$

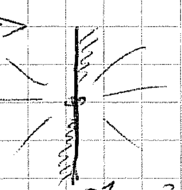


$$\sum_{\mu=1}^{4} y_\mu^2 = a \qquad a \to 0$$



conifold $\sum_{\mu=1}^{4} y_\mu^2 = 0$

(singular CY)



$\mathbb{P}^1 = S^2$

$t = t'$ Hooft coupling $=$ area of $\mathbb{P}^1$
$\qquad\quad \overset{\shortparallel}{Nx}$

$x = y_1 + i y_2 \quad y = \dots \quad u = \dots \quad v = \dots$

$\Rightarrow \; xy = uv + a$ $\qquad$ singularity $\boxed{xy = uv}$

let's put $\quad x = \lambda v \quad y = \lambda u \quad u = \lambda y$

λ inhomog. coord on $\mathbb{P}^1$ or S^2

also we may consider it as coord. transf.

$$(\lambda, x, u) \in \mathbb{C} \times \mathbb{C}^2 \Longleftrightarrow (\lambda^{-1}, v, y) \in \mathbb{C} \times \mathbb{C}^2$$

$\Rightarrow$ bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$

Yet another picture of resolved conifold

$z_1, z_2, z_3, z_4 \in \mathbb{C}^4 \qquad U(1)$ action $\quad z_1 z_4 \to e^{i\theta} z_1 z_4$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z_2 z_3 \to e^{-i\theta} z_2 z_3$

$$\mathcal{M} = \{ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = t \} / U(1)$$

$$|z_1|^2 + |z_4|^2 = t \,/ U(1) \Rightarrow \mathbb{P}^1 \qquad (\lambda = z_1/z_4)$$

$\mathbb{P}^1$ has area t $= t'$Hooft parameter of CS

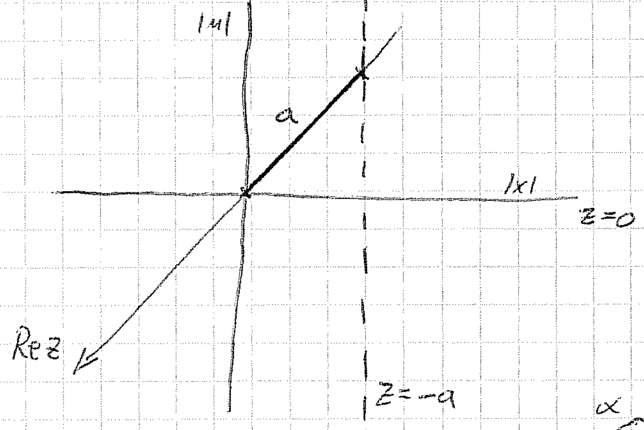
$z \equiv uv \qquad uv = z \qquad xy = a + z \quad \Rightarrow \quad T^2 \; (=U(1)_a \times U(1)_b)$ symmetry

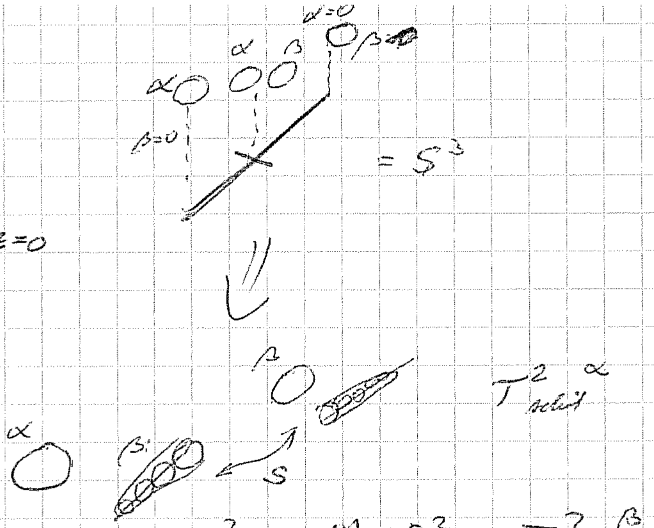
$x, y \to e^{i\theta_a} x, e^{-i\theta_a} y \qquad u, v \to e^{i\theta_b} u, e^{-i\theta_b} v$

fix $z, \theta_a, \theta_b \quad \Rightarrow \quad |u|\,|v| = |z| \qquad \theta_u + \theta_v = \theta_z$
$\qquad\qquad\qquad\qquad\qquad |x|\,|y| = |a+z| \qquad \theta_x + \theta_y = \theta_{a+z}$

$\Rightarrow \quad T^2 \times \mathbb{R} \to T^* S^3$
$\qquad\quad \downarrow \qquad\qquad \searrow z = uv$
$\qquad\; \operatorname{Im} z \qquad\qquad \mathbb{R}^3$
$\qquad\qquad\qquad\qquad (\operatorname{Re} z, |x|, |u|)$

$x = y = 0 \Rightarrow \theta_a$ collapses $\qquad\qquad u = v = 0 \quad \theta_b$ collapses

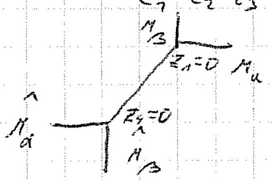




solid D^2_β $\qquad S^1\times D^2_\beta = T^{2\ \beta}_{solid}$

$\Rightarrow$ S^3 as two glued solid tori

Resolved conifold as $T^2\times\mathbb{R}$ fibration of $\mathbb{R}^3$:



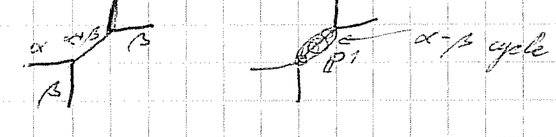
$$\mu_\alpha = |z_2|^2-|z_1|^2 \qquad \mu_\beta = |z_3|^2-|z_1|^2$$
$$\hat\mu_\alpha = |z_4|^2-|z_2|^2-t \qquad \hat\mu_\beta = |z_4|^2-|z_3|^2-t$$

in plane $\mathrm{Im}(z_1,z_2,z_3,z_4)=0$

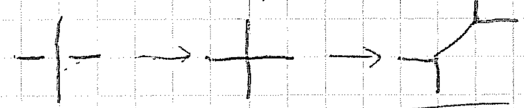
$$T^2 = U(1)_\alpha\times U(1)_\beta \qquad z_1\to e^{i\alpha}z_1 \qquad z_2\to e^{i\beta}z_2 \qquad z_3\to e^{-i(\alpha+\beta)}z_3$$

$T^2\times\mathbb{R}$: $\quad T^2=(\alpha,\beta) \qquad \mathbb{R}\ ...\ t$

α,β degenerate on:



$\Rightarrow$ conifold transition can be represented



Back to 't Hooft duality

$$\sum_{n=1}^{\infty} F_{g,n}\, t^n = F_g(t) = \frac{|B_{2g}|}{2g(2g-2)}\,\mathrm{Li}_{3-2g}(e^{-it}) \qquad \mathrm{Li}_k(x)=\sum_{d=1}^{\infty}\frac{x^d}{d^k}$$

$$t=\frac{2\pi N}{k+N}$$

$g\geq 2$ $\quad \| \quad (\ldots)$ see before notes, involves ζ, $\binom{?}{?}$ etc.

Full free energy $(it\to t)$

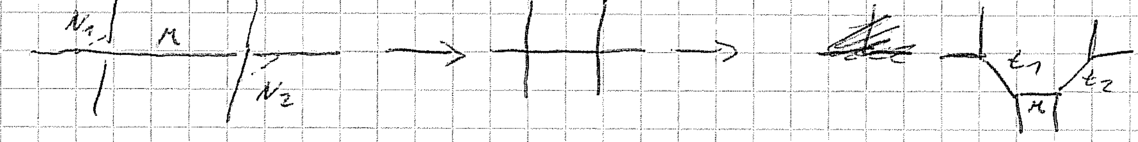
$$F(g_s) = \sum_{g=0}^{\infty} F_g(t)\, g_s^{2g-2} = \sum_{d=1}^{\infty}\sum_{\beta\in H^2(X)}\sum_{g=0}^{\infty}\frac{1}{d}\,\frac{n^\beta_g\, e^{-\beta d t}}{\left(2\sin\frac{g_s d}{2}\right)^{2-2g}}$$

$$n^1_{g=0}=1 \qquad n^\beta_{g=0}=0 \ \ \forall \beta>1$$

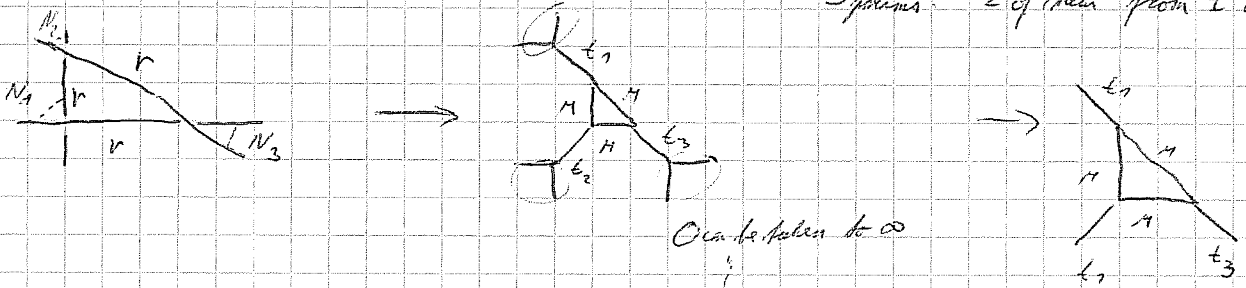
$\Rightarrow$ for the resolved conifold $\quad F(g_s)=\sum_{d=1}^{\infty}\dfrac{e^{-dt}}{2\sin\frac{g_s d}{2}}$

~~Toric of fibrations~~

Graphs of transitions of more complex geometries
(toric of fibrations & points where it degenerates)



3 params: 2 of them from t'Hooft duality

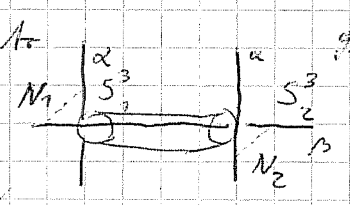


$O(3)\to \mathbb{P}^2$

O can be taken to ∞

$N_1 \to \infty$ (since $t_i = N_i g_s$)

Back to

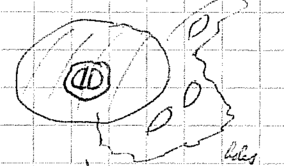


g_s, N_1, N_2 parameters

not T^*M ?

$$g_s = \frac{2\pi}{k_1+N_1} = \frac{2\pi}{k_2+N_2}$$

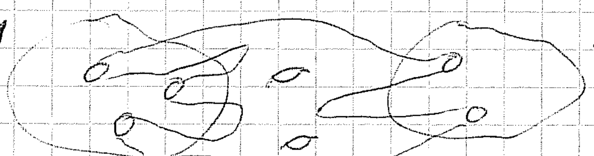
$M \subset X = T^*M$



if $X \neq T^*M$ $\Rightarrow$ the worldsheet instantons might not collapse to M

$$S(A) = S_{CS}(A) + \sum_{\text{free instantons}} \prod_{i=1}^{\text{holes}} \mathrm{Tr}\, P\exp \oint_{C_i} A$$

Even more general:



$$S = S_{CS}^{N_1}(A_1) + S_{CS}^{N_2}(A_2) + \sum_{d=1}^{\infty} \frac{1}{d} e^{-Md}\, \mathrm{Tr}_{\gamma_1} U_1^d \, \mathrm{Tr}_{\gamma_2} U_2^{-d}$$

we don't know (boundaries of annulus)

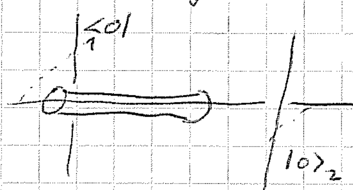
$$Z(g_s, N_1, N_2, M) = \int \mathcal{D}A_1 \mathcal{D}A_2\, e^{-S}$$

$$\exp\left(\sum_{d=1}^{\infty} \frac{1}{d} e^{-dM}\, \mathrm{Tr}\, U_1^d\, \mathrm{Tr}\, U_2^{d}\right) = \sum_R \mathrm{Tr}_R U_1\, e^{-l_R\cdot M}\, \mathrm{Tr}_R U_2$$

$$\Rightarrow Z = \int \mathcal{D}A_1 \mathcal{D}A_2\, e^{-(S_1(A_1)+S_2(A_2))} \sum_R \mathrm{Tr}_R U_1\, e^{-l_R\cdot M}\, \mathrm{Tr}_R U_2^{-1}$$

$$= \sum_R {}_1\langle 0|S|R\rangle_1\, e^{-l_R\cdot M}\, {}_2\langle R|S|0\rangle_2$$

We make two slices



$$\mathcal{H}_1(T^2) \otimes \mathcal{H}_2^*(T^2)$$
$$\ni$$
$$\sum_R |R\rangle_1\, e^{-l_R\cdot M}\, {}_2\langle R|$$

$$= {}_1\langle 0|S|0\rangle_1 \sum_R \frac{S_{0R}^{(1)}}{S_{00}}\, e^{-l_R\cdot M}\, \frac{S_{0R}^{(2)}}{S_{00}}$$

We can compute full partition function of CS on CY 3-folds!

after subst. of S_{ij} we get terms like $\frac{(1-e^{-t_1})(1-e^{-t_2})}{(2\sin\frac{g_s}{2})^2}$

also $F = F(S^3_1) + F(S^3_2) + \log\left\{\sum_R e^{-l_R M} \frac{S_{0R}^{(1)}}{S_{00}} \frac{S_{0R}^{(2)}}{S_{00}}\right\}$

P. Aspinwall

Complex Geometry

– on complex numbers

differential geometry + algebraic geometry

Differential Geometry (General Relativity)

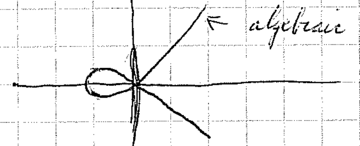
- manifold = locally $\mathbb{R}^n$, patches, transition functions C^∞ ∞-differentiable
- metric .. locally $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$
  - → Levi-Civita connection → curvature

- vector bundle $\pi : E \to M$ E, M manifolds $\pi^{-1}(x)$ is $\mathbb{R}^k$, $x \in M$
  $\pi \in C^\infty$ k .. rank of bundle
  $\forall\, x_0 \in M : \exists\, U = U^\circ, \varphi_U : \pi^{-1}(U) \cong U \times \mathbb{R}^k$, φ_U diffeomorphism

Algebraic geometry

- graphs of functions

$y^2 = x^3 + x^2$



functions on this ~~set~~ subspace of $\mathbb{R}^2$ .. sheaf ~ vector bundle

Why physicists should use algebraic geometry

a) translate the problem in diff. geometry into alg. geometry – often easier

b) more physical ? (black hole singularities etc.)

Alg. geometry is "easy" above algebraically closed fields .. $\mathbb{C}$, not $\mathbb{R}$

↑ always appear in SUSY ⇒ useful applications in physics complex manifold – manifold with an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$

s.t. $\varphi_\alpha \varphi_\beta^{-1}$ is holomorphic (analytic function of z^i only – no $\bar{z}^i$) on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$, $\forall \alpha, \beta$

E.g. z^2, $\sin(z)$ holomorphic

$Re(z) = \frac{1}{2}(z+\bar{z})$ is not holomorphic

$\frac{1}{z}$ — // — at $z=0$

If $(z^1, z^2, \dots, z^n)$ are coords on $\mathbb{C}^n$, we may put $z^j = x^j + iy^j$ to get a $2n$-dim. real manifold.

E.g. $\mathbb{C}^n$

$$\mathbb{P}^n = \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^*$$

$(\mathbb{C}^* = \mathbb{C} \setminus \{0\})$

$\mathbb{P}^n$ has homogeneous coordinates $[z_0, z_1, \dots, z_n] \cong [\lambda z_0, \lambda z_1, \dots, \lambda z_n]$ $\lambda \in \mathbb{C}^*$

where $(z_0, \dots, z_n) \neq \vec{0}$

$$\varphi_i([z_0, \dots, z_n]) = \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right) \in \mathbb{C}^n$$

$$U_i = \{[z_0, \dots, z_n] \mid z_i \neq 0\}$$

Complex vector bundles

$\pi : E \to M$ $\qquad$ M can be any manifold not necessarily complex

s.t. $\pi^{-1}(x) = \mathbb{C}^k$ $\quad \forall x \in M$

furthermore $\forall x \in M \ \exists U_x = U_x^o \ni x$, $\varphi_{U_x} : \pi^{-1}(U_x) \to U \times \mathbb{C}^k$ diffeomorphism

If U, V admit trivialisations (i.e. $\exists \varphi_U, \varphi_V$) then $g_{UV}(x) = \varphi_U \varphi_V^{-1}$ defines transition function and $g_{UV} : \varphi_V(U \cap V) \to GL(k, \mathbb{C})$

Clearly $g_{UV} g_{VU} = \mathbb{1}$, $g_{UV} g_{VW} g_{WU} = \mathbb{1}$

Conversely given an open cover $\{U_\alpha\}$ of M and C^∞ maps into $GL(k, \mathbb{C})$ satisfying $g_{UV} g_{VU} = \mathbb{1}$, $g_{UW} g_{WV} g_{VU} = \mathbb{1} \Rightarrow \exists$ a unique (up to change of coords) complex bundle with those transition functions.

Aspinwall 2

$(g_{UV}^{-1})^T$ defines the dual bundle E^*

$E \oplus F$ defined by $\begin{pmatrix} g_{UV} & 0 \\ 0 & h_{UV} \end{pmatrix}$ $\quad g_{UV}$ defines E, h_{UV} defines F

rank $E\oplus F$ = rank E + rank F

$E \otimes F$ defined by $g_{UV} \otimes h_{UV}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & \vdots \\ ag & ah & \vdots \\ \cdots & & \cdots \end{pmatrix}$

$\Lambda^n E$ antisymmetrized $\bigotimes_{i=1}^{n} E$

We can define subbundles $F \subset E$ and a quotient bundle E/F

A (global) section σ of a bundle is a "right-inverse" of the bundle map π

$$E \underset{\sigma}{\overset{\pi}{\rightleftarrows}} M \quad \text{and} \quad \pi\sigma = id_M$$

local sections over $U = U^\circ \subset M$

$E \to M$ is a <u>holomorphic vector bundle</u> $\iff$ M is a complex manifold and the transition functions are holomorphic

Equivalently $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$ is an invertible holomorphic map

<u>E.g.</u> tangent bundle

for a real manifold the tangent bundle = "space of directional derivatives" $V^\mu \frac{\partial}{\partial x^\mu}$

basis is $\left\{ \frac{\partial}{\partial x^\mu} \right\}$

for a complex manifold we have a bundle with a basis $\left\{ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j} \right\}$, $j = 1, \dots, n$

let $T_{\mathbb{C}}$ be the complex vector bundle with a basis for the fibres given by $\left\{ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j} \right\}$ ... real rank of $T_{\mathbb{C}}$ is $4n$

The basis $\left\{ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j} \right\}$ may be replaced by $\left\{ \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar z^j} \right\}$ $\quad (z = z^j = x^j + i y^j,\ \bar z^j = x^j - i y^j)$

complex manifold $\Rightarrow$ $\varphi\varphi^{-1}$ mix only $\frac{\partial}{\partial z^j}$ between themselves and $\frac{\partial}{\partial \bar z^j}$ between themselves

holomorphic functions

$\Rightarrow T_{\mathbb{C}} = T \oplus \overline{T}$ $\quad T$ spanned by $\frac{\partial}{\partial z^j}$, $\overline{T}$ spanned by $\frac{\partial}{\partial \bar z^j}$

.. holomorphic tangent bundle $\quad$ .. antiholomorphic tangent bundle

complex rank $T = n$

Cotangent bundle has basis $\{dz^j\}$ $\qquad \frac{\partial}{\partial z^j} dz^k = \delta^k_j \Rightarrow$

T^* holomorphic cotangent bundle, $\bar{T}^*$ antiholomorphic tangent bundle

$(T_{\mathbb{C}}^* = T^* \oplus \bar{T}^*)$

Diff. forms live on $\Lambda^n T_{\mathbb{C}}^* = \bigoplus_{p+q=n} (\Lambda^p T^* \otimes \Lambda^q \bar{T}^*)$

n-forms – sections of $\Lambda^n T_{\mathbb{C}}^*$, sections of $\Lambda^p T^* \otimes \Lambda^q \bar{T}^*$ – (p,q)-forms

E.g. a (2,1) form $\quad a_{ijk}\, dz^i \wedge dz^j \wedge d\bar{z}^k$, usually one writes $a_{ij\bar{k}}\, dz^i \wedge dz^j \wedge dz^{\bar{k}}$

d $\qquad$ if $\varphi = \sum a_{\mu\nu}\, dx^\mu \wedge dx^\nu \Rightarrow d\varphi = \sum \frac{\partial a_{\mu\nu}}{\partial x^\alpha} dx^\alpha \wedge dx^\mu \wedge dx^\nu$

$$\boxed{d = \partial + \bar{\partial}}$$

if $A^{p,q}$ is a space of (p,q)-forms $\quad \partial : A^{p,q} \to A^{p+1,q}$

$\bar{\partial} : A^{p,q} \to A^{p,q+1}$

$\varphi = \sum a_{ij\bar{k}}\, dz^i \wedge dz^j \wedge dz^{\bar{k}}$

$\Rightarrow \partial\varphi = \sum \frac{\partial a_{ij\bar{k}}}{\partial z^l} dz^l \wedge dz^i \wedge dz^j \wedge dz^{\bar{k}}$

$d^2 = 0$ (on real manifold) $\qquad$ If A^p is the space of p-forms $\Rightarrow$ we have a complex

$$0 \xrightarrow{d} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^3 \xrightarrow{d_3} A^4 \to \dots$$

i.e. $\quad d_p d_{p-1} = 0 \qquad$ i.e. $\boxed{\operatorname{Im} d_{p-1} \subset \operatorname{Ker} d_p}$

de Rham cohomology is defined by $\quad H^p_{dR} = \frac{\operatorname{Ker} d_p}{\operatorname{Im} d_{p-1}}$

$\bar{\partial}^2 = 0 \qquad$ Dolbeault cohomology $\quad H^{p,q}_{\bar{\partial}} = \frac{\operatorname{Ker} \bar{\partial}}{\operatorname{Im} \bar{\partial}}$

? $H^p_{dR} \leftrightarrow H^{p,q}_{\bar{\partial}}$? $\qquad$ in general no

metric $\quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad \mu,\nu = 1,..,2n$

in terms of $dz, d\bar{z}$ $\qquad ds^2 = g_{ij}\, dz^i dz^j + g_{i\bar{j}}\, dz^i dz^{\bar{j}} + g_{\bar{i}\bar{j}}\, dz^{\bar{i}} dz^{\bar{j}}$

The metric is Hermitian iff $\quad g_{ij} = g_{\bar{i}\bar{j}} = 0$.

(almost) complex structure: $\quad J^\mu_\nu$

J is a linear map (endomorphism) on the cotangent bundle (of a real manifold)

$dx^\mu \to J^\mu_\nu dx^\nu$ such that $J^2 = -1$, i.e. $J^\mu_\nu J^\nu_\xi = -\delta^\mu_\xi$

$z^k = x^k + i y^k \Rightarrow$ we may define J:

$$J\,dx^k = +dy^k, \quad J\,dy^k = -dx^k \quad \text{i.e.} \quad J\,dz^k = i\,dz^k, \quad J\,d\bar z^k = -i\,d\bar z^k$$

An almost complex structure is a complex structure (i.e. complex manifold) if the Nijenhuis tensor vanishes $N^{\mu}_{\nu\rho} = J^{\lambda}_{\nu}(\partial_\lambda J^{\mu}_{\rho} - \partial_\rho J^{\mu}_{\lambda}) - J^{\lambda}_{\rho}(\partial_\lambda J^{\mu}_{\nu} - \partial_\nu J^{\mu}_{\lambda}) = 0$

$g_{\mu\nu} J^{\mu}_{\rho} J^{\nu}_{\lambda} = g_{\rho\lambda}$ is equivalent to $g_{\mu\nu}$ is Hermitian

$\Rightarrow J_{\mu\nu} \overset{\text{def}}{=} g_{\mu\xi} J^{\xi}_{\nu} = -J_{\nu\mu}$ so $J = \frac{1}{2} J_{\mu\nu}\, dx^\mu \wedge dx^\nu$ is a (real) 2-form

In complex coordinates $J = \frac{i}{2} g_{j\bar k}\, dz^j \wedge d\bar z^{\bar k}$ (1,1)-form

A complex manifold with Hermitian metric is Kähler iff $dJ = 0$. J is called the Kähler form.

If M is Kähler $\Rightarrow H^k_{dR}(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M)$

Let $b_k = \dim H^k_{dR}$ Betti numbers

$h^{p,q} = \dim H^{p,q}_{\bar\partial}$ Hodge numbers $\quad \Rightarrow b_k = \sum_{p+q=k} h^{p,q}$

Also $\overline{H^{p,q}_{\bar\partial}} = H^{q,p}_{\bar\partial}$ i.e. $h^{p,q} = h^{q,p}$

If M is closed and compact $h^{p,q} = h^{n-p,n-q}$

Hodge diamond

$$\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
\vdots & & \vdots & & \vdots \\
h^{n,0} & \cdots & & \cdots & h^{0,n} \\
 & \ddots & & \iddots & \\
 & & h^{n,n} & &
\end{array}$$

is symmetrical w.r.t. reflection along horizontal and vertical axis

~~Shfs~~ Sheaves

Let X be a topological space (collection of open sets), connected.

A ~~sheaves~~ sheaf $\mathcal{F}$ on X does the following

a) associates an abelian group $\mathcal{F}(U)$ to any open set $U \subset X$.

b) if $U \subset V$, $U = U^\circ$, $V = V^\circ \Rightarrow \exists$ a restriction map $r_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$ group homomorphism

c) if $U \subset V \subset W$ then $r_{WU} \cong r_{VU}\, r_{WV}$, so we denote $r_{VU}(\sigma)$ by $\sigma|_U$ for $\sigma \in \mathcal{F}(V)$

a)+b)+c) pre-sheaf

d) if $U, V \subset X$ and $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that $\sigma|_{U\cap V} = \tau|_{U\cap V}$ then $\exists \varsigma \in \mathcal{F}(U\cup V)$ with $\varsigma|_U = \sigma$, $\varsigma|_V = \tau$.

e) if $\sigma \in \mathcal{F}(U\cup V)$ and $\sigma|_U = \sigma|_V = 0 \Rightarrow \sigma = 0$ (0 = unit of abelian group)

Ex: $\mathbb{Z}$ associates $\mathbb{Z}$ to every open set, $r_{VU} = \mathbb{1}$

.. constant sheaf

similarly $\mathbb{Q}, \mathbb{R}, \mathbb{C}, G$ constant sheaves
G abelian group

If X is a manifold, we have $C^\infty: C^\infty(U)$ = differentiable functions on U

$\mathcal{A}^p: \mathcal{A}^p(U)$ p-forms on U

If X is a complex manifold $\mathcal{O}$ holomorphic functions

$\mathcal{O}^*$ nowhere zero holomorphic functions

$\mathcal{A}^{p,q}$ (p,q)-forms

$\mathcal{Z}^{p,q}_{\bar\partial}$ $\bar\partial$-closed (p,q) forms

$\Omega^p = \mathcal{Z}^{p,0}_{\bar\partial}$ "holomorphic p-form"

Any vector bundle has a sheaf of sections, of holomorphic sections

Čech cohomology (sheaf cohomology)
incorrect name, another one $\exists$

Let $\underline{U} = \{U_\alpha\}$ be a cover of X. Define

$$\check{C}^0(\underline{U}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha)$$

$$\check{C}^1(\underline{U}, \mathcal{F}) = \prod_{\alpha\neq\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

$$\vdots$$

$$\check{C}^p(\underline{U}, \mathcal{F}) = \prod_{\substack{\alpha_0,\dots,\alpha_p \\ \alpha_i \neq \alpha_j\ \forall i,j}} \mathcal{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_p})$$

Coboundary $\delta : \check{C}^p(\underline{U}, \mathcal{F}) \longrightarrow \check{C}^{p+1}(\underline{U}, \mathcal{F})$: $(\delta\sigma)_{i_0 i_1 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0 i_1 \dots \hat{i}_j \dots i_{p+1}}\Big|_{U_{i_0}\cap U_{i_1}\cap\dots\cap U_{i_{p+1}}}$

it follows that $\delta^2 = 0$ $\quad 0 \longrightarrow \check{C}^1 \xrightarrow{\delta} \check{C}^2 \xrightarrow{\delta} \check{C}^3 \longrightarrow \dots$

$$H(\overline{U}, \mathcal{F}) = \frac{\operatorname{Ker} \delta_p}{\operatorname{Im} \delta_{p-1}}$$ is Čech cohomology

How to get rid of a concrete cover $\underline{U}$ ... $\exists$ limit of refinements

... stationary point ... Čech cohomology $H(X,\mathcal{F})$ is the limit of $H(\underline{U},\mathcal{F})$ for a "very refined open cover"

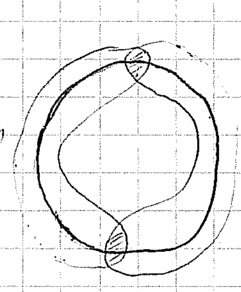
De Rham theorem $$H^p_{dR}(M) = H^p(M,\mathbb{R})$$

Dolbeault theorem $$H^{p,q}_{\bar\partial}(M) = H^q(M,\Omega^p)$$

singular and simplicial cohomology $= H^p(M,\mathbb{Z})$

For any reasonable topological space and sheaf $H^p(X,\mathcal{F})$ is non-zero only for $0 \le p \le n = \dim X$

__Ex:__ $H(S^1,\mathbb{Z})$

U_1  U_2

Let $\sigma \in \check{C}^0$

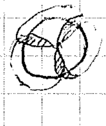
i.e. let $\sigma_1 = a,\ \sigma_2 = b \in \mathbb{Z}$

$$(\delta\sigma)_{12} = \sigma_2\big|_{U_1\cap U_2} - \sigma_1\big|_{U_1\cap U_2} = \sigma_2 - \sigma_1 = b-a$$

$\Rightarrow$ we have
$$\begin{array}{ccc} \check{C}^0 & \xrightarrow{\delta} & \check{C}^1 \\ \| & & \| \\ \mathbb{Z}^2 & \xrightarrow{b-a} & \mathbb{Z}^1 \end{array} \Rightarrow \delta \text{ is surjective}$$

$$\check{C}^2 = 0 \Rightarrow \quad \check{H}^1 = \frac{\mathbb{Z}}{\mathbb{Z}} = 0 \qquad \left(\frac{\operatorname{Ker}\delta^j}{\operatorname{Im}\delta^{j-1}}\right)$$
$$\check{H}^0 = \frac{\operatorname{Ker}\delta}{0} = \mathbb{Z}$$

$\Rightarrow$ not the right Betti numbers $\Rightarrow$ we need refinement



$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}^3 & \to & \mathbb{Z}^3 & \to & 0 \\ & & \| & & \| & & \| \\ & & \check{C}^0 & & \check{C}^1 & & \check{C}^2 \end{array}$$

$\sigma_1 = a,\ \sigma_2 = b,\ \sigma_3 = c$

$(\delta\sigma)_{12} = b-a,\ (\delta\sigma)_{23} = c-a,\ (\delta\sigma)_{13} = a-c$

$$\check{H}^1 = \frac{\mathbb{Z}^3}{\substack{\text{image given} \\ \text{by } b-a,\ c-a, \\ a-c}} \cong \frac{\mathbb{Z}^3}{a+b+c=0} = \mathbb{Z}, \quad \check{H}^0 = \mathbb{Z}$$

doesn't change with further refinement

$\underline{\underline{\mathbb{P}^n}}$ a complex manifold

admits a natural hermitian metric "Fubini - Study"

let $U \subset \mathbb{P}^n$ be an open set with $Z: U \to \mathbb{C}^{n+1}$ a lifting, i.e.

$$\begin{tikzcd} & \mathbb{C}^{n+1}\setminus\{0\} \arrow[d, "\text{quotient map}"] \\ U \arrow[ur, "Z"] \arrow[r, "\text{inclusion}"'] & \mathbb{P} \end{tikzcd}$$

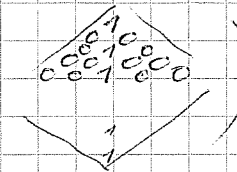
commutes

Z holomorphic map

put $\boxed{J = \frac{i}{2\pi}\partial\bar\partial \log |Z|^2}$ a $(1,1)$-form ... a Kähler form note: other choice of $z \to$ a holomorphic function of original $z \Rightarrow$ get killed by $\bar\partial \Rightarrow J$ well defined

<u>E.g.</u> $J = \frac{i}{2\pi} \partial\bar\partial \log\left(1 + \sum_{j=1}^{n} |z^j|^2\right)$ $\quad z^j = \frac{Z_j}{Z_0}$ for $Z_0 \neq 0$

J^r defines a (r,r)-form for $r \leq n$

These J^r span all the cohomology of $\mathbb{P}^n$ $\Big(\Rightarrow$  $\Big)$ (Proof using topology, cellular cohomology etc.)

e.g. $H^q(\mathbb{P}^n, \Omega^p) = \mathbb{C}^{\delta_{p,q}}$ for $0 \leq p \leq n$

in particular $H^q(\mathbb{P}^n, \mathcal{O}) = \mathbb{C}$ if $q = 0$
$\qquad\qquad\qquad\qquad = 0$ if $q \neq 0$

<u>Exact sequences</u>

$$\longrightarrow A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2}$$

a complex $d_n d_{n-1} = 0$ $\qquad$ exact $\operatorname{Ker} d_n = \operatorname{Im} d_{n-1}$

Thus

$A \xrightarrow{f} B \to 0 \quad \Leftrightarrow \quad B = \operatorname{Im} f \quad$ f surjective

$0 \to A \xrightarrow{f} B \quad \Leftrightarrow \quad \operatorname{Ker} f = 0 \quad$ f injective

$0 \to A \xrightarrow{f} B \to 0 \quad \Leftrightarrow \quad A \cong B$

<u>Short exact sequence</u>

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \Leftrightarrow \quad C \cong B/A$$

Suppose we have a short exact sequence of sheaves over the same topological space X

$$0 \to \mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G} \longrightarrow 0$$

homomorphisms of groups for each $U \subset X$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{C}^0(\mathcal{E}) & \longrightarrow & \check{C}^0(\mathcal{F}) & \longrightarrow & \check{C}^0(\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{C}^1(\mathcal{E}) & \longrightarrow & \check{C}^1(\mathcal{F}) & \longrightarrow & \check{C}^1(\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{C}^2(\mathcal{E}) & \longrightarrow & \check{C}^2(\mathcal{F}) & \longrightarrow & \check{C}^2(\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

rows are exact (check easy)

diagram commutes

the columns are in general not exact !

<u>Snake lemma</u>

$$0 \to \check{H}^0(\mathcal{E}) \to \check{H}^0(\mathcal{F}) \to \check{H}^0(\mathcal{G}) \xrightarrow{\delta} \check{H}^1(\mathcal{E}) \to \check{H}^1(\mathcal{F}) \to \check{H}^1(\mathcal{G}) \to \check{H}^2(\mathcal{E}) \to \dots$$

is exact

constructed in the following way: $\check{C}^k(\mathcal{G}) \Rightarrow$ (surjective) $\exists$ its preimage in $\check{C}^k(\mathcal{F})$, we go ~~back~~ down to $\check{C}^{k+1}(\mathcal{F})$ and then to $\check{C}^{k+1}(\mathcal{E})$ (use injective). One should check that it is well defined

$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \dots \to 0 \Rightarrow$ spectral sequence (a bookkeeping method)

Holomorphic line bundle on $\mathbb{P}^n$ — specified by the transition functions - elements of $\check{C}^1(\mathbb{P}^n, \mathcal{O}^*)$

complex — rank 1

nonvanishing holomorphic function, group operation - multiplication pointwise

consistency requires $g_{\nu\nu} g_{\upsilon\nu} g_{\mu\nu} = 1$ on triple overlaps $\Leftrightarrow \delta\sigma = 0$

If $\sigma = \delta\tau$ then σ can be trivialised as a bundle by a change of coordinates

$\Rightarrow$ Line bundles are classified by the $H^1(\mathbb{P}, \mathcal{O}^*)$

$\Rightarrow$ Line bundles form a group under $\otimes$ .. "Picard Group"

$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow[\exp(2\pi i\cdot)]{} \mathcal{O}^* \to 0$ is a short exact sequence of sheaves

$\Rightarrow$

$$\to H^1(\mathbb{P}^n, \mathbb{Z}) \to \overset{0}{\overset{\|}{H^1(\mathbb{P}^n, \mathcal{O})}} \to H^1(\mathbb{P}^n, \mathcal{O}^*) \to H^2(\mathbb{P}^n, \mathbb{Z}) \to \underset{0}{\underset{\|}{H^2(\mathbb{P}^n, \mathcal{O})}} \to$$

$$\Rightarrow \quad 0 \to H^1(\mathbb{P}^n, \mathcal{O}^*) \to H^2(\mathbb{P}^n, \mathbb{Z}) \to 0$$

$$\Rightarrow \quad H^1(\mathbb{P}^n, \mathcal{O}^*) \cong \underset{"c_1"}{\mathbb{Z}}$$

"c_1" 1st Chern class of the bundle

Let H be the line bundle corresponding to $1 \in \mathbb{Z}$ so all line bundles are isomorphic to $H^{\otimes n}$ or $(H^*)^{\otimes n}$.

Let the sheaf of holomorphic sections of $H^{\otimes n}$ be denoted by $\mathcal{O}(n)$, $\mathcal{O}(0) = \mathcal{O}$

H is the "hyperplane" bundle — admits a global section that vanishes to first order along a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ E.g. $Z^0 = 0$

H^* admits no global section

A global holomorphic section of $H^{\otimes k}$ is an element σ of $\check{C}^0(\mathbb{P}^n, \mathcal{O}(k))$ (as a union of sections above patches) such that $\delta\sigma = 0 \Rightarrow [\sigma] = 0$ global sections correspond to $H^0(\mathbb{P}^n, \mathcal{O}(k))$

compatibility on overlaps

E.g. global sections of H are given by $H^0(\mathbb{P}^n, \mathcal{O}(1))$. Such sections are also sheaf maps

$\mathcal{O} \xrightarrow{f} \mathcal{O}(1)$

If $a \in \mathcal{O}(U)$ then $fa \in \mathcal{O}(1)(U)$
(a holomorphic function over U)

By definition of H f is a linear function of $Z_0, Z_1, \dots, Z_n$ so $H^0(\mathbb{P}^n, \mathcal{O}(1)) = \mathbb{C}^{n+1}$

Similarly

$H^0(\mathbb{P}^n, \mathcal{O}(k)) =$ ~~the~~ set of deg k polynomials in $(n+1)$ variables

$= \mathbb{C}^{\dots}, k>0 \; (= 0 \text{ if } k<0)$

$H^r(\mathbb{P}^n, \mathcal{O}(k)) = 0$ for $r \neq 0 \text{ and } \neq n, \forall k$ without proof (see Kodaira Vanishing Theorem)

$H^n(\mathbb{P}^n, \mathcal{O}(k)) = H^0(\mathbb{P}^n, \mathcal{O}(-k-n-1))$ (see Serre duality)

<u>Tangent sheaf</u>

holomorphic tangent bundle T for $\mathbb{P}^n$

$T(\mathbb{C}^{n+1})$ has a basis $\frac{\partial}{\partial Z_i}$ $i = 0, \dots, n$

i.e. a vector field is $f^i(Z) \frac{\partial}{\partial Z_i}$

For $\mathbb{P}^n$ we insist that f^i are linear $\Rightarrow H^{\oplus(n+1)} \longrightarrow T \longrightarrow 0$

$\Rightarrow \quad \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow \mathcal{T} \longrightarrow 0$ as sheafs

Let x^i be affine coords for $z_0 \neq 0$ $\quad x^i = \frac{Z_i}{Z_0} \quad \Rightarrow \quad \frac{\partial}{\partial z_0} = -\frac{Z_i}{(Z_0)^2} \frac{\partial}{\partial x_i}, \; \frac{\partial}{\partial Z_i} = \frac{1}{Z_0} \frac{\partial}{\partial x_i}$

$\Rightarrow \sum_i Z_i \frac{\partial}{\partial Z_i} = 0 \quad \Rightarrow$ map above is not injective, the kernel is $f \sum_i Z_i \frac{\partial}{\partial Z_i} = 0$

$0 \rightarrow \mathcal{O} \xrightarrow{Z_i} \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow \mathcal{T} \longrightarrow 0$ short exact sequence

Let's define $\mathcal{T}(k) = \mathcal{T} \otimes \mathcal{O}(k)$

<u>Algebraic variety</u> (projective)

Let $f(Z_i)$ be a homogeneous function of $\mathbb{P}^n$ $\quad f(\lambda Z_i) = \lambda^d f(Z_i)$

Then $f = 0$ defines a hypersurface X in $\mathbb{P}^n$. Implicit function theorem implies that X is a manifold if $f = df = 0$ has no solution.

We can have several homogeneous functions f_j and find $(f_1 = 0) \cap (f_2 = 0) \cap \dots \cap (f_n = 0)$ .. algebraic variety

If $f = df_1 \wedge df_2 \wedge df_3 \wedge \dots = 0$ has no solution on $X \Rightarrow X$ is a manifold and is called a "complete intersection".

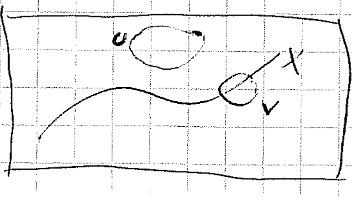
But $\mathbb{P}^1 \times \mathbb{P}^2$

$[a,b]$ $\quad [c,d,e]$ $\quad x_1 = ac, \; x_2 = ad, \; x_3 = ae, \; x_4 = bc, \; x_5 = bd, \; x_6 = be$ $\quad x_i$'s give homogeneous coords of $\mathbb{P}^5$

$$\mathbb{P}^1 \times \mathbb{P}^2 = \{ [x_1, \dots, x_6] \in \mathbb{P}^5 \mid \underbrace{x_1 x_5 = x_2 x_4, \; x_1 x = \dots, \; x_1 x = \dots}_{3 \text{ eqns., independent}} \}$$

$\Rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is a manifold, i.e. smooth, but not a complete section

<u>structure sheaf</u> $\mathcal{O}_X$



$\mathbb{P}^n \quad U \subset \mathbb{P}^n$ $\quad U \rightarrow \{0\}$

$V \rightarrow$ functions on $V \cap X$

$\Rightarrow$ not a fibre bundle disguised as a sheaf $\mathcal{O}^V$

$\to \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$

Aspinwall 6

we assume f is of degree d

since $f \neq 0$ wherever I am outside X, $\mathcal{O} \xrightarrow{f} \mathcal{O}(d)$

If $X = (f_1 = 0) \cap (f_2 = 0)$

$\to \mathcal{O}(-d_1-d_2) \xrightarrow{(f_2, -f_1)} \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \xrightarrow{(f_1, f_2)} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$

The Kähler form on $\mathbb{P}^n$ induces (by restriction) a Kähler form on any algebraic variety

vice versa

Kodaira Embedding Theorem:

If X is a Kähler manifold with $J \in H^2(X, \mathbb{Q})$ then there is an embedding $X \subset \mathbb{P}^N$.

take basis of $H^2(X, \mathbb{R})$ such that integrals over cycles are integers, then J is decomposable into this basis with integer coefficients

Ex: Calabi-Yau manifolds compact complex and Kähler n-fold (sometimes impose $b_1 = 0$) with any of the following equivalent conditions

a) admits a Ricci-flat metric (i.e. $SU(n)$-holonomy) & $c_1(T_X)$ is torsion free

b) $c_1(T_X) = 0$ (1st Chern class)

c) $K = \Omega^n$ (canonical line bundle (sheaf)) $= \mathcal{O}_X$ (is trivial)
highest exterior power of holomorphic cotangent sheaf

d) $H^n(X, \mathcal{O}_X) = \mathbb{C}$

Let f have degree d in $\mathbb{P}^n$, $d > 0$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

previously unknown

H^0	$0 \longrightarrow$	$\mathbb{C} \longrightarrow$	$\Rightarrow \mathbb{C}$
H^1	$0 \longrightarrow$	$0 \longrightarrow$	$\Rightarrow ?$
H^2	$0 \longrightarrow$	$0 \longrightarrow$	$?$
$\vdots$		$0 \longrightarrow$	$\Rightarrow \mathbb{C}^{\binom{d-1}{n+1}}$
H^{n+1}	$\mathbb{C}^{\binom{d-1}{n+1}} \longrightarrow$	$0 \longrightarrow$	$\Rightarrow ?$

so $H^n(X, \mathcal{O}_X) = \mathbb{C}^{\binom{d-1}{n+1}}$

$\Rightarrow$ X is Calabi-Yau manifold if $d-1 = n+1$, i.e. $d = n+2$

e.g.

n	d	Geometry
1	3	2-torus (elliptic curve) ($b_1 = 2 \Rightarrow$ may not be depending on def. of CY)
2	4	K3-surface
3	5	Quintic 3-fold

More f's:

If we have k equations of degree $d_1, d_2, \dots, d_k$ in $\mathbb{P}^{n+k}$ then we have a CY complete intersection only if $d_1 + d_2 + \dots + d_k = n+k+1$

If $n=1$ we always get 2-torus

$n=2$ 4-torus, K3-surface nothing else

$n=3$ 10000s known different topologies, no classification, not even known whether the number is finite

---

Adjunction formula

Let X be a hypersurface in $W \equiv \mathbb{P}^n$ defined by f which is a section of the line bundle $L \to W$ (i.e. if f is of degree d, L is associated to $\mathcal{O}(d)$) then

$$0 \to T_X \to T_W|_X \to \underset{\underset{L|_X}{\parallel}}{N} \to 0$$ is a short exact sequence of bundles on X

N.. normal bundle

In sheaf language $0 \to \mathcal{T}_X \to \mathcal{T}_{W|X} \to \mathcal{O}(d)|_X \to 0$.

Note: on a CY manifold $\Lambda^n T^*$ is trivial $\Rightarrow \Lambda^d T_X$ is isomorphic to $\Lambda^{n-d}(T_X^*)$.

Calculation of Hodge numbers $h^{1,2} = \dim H^2(X, \Omega^1)$ $\qquad \mathcal{T}_X = \Omega^2$ (since $\Omega^3 = \mathcal{O}$)

$$0 \to \mathcal{T}_X \to \mathcal{T}_{W|X} \to \mathcal{O}(5)|_X \to 0$$

1) Find cohomology of $\mathcal{T}_W$ $\Leftarrow$ $0 \to \mathcal{O}_W \to \mathcal{O}_W(1)^{\oplus 5} \to \mathcal{T}_W \to 0$

$$\begin{array}{cccccc}
H^0 & \mathbb{C} & \to & \mathbb{C}^{25} & \to \Rightarrow & \mathbb{C}^{24} \\
H^1 & 0 & \to & 0 & \to & 0 \\
H^2 & 0 & \to & 0 & \to & 0 \\
H^3 & 0 & \to & 0 & \to & 0 \\
H^4 & 0 & \to & 0 & \to & 0
\end{array}$$

$\Rightarrow H^*(W, \mathcal{T}_W) = \mathbb{C}^{24}, 0, 0, 0, 0$

$$0 \to \mathcal{T}_W(-5) \xrightarrow{f} \mathcal{T}_W \to \mathcal{T}_{W|X} \to 0 \qquad\qquad 0 \to \mathcal{O}_W(-5) \to \mathcal{O}_W(-4)^{\oplus 5} \to \mathcal{T}_W(-5) \to 0$$

$$\begin{array}{ccc}
0 & \mathbb{C}^{24} & \mathbb{C}^{24} \\
0 & 0 & 0 \\
0 & 0 & \mathbb{C} \\
\mathbb{C} & 0 & 0 \\
0 & 0 & 0
\end{array}
\qquad \Longleftarrow \qquad
\begin{array}{cccc}
H^0 & 0 & 0 & 0 \\
H^1 & 0 & 0 & 0 \\
H^2 & 0 & 0 & 0 \\
H^3 & 0 & 0 & \mathbb{C} \\
H^4 & \mathbb{C} & 0 & 0
\end{array}$$

$\Rightarrow H^*(W, \mathcal{T}_W(-5)) = 0, 0, 0, \mathbb{C}, 0$

$\Downarrow$

$H^*(W, \mathcal{T}_{W|X}) = H^*(X, \mathcal{T}_{W|X}) = \mathbb{C}^{24}, 0, \mathbb{C}, 0, 0$

$0 \to \mathcal{O} \xrightarrow{f} \mathcal{O}_W(5) \to \mathcal{O}_W(5)|_X \to 0$ $\boxed{\text{Aspinwall 7}}$

$$\begin{array}{ccc} \mathbb{C} & \mathbb{C}^{126} & \mathbb{C}^{125} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \qquad 126 = \binom{9}{4}$$

$\Rightarrow H^*(W, \mathcal{O}(5)|_X) = \mathbb{C}^{125}, 0, 0, 0, 0$

$\Rightarrow$ Finally $0 \to T_X \to T_W|_X \to \mathcal{O}_W(5)|_X \to 0$

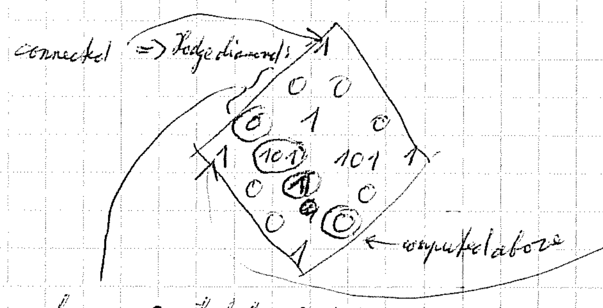
$$\begin{array}{ccccc} ? & \to & \mathbb{C}^{24} & \to & \mathbb{C}^{125} \\ ? & \to & 0 & \to & 0 \\ ? & \to & \mathbb{C} & \to & 0 \\ ? & \to & 0 & \to & 0 \\ ? & \to & 0 & \to & 0 \end{array}$$

$\Rightarrow$ we need exact form of $\mathbb{C}^{24} \to \mathbb{C}^{125}$ .. it is injective

$\Rightarrow \boxed{H^*(X, T_X) = 0, \mathbb{C}^{101}, \mathbb{C}, 0, 0}$

$\dim H^0(X, T_X) = h^{2,0} = 0$ $\dim H^1(X, T_X) = h^{2,1} = 101$

$\dim H^2(X, T_X) = h^{2,2} = 1$ $\dim H^3(X, T_X) = h^{3,2} = 0$



$H^0(X, \Omega^3) = \mathbb{C}$

always $= 0$ if hol $= SU(3)$

the rest by symmetries (,) of the Hodge diamond

<u>Deformations of complex structure</u>

J^μ_ν satisfies $J^\mu_\nu J^\nu_\rho = -\delta^\mu_\rho$ $N^\mu_{\nu\rho} = 0$

in the complex coordinates $J^j_k = i\delta^j_k$ $J^{\bar j}_{\bar k} = -i\delta^{\bar j}_{\bar k}$ $J^j_{\bar k} = J^{\bar j}_k = 0$

$J^\mu_\nu \to J^\mu_\nu + \tau^\mu_\nu$ τ^μ_ν is „small"

$J^2 = -1 \Rightarrow \tau^j_k = \tau^{\bar j}_{\bar k} = 0$ leaving $\tau^j_{\bar k}$ (and its complex conjugate $\tau^{\bar j}_k$)

Let $\tau = \tau^j_{\bar k}\, dz^{\bar k} \frac{\partial}{\partial z^j}$ (0,1)-form valued in T_X

$N^\mu_{\nu\rho} = 0 \Rightarrow \bar\partial \tau = 0$

If $\tau = \bar\partial v$, for v valued in T_X then this corresponds to a reparametrisation $z^j \to z^j + v^j$

$\Rightarrow$ deformations of complex structure correspond to $H^1(X, T_X)$ (to first order)

In general these deformations can be obstructed. For a CY $(3?)_{n\text{ only}?}$-fold the obstructions do not exist.

$\mathbb{C}^{101} = ..$

$\Rightarrow$ 101 different deformations exist

X is defined by f .. a degree 5 polynomial. Such polynomials are sections of $\mathcal{O}_{\mathbb{P}^4}(5)\big|_X$

$$\ldots \mathbb{C}^{125}$$
$$\mathbb{C}^{24} \ldots \quad GL(5,\mathbb{C}) \text{ acting on } z_0, \ldots, z_4 \ \ldots \text{ change of coords.}$$
$$\Rightarrow \mathbb{C}^{125}/\mathbb{C}^{24} = \mathbb{C}^{101}$$

All deformations of complex structure are deformations of $f: z_0 z_1 z_2 z_3 z_4 + z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$

K3-surface Quartic in $\mathbb{P}^3$

$$0 \to T_X \to T_{\mathbb{P}^3}|_X \to \mathcal{O}_{\mathbb{P}^3}(5)\big|_X \to 0$$

$$\begin{array}{ccccc} 0 & \to & \mathbb{C}^{15} & \xrightarrow{\text{injective}} & \mathbb{C}^{34} \\ \mathbb{C}^{20} & \to & \mathbb{C} & \to & 0 \\ 0 & \to & 0 & \to & 0 \\ 0 & \to & 0 & \to & 0 \end{array}$$

$H^1(X, T_X) = \mathbb{C}^{20}$

$\Rightarrow$ 20 indep. deformations

but only 19 deformations can be obtained for quartic defining equation.

There are deformations of K3-surface which stop being embedded in $\mathbb{P}^3$.

Ex: intersect 2 cubics $\mathbb{P}^5$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 73 & & 73 & & 1 \end{array}$$

Calabi-Yau 3-fold

Ex: a bicubic in $\mathbb{P}^2 \times \mathbb{P}^2$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 83 & & 83 & & 1 \end{array}$$

<u>Orbifolds</u>

Take quintic $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$ (Fermat quintic)

admits a $\mathbb{Z}_5$-symmetry $[z_0, \ldots, z_4] \to [z_0, \alpha z_1, \alpha^2 z_2, \ldots, \alpha^4 z_4]$ $\quad \alpha = e^{\frac{2\pi i}{5}}$

.. fixed point free $\Rightarrow$ $X/\mathbb{Z}_5$ is another CY 3-fold
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 41 & & 41 & & 1 \end{array}$$

An ~~diff~~ orbifold is like a manifold except the fact that the patches ~~that~~ look like subsets of $\mathbb{C}^n$ or $\mathbb{C}^n/G$ where $G \subset GL(n,\mathbb{C})$, G discrete, usually finite

Orbifold singularities can sometimes be resolved keeping the CY condition

EX: Consider the bundle corresponding to $\mathcal{O}(-3)$ on $\mathbb{P}^2$. This may be defined as $\{[z_0, z_1, z_2, y) \in \mathbb{C}^4 \mid (z_0, z_1, z_2) \neq (0,0,0)\}$ and

$[z_0, z_1, z_2, y] = [\lambda z_0, \lambda z_1, \lambda z_2, \lambda^{-3} y]$ $\forall \lambda \in \mathbb{C} \setminus \{0\}$

... an example of toric geometry

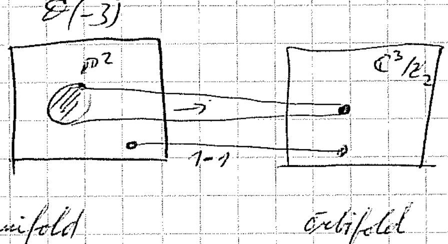
define $x_0 = z_0 y^{1/3}$ $x_1 = z_1 y^{1/3}$ $x_2 = z_2 y^{1/3}$ Aspinwall 8

This gives a map $\pi : \underset{\uparrow}{\mathcal{O}}(-3) \to \mathbb{C}^3/\mathbb{Z}_3$ where $\mathbb{Z}_3$ is generated by $(x_0, x_1, x_2) \to (\omega x_0, \omega x_1, \omega x_2)$

should be the corresp. line bundle $\omega = e^{\frac{2\pi i}{3}}$

$\Rightarrow$ fixed point of π at origin $\Rightarrow$ singularity of the group action

π is 1-1 away from $y=0$, if $y=0$, the whole $\mathbb{P}^2$ (given by $y=0$) maps to the origin of $\mathbb{C}^3/\mathbb{Z}_3$



in general
$I = \mathbb{C}^3/G$ $\Rightarrow$ Then I can blow-up to give a CY 3-fold
$G \subset\subset SU(3)$, G finite

Ex: $(z_0, z_1, z_2) \to (\alpha z_0, \alpha z_1, \alpha^3 z_3)$ $\alpha = e^{\frac{2\pi i}{7}}$ must be blown-up here

---

Let $W = \{(a,b,c,d) \in \mathbb{C}^4 \mid (a,b) \neq (0,0),\ (a,b,c,d) \cong (\lambda a, \lambda b, \lambda^{-1} c, \lambda^{-1} d),\ \forall \lambda \in \mathbb{C}^*\}$

$x = ac$ $y = bd$ $z = ad$ $w = bc$ $\Rightarrow$ $xy = wz$

$\Rightarrow W \to Y$ Y is hypersurface $xy = wz$ in $\mathbb{C}^4$, singular at the origin

-- "conifold" but W is smooth $\Rightarrow$ blow-up of a conifold

a $\mathbb{P}^1 \subset W$ is mapped to origin in Y ... "small resolution"
↑ 1dim object, above 2-dim

$W' = \{(a,b,c,d) \in \mathbb{C}^4 \mid (c,d) \neq (0,0),\ (a,b,c,d) \cong (\lambda a, \lambda b, \lambda^{-1} c, \lambda^{-1} d),\ \forall \lambda \in \mathbb{C}^*\}$

This also resolves the conifold but with a different $\mathbb{P}^1$.

On general algebraic CY variety we can resolve the conifold locally $\Rightarrow$ the global geometry may make W vs. W' resolution different. $W \leftrightarrow W'$ .. flop

Special holonomy manifolds in String theory and M-theory $\boxed{\text{Acharya } 1}$

SUSY – gives reasons why Higgs should have reasonable mass $\lesssim 1\,TeV$ (without it its mass should be of the order of fundamental energy scale – Planck scale)

superstring theory $\underset{d=2}{\underset{\uparrow}{\Sigma}} \to \underset{dim=10}{\underset{\uparrow}{M}}$ $\Rightarrow$ CFTs on Σ – can be reinterpreted in terms of spacetime physics on M

mass scale of string theory $\sim T$ $\Rightarrow$ massless excitations (finite number) + infinite tower of massive states

$T \to \infty \Rightarrow$ we can integrate out massive states $\Rightarrow$ massless states described by local SUSY theory

pre 1995 – five superstring theories (differ by fermionic partners to bosonic degrees of freedom)

Type I, Type IIA, Type IIB, ~~Type~~ Heterotic $E_8 \times E_8$ or $SO(32)$ – M-theory

closed string theories – Σ_g worldsheet Riemann surface of genus g $\Rightarrow$ perturbation ~~theory~~ expansion – summation over ~~R~~ Σ_g

M-theory – out of nonperturbative results, mainly BPS-branes – extended solutions – break Lorentz invariance $SO(9,1) \to SO(9-p) \times SO(p,1)$

~~shorte~~ multiplets of SUSY algebra $(M \propto Q)$ e.g. $M = \frac{Q}{g_s}$

in 11 dimensions, in different limits approximated by 10d-string theories

$\Rightarrow$ lot of results using effective field theory in 10 and 11-dim spacetime

SUGRA 10 dim: g_{ij} metric, $\psi_{i\alpha}$ gravitino, $C_{i_1 \dots i_p}$ p-form fields, λ superpartners of, A gauge fields, χ their superpartners

low energy M-theory $\to$ 11 dim SUGRA essentially unique

$g_{ij}, \psi_{i\alpha}, C_3$ $\qquad G = dC$

$$S = \frac{1}{l_p^9}\int_M \sqrt{g}\, R - \frac{1}{2} G \wedge {*G} - \underbrace{\frac{1}{6} C \wedge G \wedge G}_{\text{topological, i.e. doesn't depend on metric – Chern-Simons term}} + \text{fermionic (gravitino) terms}$$

, l_p – Planck length

eq. of motion C: $d{*G} = G \wedge G$ $\qquad g$: $R_{ij} - \frac{1}{2} g_{ij} R = T_{ij}$

Local Lorentz & local SUSY invariance

SUSY transf. $\delta_\eta(\text{metric}) = f(\eta, \psi_{i\alpha})$ $\quad \delta_\eta C = h(\eta, \psi, g)$

$\eta \in S(M)$

32-component real spinor

$$\delta \psi_{i\alpha} = \underset{\uparrow}{\nabla_i} \eta_\alpha + \frac{1}{288}\left(\Gamma_{i\alpha\beta}{}^{jklm} G_{jklm} - 8\Gamma_{\alpha\beta}{}^{jkl} G_{ijkl}\right)\eta^\beta$$

covariant Levi-Civita derivative

$$\Gamma^{i_1 \dots i_k} = \frac{1}{k!}\Gamma^{[i_1}\Gamma^{i_2}\dots\Gamma^{i_k]}$$

? field configurations invariant w.r.t. SUSY – SUSY field configurations, we also want to preserve some degree of Lorentz symmetry, e.g. $M = X_7 \times \Sigma^{3,1}$

if we want local Lorentz invariance on $\Sigma^{3,1}$, we are forced to set $\psi_{i\alpha} = 0$
$\Rightarrow \delta_\eta g = \delta_\eta C = 0$

but still we have to solve $\delta\psi_{i\alpha} = 0$ $\quad$ 1) $G = 0$, i.e. $C = $ const.

"zero flux" configuration

$\Rightarrow \boxed{\nabla_i \eta_\alpha = 0}$

2) $G \neq 0$ still unclear

$\Rightarrow$ look for 11-d metric g s.t. $\exists$ a spinor field η which is parallel (covariantly constant)

similarly in $\dim = 10$ $\quad \psi = \chi = \lambda = 0$, $A = $ const., $G = 0 \Rightarrow$? manifolds allowing parallel spinors

In curved space generic fields get non-trivially parallel transported. We look for η_α s.t. (when transported along contractible loops) are invariant in <u>curved space</u>

The <u>linear holonomy group</u> of the manifold ... given metric $\rightarrow$ connection $\rightarrow$ parallel transport along contractible loops $\rightarrow$ objects (e.g. vector fields) get multiplied by action of the subgroup of Lorentz group $SO(10,1)$ .. <u>the linear holonomy group</u> $Hol(g)$

definition may depend on objects considered (vector fields, spinor fields) but in our cases will be the same

for generic g $\quad Hol(g) \cong SO(10,1) \Rightarrow$ our g has to have special properties (in order to be SUSY)

We will take the ansatz $\boxed{M = X_n \times \Sigma^{10-n,1}}$ $\quad \boxed{g(M) = g(X) + g_2(\Sigma)}$ typically static spacetime

i.e. $ds^2 = g^{(x)}_{ij} dx^i dx^j + g(y)_{\mu\nu} dy^\mu dy^\nu$

we assume $\quad$ small $\quad$ big, flat (we don't consider cosmology)

$SO(10,1) \rightarrow SO(n) \times SO(10-n,1)$ $\quad \Rightarrow \delta_\eta \chi = \nabla_\mu \chi$, spinor on Σ

$Spin(10,1) \rightarrow Spin(n) \times Spin(10-n,1)$

?.. 32×32 real matrices $\rightarrow S_n \otimes S_{10-n,1} \quad (=(S_n, S_{10-n,1}))$ $\quad$ (S_n, $S_{10-n,1}$ representations of S^n) $\Rightarrow$ $\boxed{\text{What } Hol(g_n(X)) \text{ admit parallel spinors?}}$

$SO(10,1) = SO(n) \times SO(10-n,1)$ [Lecture 2]

$32 = (S_n, S_{10-n,1})$ no trivial repr. ($\mathbb{1}$) on the RHS (in the decomposition)

Ex: $n=3$

$Spin(3) \times Spin(7,1) = SU(2) \times Spin(7,1)$

$32 = (2, 8_+) + (2, 8_-)$ 2 fundamental repr. of $SU(2)$

$Spin(\cancel{even}^{2k}) \Rightarrow 2$ different spinor repr.

S_+, S_- differing by chirality

$\dim S_\pm = 2^{k-1}$

$\gamma_i : \quad \gamma_{i2k} \psi^\pm = \pm\psi^\pm$

$(i = 0,..,2k-1)$

We would like to interpret $SO(n)$ as the holonomy group $\to$ but there is no trivial repr. $\Rightarrow$

no invariant ~~space~~ nontrivial spinors $\Rightarrow$ we must consider subgroups of it

$SO(n)$ is compact $\Rightarrow$ holonomy group will be compact, linear holonomy group its connected component of unity

Ex: $n=3$ $SU(2)$ has only one compact connected subgroup $U(1)$

$SU(2) \to U(1)$

$2 \to 1^\pm$ $\quad 2 \to 1^+ + 1^-$ $\Rightarrow$ conserved because of charges

$\Rightarrow$ no SUSY unless X is flat (R compact & small)

$\Rightarrow$ 3-dim torus $\Rightarrow Hol(g(X))$ trivial $\Rightarrow$ maximum number of parallel spinors

$ds^2_{Torus} = dx_1^2 + dx_2^2 + dx_3^2, \quad x_i \sim x_i + 2\pi R$

$\Leftrightarrow \nabla_i = \partial_i \Rightarrow \underbrace{\text{parallel spinors}}_{\Rightarrow \text{ 32 of them}}$ = constant spinors in such coords

$\Rightarrow$ 32 supersymmetries

$n=4$

$Spin(4) \times Spin(6,1)$

$= (SU(2)_L \times SU(2)_R) \times Spin(6,1) \Rightarrow (2,1,8) + (1,2,8)$

if $Hol(g(x)) = SU(2) \times SU(2) \Rightarrow$ parallel nontrivial spinors ~~doesn't~~ don't exist

<u>but</u> we can make e.g. $SU(2)_R$ trivial $\to SU(2)_L \times 1 \times Spin(6,1)$, $(2,8) + \underbrace{(1,8) + (1,8)}$

should allow parallel spinors (16 of them)

Can one prove: If (X,g) is a compact Riemannian manifold with $SU(2)$ holonomy $\Rightarrow X$ is K3-surface (an example of Calabi-Yau manifold)

$n=4 \Rightarrow g(X_5) = g(X_4) + dy^2, \quad X_5 = \frac{T_4 \times S}{\Gamma}$ ← discrete isometry group

because $Spin(5) \times Spin(5,1)$

$32 = (4, 4^+) + (4, 4^-)$ → we have to reduce $Spin(5)$ to $Spin(4)$ in order to get a chiral representation

$n=5 \quad Spin(6) \times Spin(4,1) = SU(4) \times Spin(4,1)$

$32 = (4,4) + (\bar{4}, 4) \qquad$ (as $Spin(6) \quad (4^+,4) + (4^-,4)$)

$SU(4)$ = unitary $\det = 1$ $\mathbb{C}$ 4×4 matrices

if we want e.g. a_4 invariant $\Rightarrow \begin{pmatrix} & & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{pmatrix} \Rightarrow SU(3)$, $? \ \exists g(x): Hol(g(x)) = SU(3)$?

$(4,4) + (\bar{4},4) \longrightarrow \underbrace{(1,4)}+(3,4)+\underbrace{(1,4)}+(\bar{3},4)$

$4+4=8$ susy, minimal SUGRA in 5 dimensions

$n=7 \quad Spin(7) \times Spin(3,1) = Spin(7) \times SL(2,\mathbb{C}) \qquad SL(2,\mathbb{C}) \quad 2\times2 \ \mathbb{C}$ matrices with $\det$

$32 = (8,2) + (8,\bar{2})$

NB: Spin representations ... rotations of spinors

$\Gamma_i \qquad \{\Gamma_i, \Gamma_j\} = 2g_{ij} \qquad \Gamma_{ij} = \frac{1}{2}\Gamma_i\Gamma_j - \frac{1}{2}\Gamma_j\Gamma_i$ generate infinitesimal rotations of spinors (Lie algebra of $Spin(n)$ .. $\mathcal{L}(Spin(n))$)

$$\mathcal{L}(Spin(n)) \cong \mathcal{L}(SO(n)) \cong \Lambda^2(\mathbb{R}^n)^*$$

$$e^{\alpha^{ij}\Gamma_{ij}} = h(\alpha) \in Spin(n)$$

$\Rightarrow$ we have condition for parallel spinors $\quad e^{\alpha^{ij}\Gamma_{ij}}\eta = \eta \iff \alpha^{ij}\Gamma_{ij}\eta = 0$

$n=7 \Rightarrow \Gamma_i \in Mat(n, \mathbb{C}) \Rightarrow \ ?g \in Spin(7): \begin{pmatrix} & \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_8 \end{pmatrix}$

Γ_i: $\quad \Gamma_\mu = \gamma_\mu \otimes \mathbb{1}_{2\times2}\,\sigma_2, \quad \Gamma_\alpha = \gamma_5\mathbb{1}_{4\times4} \otimes \sigma_\alpha$ (probably, one should check the anticommutators)

$Spin(7) \supset \underset{\gamma_\mu}{Spin(4)} \times \underset{\sigma_\alpha}{Spin(3)} = (SU(2))^3 \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$

$\Rightarrow \alpha^{ij}\Gamma_{ij}\eta = 0$ form 7 independent eqns. (i.e. 7 pairs of indices i,j are interesting) $\Rightarrow \mathcal{L}(Spin(7)) = \mathcal{L}(hol(g(X_7)))$

+ orthogonal complement
$\parallel$
representation of $\mathcal{L}hol(g(X_7))$ ... 7 dimensional

$\Rightarrow \quad 21 = 14 + 7$

$\Rightarrow$ after exponentiation $hol(g(X))$ a compact 14-dim. Lie group .. G_2

$\Rightarrow Spin(8) \to G_2$ maximal possible $hol(g(X))$ allowing parallel spinors .. $N=1$ susy on $\mathbb{R}^{3,1}$
$\quad 8 \to 1+7$

$n=8$ $\quad Spin(8) \times Spin(2,1)$ $\quad 32 = (8^+,2) + (8^-,2)$

we have to reduce to $Spin(7) \times Spin(2,1)$ $\quad 32 = (1,2) + (7,2) + (8,2)$

dim X	$Hol(g(X))$	# SUSY	Notation
8	$Spin(7)$	2	$d=3, N=1$
7	G_2	4	$d=4, N=1$
6	$SU(3)$	8	$d=5, N=1$
5	$SU(2) \times \mathbb{1}$	16	$d=6, N=1$
4	$SU(2)$	16	$d=7, N=1$
$1,2,3=n$	$\mathbb{1}$	32	$d=11-n, N=2, n\geq 1$; $N=1, n=0$

Maximal possible holonomy groups preserving supersymmetry and fulfilling our ansatz.

We may reduce further up to flat case $Hol = \mathbb{1}$

Possible connected subgroups of $Hol(g(X))$

$n=8$ $\quad Spin(6) = SU(4)$ ← $N=2$ $d=3$ SUSY
$\quad Spin(5) = Sp(2)$ ← $N=3$ $d=3$ SUSY
} ⇒ irreducible manifolds – not a local product, groups preserve more SUSYs
$\quad Spin(4) = SU(2) \times SU(2)$ → product of 2 K3-surfaces

$n=7$ $\quad SU(3) \times$ trivial $\quad$ etc. we move down in the list

Note: $SU(K)$ holonomy manifolds $\exists \, \forall K$ .. $n=2K$ $\quad$ Calabi-Yau n-folds

$Sp(K) \Rightarrow n = 4K$ $\quad$ HyperKähler manifolds

$Sp(n) \subset SU(2n) \Rightarrow$ HyperKähler manifolds are special Calabi-Yau manifolds

$SU(2) = Sp(1) \Rightarrow$ K3 is HyperKähler

noncompact cases ⇒ only other possible holonomy groups & manifolds are

$U(K)$ $\quad n=2K$ $\quad$ Kähler $\quad$ ⇒ Calabi-Yau are special Kähler manifolds

$\dfrac{Sp(n)\,Sp(1)}{\mathbb{Z}_2}$ $\quad n=4n$ $\quad$ Quaternionic Kähler

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Eqns of motion $\quad R_{ij} - \frac{1}{2} g_{ij} R = T_{ij} = 0 \leftarrow \quad d*G = \frac{1}{2} G \wedge G \quad \checkmark \;(\Leftarrow G=0)$

$$0 = [\nabla_i, \nabla_j]\eta = \frac{1}{4} R_{ijkl} \underset{\uparrow}{\Gamma^{kl}} \eta \qquad R_{ijkl} = R_{klij},\; R_{ijkl} = -R_{jikl} \text{ etc.}$$

generators of $\mathcal{L}(Spin(n))$

(before we have $\alpha^{ij}\Gamma_{ij}\eta = 0 \Rightarrow$ we have also the eqn $R\Gamma\eta = 0$ without need to commute $[\nabla,\nabla]$

$\Gamma^j \Rightarrow \quad R_{ijkl}\Gamma^{jkl}\eta = 0 \Rightarrow R_{ij}\Gamma^j\eta = 0 \Rightarrow R_{ij} = 0$

$\not\nabla \equiv \Gamma^i \nabla_i \quad \Rightarrow \not\nabla^2 \eta = (\pm)\nabla_i^2 \eta - \underbrace{\frac{1}{4} R\eta}_{=0} \quad \Rightarrow \not\nabla^2 \eta = \nabla_i^2 \eta$

$\Rightarrow \int_X \eta^+ \not\nabla^2 \eta = \int_X \eta^+ \nabla_i^2 \eta \Rightarrow$

$\int_X (\not\nabla \eta, \not\nabla \eta) = \int_X (\nabla_i \eta, \nabla_i \eta) \Rightarrow \not\nabla \eta = 0 \Leftrightarrow \nabla \eta = 0$

$\not\nabla : S^+ \to S^-, \; n = 2k \qquad \not\nabla : S \to S, \; n = 2k+1$

$\mathrm{Ind} \not\nabla$ = # zero modes of $\not\nabla$ − # zero modes of $\not\nabla^+$ $\qquad n = 2k+1 \Rightarrow \mathrm{ind} \not\nabla = 0$

$\Rightarrow$ we can compute the index of $\not\nabla$ on manifolds of interest

$\Rightarrow$ Exercise Compute $\mathrm{Ind}(\not\nabla)$ on $Spin(7)$, $SU(4)$
$Sp(2)$
$Sp(1) \times Sp(1)$
$SO(3)$
$SU(2)$

$SO(n) \quad \Lambda^p(T^*X)$ ... under $SO(n)$ irreps of dim $\binom{n}{p} = \frac{n!}{p!(n-p)!}$

under $Hol(g(X)) \quad = \sum_i R_i \qquad R_i$ irreps of $Hol(g(X))$

any R_i trivial $\to$ covariantly constant p-form

from another point of view $\quad \alpha_{i_1 \dots i_p} = \eta^+ \Gamma_{i_1 \dots i_p} \eta$ defines a ↑ whenever $\nabla_i \eta = 0$

$n=4 \quad SU(2)$ holonomy

irreducible manifolds $\Rightarrow \{dx^i\}$ 1-forms form irreps

---

Hodge star: $\quad SO(n)$ preserves $\varepsilon_{i_1 \dots i_n}$ ... invariant tensor

$\Rightarrow \quad \beta_{i_1 \dots i_p} \longrightarrow \varepsilon_{i_1 \dots i_{n-p}}{}^{i_{n-p+1} \dots i_n} \beta_{i_{n-p+1} \dots i_n} = (*\beta)_{i_1 \dots i_{n-p}}$

$\Rightarrow$

$\Lambda^p \cong \Lambda^{n-p}$, we shall not consider Λ^{n-p} if we already know Λ^p

---

$\Lambda^2(\mathbb{R}^4) \cong \mathcal{L}(SO(4)) = (3,1) + (1,3) \longrightarrow \underset{\sim}{3} + 3 \cdot \mathbb{1}$

$*\alpha = \alpha \qquad *\alpha = -\alpha \qquad \Downarrow$

3 covariantly constant 2-forms $\omega_I, \omega_J, \omega_K$

If we write $\mathbb{R}^4 \cong \underset{z_1}{\mathbb{C}^2} + \underset{z_2}{\mathbb{C}^2}$ then $\omega_I = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$, $\omega_J + i\omega_K = dz_1 \wedge dz_2$

I, J, K generate $\mathcal{L}(SU(2)) \qquad IJ = K$ etc.

$n=6 \quad \Lambda^2(\mathbb{R}^6) = \overset{SO(6)}{15} = \overset{SU(3)}{\underset{\Lambda^{(1,1)}}{1+8} + \underset{\Lambda^{(0,2)}}{3} + \underset{\Lambda^{(2,0)}}{\bar{3}}} \Rightarrow 1$ .. Kähler form

tangent vectors can be decomposed into holom. and antiholomorphic $\quad 6 = 3 + \bar{3}$

$\Rightarrow \quad \Lambda^2(\mathbb{R}^6) = \Lambda^{(1,1)}(\mathbb{R}^6) + \Lambda^{(0,2)}(\mathbb{R}^6) + \Lambda^{(2,0)}(\mathbb{R}^6)$

$SO(6) \to SU(3)$

$\Lambda^3(\mathbb{R}^6) = 20 = \Lambda^{(3,0)} + \Lambda^{(2,1)} + \Lambda^{(1,2)} + \Lambda^{(0,3)}$ Acharya 4

$= 1 + 3\otimes 3 + \bar{3}\otimes\bar{3} + 1$

$\Rightarrow$ 2 covariantly constant 3-form $\Rightarrow$ 1 complex 3-form & its conjugate
$dz_1 \wedge dz_2 \wedge dz_3$

$\underline{n=7}$ $SO(7) \to G_2$

$\Lambda^2(\mathbb{R}^7)$ $21 = 14 + 7$ $\Rightarrow$ no covariantly constant 2-forms

$\Lambda^3(\mathbb{R}^7)$ $35 = 1 + 7 + 27$ $\Rightarrow$ a covariantly constant 3-form φ

G_2 is automorphism group of octonions $\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$

$\underline{n=8}$ $SO(8) \to Spin(7)$

only singlets in $\Lambda^4(\mathbb{R}^8) = 35^+ + 35^- = (1+7+27) + 35$
antiself-dual 4-forms

$\Rightarrow$ 1 covariantly constant 4-form

ik: $\mathbb{R}^7 = \mathbb{C}^3 \times \mathbb{R} \Rightarrow \varphi = Re(e^{i\theta}\,\Omega^{3,0}) + \frac{1}{2}\omega \wedge dy)$

Similarly $\mathbb{R}^8 = \mathbb{R}^7 \times \mathbb{R}_2$ $\Omega = -\varphi \wedge dz + *\varphi$, $(\Rightarrow *\Omega = \Omega)$

Calabi-Yau 2K-fold admits 2 covariantly constant forms ω, $\Omega^{k,0} = dz^1 \wedge \ldots \wedge dz^k$

$\omega = \sum dz^i \wedge d\bar{z}^{\bar{i}}$

Nt: sometimes the character of object is less important than the irrep under which it transforms

e.g. $\mathbb{R}^7$ .. $h_{ij} = h_{ji}$ ... 27 & $\tau_{ijk} \in 27 \subset \Lambda^3(\mathbb{R}^7) \Rightarrow *\tau \in \Lambda^4(\mathbb{R}^7)$

$h_{ij} = (*\tau)_{i}{}^{klm}(*\varphi_{jklm})$

Theorem: Compact ~~irr~~ irreducible manifolds admitting parallel spinors have finite fundamental group $\pi_1(X)$.

$\Longrightarrow H^1(X,\mathbb{Z})$ is torsion $\Rightarrow b^1 = 0$

$b^p = \sum b^p_{R_i}$ $\Rightarrow$ $b^1_7 = 0$ $b^3_7 = 0$

$\Lambda^p = \sum R_i$

$\Rightarrow$ possible solutions of $\nabla\xi = 0$ for ξ any object (form, spinor) have to be in the 1 irrep. of holonomy group

if $\nabla\xi = 0 \Rightarrow d\xi = d*\xi = 0$ harmonic form $\leftrightarrow$ definitions of Calabi-Yau, hyperKähler etc. manifolds

by antisymmetrization / divergence free

$(\Leftrightarrow) \exists$ the appropriate invariant closed and coclosed forms

Given an (X,g) of special holonomy, when does (X,g') have special holonomy.

a) perturbation $g \to g+\delta g = g' \Rightarrow \nabla_g \to \nabla_{g+\delta g}$, $\varsigma_g \to \varsigma_{g+\varepsilon_g}$, look if $\nabla_{g+\delta_g} \varsigma_{g+\delta_g} = 0$

$\Rightarrow$ in certain sense $(X, g+\delta g)$ will have special holonomy $\Leftrightarrow$ $\delta\varsigma$ is harmonic

$$R_{ij}(g)=0 \longrightarrow R_{ij}(g+\delta g) = \underset{\text{Lichnerowicz operator}}{\Delta_L}\,\delta g + \mathcal{O}(\delta g)^2 = 0$$

$$\shortparallel \quad -\nabla^2 \delta g_{ij} - 2R_{imjn}\,\delta g^{mn} + 2R^k{}_{(i}\,\delta g_{j)k} \qquad (???)$$

$$\delta g \equiv h \Rightarrow \text{eqn. for } T_{jklm} \quad \text{"}h = \varsigma T\text{"} \quad \Rightarrow \quad \Delta T = 0$$

one assumes gauge condition, preferably $\nabla^i \delta g_{im} = 0$, — Laplacian in the original metric g

δg is also assumed symmetric traceless — $\Rightarrow$ we add to original harmonic form small harmonic pieces

$\Rightarrow$ in the Calabi–Yau case the moduli space of non-equivalent metrics is

$$H^{(1,1)}(X,\mathbb{R}) + H^{(k-1,1)}(X,\mathbb{C})$$

different ω — different $\Omega^{k,0}$

$$H^2(X) = H^{(1,1)}(X) = H^{(1,1)}_1(X) + H^{(1,1)}_{\varphi}(X)$$

$\omega \to \lambda\omega = \omega'$

G_2-case $\quad H^3(X,\mathbb{R})$

$Spin(7)$ case $\quad H^{4-}_{35}(X,\mathbb{R}) + H^4_1(X,\mathbb{R})$

$$H^4(X) = H^{(k,0)}(X) + \underline{H^{(k-1,1)}(X)} + \dots$$

only this piece contributes to deformations

These groups tend to be nontrivial $\Rightarrow$ zero potential for $g \to g+\delta g$ (.. flat directions) $\Rightarrow$ moduli space of vacua

.. space of vacua, i.e. different classical solns. of equations of motion with $G = \psi = 0$

.. several (or lot of) continuous params., physical quantities may depend on them

... __vacuum degeneracy problem in string theory__

Other fields: $\quad M = \underset{\text{"}x\text{"}}{X_n} \times \underset{\text{"}y\text{"}}{\mathbb{R}^{10-n,1}} \leftarrow$ Minkowski

C-field

Kaluza–Klein harmonic ansatz $\quad C = \sum_{I=1}^{b_3(X)} \alpha^3_I(x)\,\underbrace{\varphi^I(y)}_{\text{scalar on } \mathbb{R}^{10-n,1}} + \sum_{J=1}^{b_2(X)} \beta^2_J(x) \wedge \underbrace{A^J(y)}_{\text{1-form on } \mathbb{R}^{10-n,1}} + \sum_{k=1}^{b_1(X)} \gamma^1_k(x) \wedge B^k(y) + C(y)$

α^3_I basis of harmonic 3-forms
β^2_J — " — 2 "
γ^1_K — " — 1 "

in most cases $G \wedge G = 0 \Rightarrow d\star G = 0 = dG \Rightarrow d\star d\,C = 0 \Rightarrow \varphi, A, B$ must be massless
(quadratic $\Rightarrow$ don't contribute to massless) $d\star G \sim G\wedge G$ — Bianchi

since α, β, γ are zero modes of $d\star d$
e.g. $d\star G = \alpha\, d\star d\varphi = 0 \Rightarrow \boxed{d\star d\varphi = 0}$
$\Rightarrow \varphi$ zero mode $\Rightarrow$ massless

$\Rightarrow \varphi, A, B$ superpartners of variations of metric

$\Rightarrow$ Spectrum of low-energy M-theory compactified on Calabi–Yau $\quad M = X_6 \times \mathbb{R}^{4,1}$

$\delta g \to$ $h^{1,1}+h^{2,1}$: e_a ($a = 1..h^{1,1}$) real, $\Phi_{m=1,..,h^{2,1}}$ complex

$\delta C \to \varphi^{I=1..b_3}$, $A_\mu^{J=1..h^{1,1}}$, $C(y)$
real scalar fields, $b_3(X) = 2+2h^{2,1}$ — gauge fields — 3-form field

no 2-form ($\Leftarrow$ no $\gamma^1(x)$ on Calabi–Yau)

$G(y) = dC(y) \Rightarrow dG(y) = 0 = d\star_5 G(y)$

$\star_5 G(y) \cong \tilde{G}(y) \Rightarrow$ locally $\tilde{G} = d\psi$, ψ scalar

$\Rightarrow$ in fact $\varphi^{I=1..b_3+1}$ $\qquad \Rightarrow$ E–M duality $\quad C(y) \leftrightarrow \psi(y)$

$d=5, N=1$ superfields (repr. of SUSY algebra) $\boxed{\text{Acharya } 5}$

Vector multiplets $\begin{pmatrix} A \\ \varphi \end{pmatrix}$

Hyper multiplets $Q = a_1 + i a_2 + j a_3 + k a_4$

$\Rightarrow$ A^J_μ should combine with e_a to give vector multiplets ($h^{(1,1)}$ of them)

Φ_M — " ——— & φ^I — " — hyper — " — ($h^{(2,1)}$ of them)

$\Rightarrow$ these scalar fields we haven't accounted for yet

Gravity $\Rightarrow$ SUGRA multiplet $g^5_{\mu\nu}$, 3 scalars

$\Rightarrow$ at all $N=1$ $d=5$ SUGRA with $h^{(1,1)}$ vector & $h^{(2,1)}$ hyper multiplets

$\boxed{G_2}$ $\Rightarrow$ $N=1$ $d=4$ SUGRA $b^2(X)$ vector & $b^3(X)$ chiral multiplets ("neutral")

$\boxed{Spin(7)}$ $\Rightarrow$ $N=1$ $d=3$ SUGRA after dualization $b^2(X) + b^3(X) + b^4_{35}(X) + 1$ scalar multiplets

Topics in String theory and Gauge theories

$\mathcal{N}=2$ SUSY Gauge theory $G=U(N)$

... mathematics of equivariant cohomology

$H^*(X)$ X manifold, de Rham complex $\omega = \omega_{i_1 \dots i_k}(x)\, dx^{i_1}\wedge \dots \wedge dx^{i_k}$

$d\omega = 0$ mod $\omega \sim \omega + d\chi$

$d^2=0$ $d\omega = (\partial_j \omega_{i_1 \dots i_k})\, dx^j \wedge dx^{i_1}\wedge \dots \wedge dx^{i_k}$

G symmetry group, acts on X

morally speaking one wants $H^*(X/G)$ ← very often singularities → more refined structure needed

.. G-equivariant cohomology

$\Omega^*(X)$ differential forms on X $\Omega^*(X) = \bigoplus_{k=0}^{\dim X} \Omega^k(X)$

$\mathfrak{g} = \mathrm{Lie}(G)$ = Lie algebra of G

$\Omega^*_G(X) = \{$ functions on $\mathfrak{g}$ with values in $\Omega^*(X)$ | $\underbrace{\omega(g^{-1}\phi g^{+1}) = g^*\omega(\phi)}_{G\text{-equivariance}}$ $\}$

g acts on X $g: X \to X$

$\phi \in \mathfrak{g} \rightarrow \omega(\phi; x)_{i_1 \dots i_k}\, dx^{i_1}\wedge \dots \wedge dx^{i_k}$

$d \to D = d_{p}$ + contraction of $V(\phi)$ with diff. form

odd in de Rham of X

$\forall \phi \in \mathfrak{g}\ \exists$ a vector field $V(\phi) \in \mathrm{Vect}(X)$

$V(\phi) = V(\phi)^\mu \frac{\partial}{\partial x^\mu}$

$\Rightarrow V(\phi)+V(\psi) = V(\phi+\psi)$

$[V(\phi), V(\psi)] = V([\phi,\psi])$

i.e. $V(\phi)^\mu \partial_\mu V(\psi)^\nu - V(\psi)^\mu \partial_\mu V(\phi)^\nu = V([\phi,\psi])^\nu$

$$\boxed{\begin{aligned}(D\omega(\phi;x)_{i_1\dots i_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}) &= (\partial_j \omega(\phi;x)_{i_1\dots i_p})\, dx^j \wedge dx^{i_1}\wedge\dots\wedge dx^{i_p} \\ &+ V(\phi)^{i_1}\omega(\phi;X)_{i_1 i_2 \dots i_p}\, dx^{i_2}\wedge\dots\wedge dx^{i_p}\end{aligned}}$$

$\Rightarrow$ a combination of $p+1$ & $p-1$ form on X, we define

if $\omega(\phi; X)$ is a polynomial in ϕ of degree d then $\boxed{\underline{\deg}\, \omega(\phi) = p+2d}$

$\Rightarrow D$ raises $\underline{\deg}\, \omega(\phi)$ by 1

$$D^2 = \mathrm{Lie}_{V(\phi)} = 0\Big|_{G\text{-equivariant forms}}$$

$\Rightarrow$ one can define $H^*_G(X) = \mathrm{Ker}\, D / \mathrm{Im}\, D$

$= \bigoplus_{k=0}^{\infty} H^k_G(X)$

If G acts freely (no fixed points) $\Rightarrow H^*(X/G) = H^*_G(X)$

Ex: Equivariant cohomology of a point

$X = \text{point} \Rightarrow H^*_G(\text{point}) = $ the space of G-invariant functions on $\mathfrak{g}$

Given any $\omega(\phi) \in \Omega^*_G(X)$ and invariant polynomial $f(\phi) = f(g^{-1}\phi g)$ $\forall g$

$\Rightarrow f(\phi)\cdot\omega$ is a new equivariant form, if $D\omega = 0 \Rightarrow D(f(\phi)\omega) = 0$

If X/G exists then $f(\phi)\omega \in H^*_G(X)$ translates to $f(F)\tilde{\omega} \in H^*(X/G)$

where $\omega \in H^*_G(X) \to \tilde{\omega} \in H^*(X/G)$

curvature of some connection on $X \to X/G$

G-bundle
$$\begin{array}{c} X \\ \downarrow \\ X/G \end{array}$$

We shall study gauge theory on $\mathbb{R}^4$, namely instantons, i.e. gauge fields $A = A_\mu dx^\mu = \| A_\mu^{lm} \| dx^\mu$

Ψ (x^1, x^2, x^3, x^4), $ds^2 = \sum_{\mu=1}^{4} (dx^\mu)^2$

$l, m = 1, .., N$

$A_\mu^+ = -A_\mu$, i.e.

$A_\mu^{lm} = -\bar{A}_\mu^{ml}$

A satisfies:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \Rightarrow \underbrace{F_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}}$$

instantons minimize the Yang-Mills action $-\frac{1}{4}\int \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} d^4x$ in a given topological class $\quad \overset{\shortparallel}{S}$

$X = \{$all instantons on $\mathbb{R}^4$ with finite action $S\}/g_\infty$ .... moduli space of framed instantons

$\hookrightarrow F_{\mu\nu} \to 0$ as $\frac{1}{r^3}$, $r \to \infty$

$$g(x): A_\mu(x) \to g^{-1}(x) A_\mu(x) g(x) + g^{-1}(x)\partial_\mu g(x), \quad g_\infty = \{\underset{U(N)}{\underset{\shortparallel}{g(x)}} \mid g(x) \to \mathbb{1}, x \to \infty\}$$

G the symmetry group of X $\quad G = \underset{\text{gauge transf. at } \infty}{\underset{\uparrow}{U(N)}} \times \underset{\text{rotation group (Euclidean version of Lorentz group)}}{\underset{\uparrow}{SO(4)}}$

partition function

$$Z(\phi) = \int_X e^{\tilde{\omega}} \in \overset{G\text{-invariant}}{\text{functions on } \mathfrak{g}}, \quad \tilde{\omega} \in \Omega^*_G(X) \qquad e^{\tilde{\omega}} = \sum_{p=0}^{\infty} \frac{1}{p!} \overbrace{\tilde{\omega}(\phi) \wedge \dots \wedge \tilde{\omega}(\phi)}^{p\text{-times}}$$

and we integrate only the piece that can be integrated, i.e. has suitable degree, throw out the others

By doing $\phi \to g^{-1}\phi g$ bring $\phi = \mathrm{diag}(a_1, .., a_N) \cdot \begin{pmatrix} 0 & \varepsilon_1 & & \\ -\varepsilon_1 & 0 & & \\ & & 0 & \varepsilon_2 \\ & & -\varepsilon_2 & 0 \end{pmatrix}$

X is infinite dimensional, $X = \bigsqcup_{k=0}^{\infty} X_k \qquad k = -\frac{1}{8\pi^2}\int \mathrm{Tr}\, F \wedge F \qquad \dim X_k = 4k \cdot N$

$$\Rightarrow Z(\phi) = \sum_{k=0}^{\infty} \int_{X_k} e^{\tilde{\omega}(\phi)}$$

needed to model equivariant de Rham complex $\Omega^*_{g_\infty}(\mathcal{A})$

The claim

$$Z(\phi) = \int_{\mathcal{A}} \mathcal{D}A(x)\, e^{\frac{1}{4g^2}\int \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \text{other stuff}}$$

$\mathcal{A}$ = all gauge fields

altogether fit into field content of $\mathcal{N}=2$ SUSY gauge theory on $\mathbb{R}^4$

Reminder on $\mathcal{N}=2$ SYM

Nekrasov 2

$A_\mu, \phi, \bar\phi$ $\quad\phi$ complex Higgs field in adjoint representation of ~~...~~ $U(N)$ (scalar)

$\alpha=1,2,\ \dot\alpha=\dot1,\dot2,\ i=1,2$

$\lambda_{\alpha i},\ \bar\lambda^{\dot\alpha}{}_i$ two sets of Weyl fermions in adjoint of $U(N)$ ... gluinos

$$\Big(\underset{}{Spin(4)} = \underset{(2,1)}{SU(2)_L} \times \underset{(1,2)}{SU(2)_R} \Big)$$

One of the supercharges will become D

First assume $\varepsilon=0$

$$S_{YM}^{\mathcal{N}=2} = \int -\frac{1}{4g_0^2}\mathrm{Tr}\,F_{\mu\nu}^2 + \mathrm{Tr}\,D_\mu\phi D^\mu\bar\phi + \mathrm{Tr}[\phi,\bar\phi]^2$$
$$+\sum_{i=1}^{2}\mathrm{Tr}\,\bar\lambda_i \not{D}\lambda^i + \mathrm{Tr}\,\phi[\bar\lambda,\bar\lambda] + \mathrm{Tr}\,\bar\phi[\lambda,\lambda]$$
$$[\lambda,\lambda]\equiv[\lambda_{\alpha i},\lambda_{\beta j}]\,\varepsilon^{\alpha\beta}\varepsilon^{ij}$$

$\varepsilon\neq0 \qquad \Omega_{\mu\nu}=-\Omega_{\nu\mu} \qquad \Omega=\begin{pmatrix}0 & \varepsilon_1 & 0 & 0\\ -\varepsilon_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \varepsilon_2\\ 0 & 0 & -\varepsilon_2 & 0\end{pmatrix}$

$$\Rightarrow S_{YM}^{\mathcal{N}=2}(\varepsilon) = \int\Big(-\frac{1}{4g_0^2}\mathrm{Tr}\,F_{\mu\nu}^2 + \mathrm{Tr}\big(D_\mu\phi - \Omega^{\tilde\mu}{}_{\tilde\nu}x^{\tilde\nu}F_{\mu\tilde\mu}\big)\big(D_\mu\bar\phi - \bar\Omega^{\tilde\mu}{}_{\tilde\nu}x^{\tilde\nu}F_{\mu\tilde\mu}\big)$$
$$+\frac12\mathrm{Tr}[\phi,\bar\phi]^2 + \mathrm{Tr}\,\bar\lambda\not{D}\lambda + \mathrm{Tr}\,\phi[\bar\lambda,\bar\lambda] + \mathrm{Tr}\,\bar\phi[\lambda,\lambda]$$
$$+\mathrm{Tr}\,\lambda_{\alpha i}\bar\Omega^\mu{}_\nu x^\nu\partial_\mu\lambda^{\alpha i} + \mathrm{Tr}\,\bar\lambda_{\dot\alpha i}\Omega^\mu{}_\nu x^\nu\partial_\mu\bar\lambda^{\dot\alpha i}$$
$$+\mathrm{Tr}\,\bar\Omega\lambda\lambda + \mathrm{Tr}\,\Omega\bar\lambda\bar\lambda\Big) + \frac{\vartheta_0}{2\pi}\int\mathrm{Tr}\,F\wedge F$$

related to equivariant cohomology

Partition function $\quad Z(a,\varepsilon)=\int\mathcal{D}A\,\mathcal{D}\phi\,\mathcal{D}\bar\phi\,\mathcal{D}\lambda\,\mathcal{D}\bar\lambda\,\exp\big(-S_{YM}^{\mathcal{N}=2}(\varepsilon)\big)$

$$\phi \xrightarrow[x\to\infty]{} \mathrm{diag}(a_1,\dots,a_N)$$

$$\tau_0=\frac{\vartheta_0}{2\pi}+\frac{4\pi i}{g_0^2}$$

in computing $Z(a,\varepsilon)$ we need to regularise $\to$ a cutoff μ .. high energy scale, we then want to send $\mu\to\infty$

$$\Lambda=\mu\, e^{\frac{2\pi i\tau_0(\mu)}{N}} \quad \text{finite as } \mu\to\infty \ \Rightarrow\ Z(a,\varepsilon;\Lambda)$$

ε-deformed supersymmetry guarantees that Z is $\bar\tau$ independent.

Take $g_0\to0 \Rightarrow Z(a,\varepsilon,\Lambda)\to \underset{\text{minima of } S_{YM} \text{ up to the gauge transf.}}{\int\dots} = \sum_{k=0}^{\infty}\int_{X_k} e^{\omega(\phi=(a,\varepsilon))}\,\Lambda^{kN}$

ϖ in general $\quad X, G$

suppose $g_{\mu\nu}$ is G-invariant metric on X $\quad \bar{\Phi} \in \mathfrak{g} \quad [\bar{\Phi}, \phi] = 0$

$$\varpi(\phi) = D\left(g_{\mu\nu} V(\bar{\Phi})^\mu dx^\nu\right) = -g_{\mu\nu} V(\phi)^\mu V(\bar{\Phi})^\nu + d\left(g_{\mu\nu} V(\bar{\Phi})^\mu dx^\nu\right)$$

Varying $\bar{\Phi}$ doesn't change $\int_X e^{\varpi(\phi)}$

$\int_X e^{\varpi(\phi)}$ would be trivial on compact X but in gauge theory X is non-compact

$\lim_{\bar{\Phi}\to\infty} \Rightarrow \int_X e^{\varpi(\phi)}$ localizes near points x where $V(\phi)(x) = 0$ .. fixed points of the group action

$\Rightarrow$ precisely given by saddle point contributions of the fixed points

$\Rightarrow$

$$Z(\phi) = \sum_{\substack{x \in X \\ V(\phi)(x)=0}} \frac{1}{\prod_i \text{weights}_i(\phi)} = \sum_{\dots} \frac{1}{\prod_i \left(\sum_{k=1}^m n_k^i \phi_k\right)}$$

$T_x X$ representation of maximal torus T of G $\quad (\exp \phi \in T \subset G)$

$\Rightarrow T_x X = \bigoplus_{i=1}^{\dim X} \mathcal{L}_i \quad$ 1-dim repr. $\qquad T \ni (e^{i\vartheta_1}, .., e^{i\vartheta_m}) = \eta \Rightarrow \eta$ acts on $\mathcal{L}_i$ $\;$ $m = \dim T$

irrep $\mathcal{L}_i$ defines $(n_1^i, .., n_m^i) \longleftarrow \Big|_{\text{by}} \left(e^{i\sum_{k=1}^m n_k^i \vartheta_k}\right)$.

<u>Example</u>: $\quad X = \mathbb{R}^4 \quad ds^2 = \sum_{\mu=1}^4 dx^\mu dx^\mu \qquad G = SO(4)$

$$Z(\phi) = \int_{\mathbb{R}^4} e^{\varpi(\phi)} \qquad \varpi(\phi) = D\left(g_{\mu\nu} V^\mu(\bar{\phi}) dx^\nu\right) \qquad \phi = \begin{pmatrix} 0 & \varepsilon_1 & 0 & 0 \\ -\varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & -\varepsilon_2 & 0 \end{pmatrix} = \Omega_{\mu\nu}$$

$$= \sum_{\text{fixed points}} \frac{1}{\prod \text{weights}}$$

only 1 fixed point $x = 0$ $\qquad V(\phi) = \Omega_{\mu\nu} x^\nu \frac{\partial}{\partial x^\mu} \Rightarrow g_{\mu\nu} V^\mu(\bar{\phi}) dx^\nu = \bar{\Omega}_{\mu\nu} x^\mu dx^\nu$

$$\varpi(\phi) = D\left(\bar{\Omega}_{\mu\nu} x^\mu dx^\nu\right) = \bar{\Omega}_{\mu\nu} dx^\mu \wedge dx^\nu + \Omega_{\mu\nu} x^\nu \bar{\Omega}_{\mu\lambda} x^\lambda$$

↳ called regulator form

$$\int_{\mathbb{R}^4} e^{\varpi(\phi)} = \int_{\mathbb{R}^4} e^{\bar{\Omega}_{\mu\nu} dx^\mu \wedge dx^\nu} e^{-\Omega_{\mu\nu} x^\nu \bar{\Omega}_{\mu\lambda} x^\lambda} = \frac{1}{2} \int_{\mathbb{R}^4} \bar{\Omega}_{\mu\nu} dx^\mu \wedge dx^\nu \, \bar{\Omega}_{\lambda\sigma} dx^\lambda \wedge dx^\sigma \cdot e^{-\Omega_{\mu\nu} x^\nu \bar{\Omega}_{\mu\lambda} x^\lambda}$$

$$= \frac{1}{2} \bar{\varepsilon}_1 \bar{\varepsilon}_2 \int d^4x \, e^{-\left(\varepsilon_1 \bar{\varepsilon}_1 (x_1^2 + x_2^2) + \varepsilon_2 \bar{\varepsilon}_2 (x_3^2 + x_4^2)\right)} = \frac{\text{factors like } 2\pi}{\varepsilon_1 \varepsilon_2}$$

$SO(4)$ .. $T = U(1) \times U(1) \qquad T_0 \mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$
$\qquad (n_1^1, n_2^1) = (1, 0) \quad (0, 1) = (n_1^2, n_2^2)$

$Z(a_1, .., a_N, \varepsilon_1, \varepsilon_2, \Lambda)$ .. partition function of $\mathcal{N} = 2$ SYM on $\mathbb{R}^4$ in the presence of ε-coupling with the boundary condition $\Phi \xrightarrow[x\to\infty]{} \text{diag}(a_1, .., a_N)$

eff. energy scale (dimensionally transmuted) gauge coupling

Nekrasov 3

$$Z(a_1,\dots,a_N,\varepsilon_1,\varepsilon_2,\Lambda) = \sum_{k=0}^{\infty} \Lambda^{2kN} \int_{X_k} e^{\mathcal{O}(\phi)}$$

$$= \sum_{\substack{\text{fixed points} \\ \text{of } U(N)\times SO(4) \\ U(1)^N \times U(1)^2 \text{ action on } X_k}} \Lambda^{2kN} \frac{1}{\prod_{i=1}^{2kN} \text{weights}(a,\varepsilon)} = \exp\Big[\frac{1}{\varepsilon_1\varepsilon_2} \underbrace{\mathcal{F}(a,\varepsilon_1,\varepsilon_2,\Lambda)}_{\text{power series in } \varepsilon_1,\varepsilon_2}\Big]$$

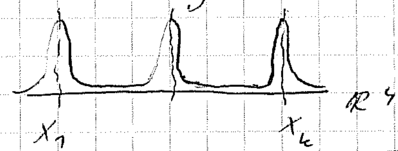
$\dim X_k = 4kN$

$\mathcal{F}_0(a,\Lambda) + (\varepsilon_1+\varepsilon_2)\mathcal{F}_1(a,\Lambda) + \dots$

assign $[a]=[\varepsilon]=[\Lambda]=[\text{energy}] \Rightarrow Z$ dimensionless

X_k parametrise solutions to $F_A^+ = 0$ with $-\frac{1}{g_0^2}\int \mathrm{Tr}\, F \wedge * F = k$

for a typical sol.



$$\Rightarrow \{X_k\} \supset \{(\overbrace{X_1 \times \dots \times X_1}^{k})/S_k\}$$

if X_k is of the form $\int_{X_k} \sim \frac{1}{k!}\int X_1 \Rightarrow Z \propto \exp \int_{X_1} e^{\mathcal{O}(\phi)}$

$$X_1 = \mathbb{R}^4 \times X_1^{\text{central}} \qquad \Rightarrow \int_{X_1} = \int_{\mathbb{R}^4}\int_{X_1^{\text{central}}} = \frac{1}{\varepsilon_1\varepsilon_2}\int_{X_1^{\text{central}}}$$

$\mathbb{R}^4$: position of the centre of instanton, doesn't care about $U(1)^N$

$\int_{X_1^{\text{central}}}$: nonsingular in $\varepsilon_1,\varepsilon_2$

$\Rightarrow$ the form of $Z(\dots)$ above

$\mathcal{F}_0(a,\Lambda)$ "prepotential" of the gauge theory, could be defined purely in $\mathcal{N}=2$ SUSY terms (without ε-deformation)

$\mathcal{F}_0(a,\Lambda)$ is also a prepotential (in $\mathcal{N}=2$ sense) of the low-energy effective theory $U(1)^N$ with Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{\partial^2 \mathcal{F}}{\partial a_e \partial a_m} F^-_{e\mu\nu} F^-_{m\mu\nu} + \frac{\partial^2 \bar{\mathcal{F}}}{\partial \bar{a}_e \partial \bar{a}_m} F^+_{e\mu\nu} F^+_{m\mu\nu}$$

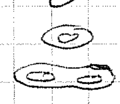
$$+ \operatorname{Im} \frac{\partial^2 \mathcal{F}}{\partial a_e \partial a_m}\, \partial_\mu a_e\, \partial_\mu \bar{a}_m + \text{fermions}$$

where $(A_{\mu e}, a_e, \bar{a}_e)$ is the bosonic field content

$A_{\mu e}$: abelian photons

Special case $\varepsilon_1 = -\varepsilon_2 = \hbar$ ($\sim$ string coupling of the dual string theory)

$$Z(a_1,\dots,a_N,\hbar,\Lambda) = \exp\Big(-\sum_{g=0}^{\infty} \mathcal{F}_g(a,\Lambda)\,\hbar^{2g-2}\Big) \quad \text{string partition function}$$



$\mathcal{F}_0$
$\mathcal{F}_1$
$\mathcal{F}_2$
$\vdots$

Seiberg & Witten (1994) predicted that

$$w + \frac{\Lambda^{2N}}{w} = x^N + u_1 x^{N-1} + \dots + u_N \qquad \Sigma \subset \overset{x}{\mathbb{C}} \times \overset{w}{\mathbb{C}^*}$$

$\text{genus}(\Sigma_u) = N-1 \qquad dS = x\frac{dw}{w}$ meromorphic differential

$\to$ one can define a set of N A-cycles A_e, N B-cycles B_e on Σ_u s.t. $a_e = \oint_{A_e} dS$, $\frac{\partial \mathcal{F}_0}{\partial a_e} = \oint_{B_e} dS$

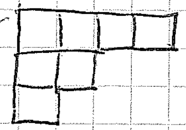
$(2(N-1)$ compact cycles $+$ 1 around the pole of dS $+$ 1 non-compact$)$

Fixed points on instanton moduli space

.. labeled by coloured partitions

<u>partition</u> of k: $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_n \geq k_{n+1} = 0 = \ldots \qquad k = \sum_{i=1}^{n} k_i$

Young diagram — n – the length of partition



k_1 boxes
k_2 —"—
k_3 —"— etc.

Choose a box (i,j) $\quad i \leq n,\ 1 \leq j \leq k_i \Rightarrow \text{arm}(i,j) = k_i - j \qquad \text{hook}(i,j) = k_i - j + k_j' - i + 1$

$\text{leg}(i,j) = k_j' - i$

where $k_1' \geq \ldots \geq k_{k_n}' > 0 = \ldots$ — dual partition

<u>coloured partition</u> N-tuple of partition

$\underset{\sim}{k}_1, \ldots, \underset{\sim}{k}_N \qquad \underset{\sim}{k}_l = (k_1 \geq k_2 \geq \ldots)$

fixed points on $X_k \longleftrightarrow (\underset{\sim}{k}_1, \ldots, \underset{\sim}{k}_N),\ \sum_{l=1}^{N} |\underset{\sim}{k}_l| = K$

$$\sum_{k=0}^{\infty} (\#\text{ of fixed points on } X_k)\, q^k = \prod_{n=0}^{\infty} \frac{1}{(1-q^n)^N} \quad (\sim e^{\#\sqrt{N}})$$

Weight of a given fixed point

$$\frac{1}{\prod \text{weights}(a,k)} = \prod_{\substack{(l,i) \neq (n,j) \\ l,n = 1,\ldots,N \\ i,j = 1,\ldots,\infty}} \frac{a_l - a_n + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_l - a_n + \hbar(j-i)} \underset{\text{actually}}{=} \frac{1}{\prod (2kN \text{ factors})}$$

<u>Example: $N=1$</u>

$$\prod_{i \neq j} \frac{k_i - k_j + j - i}{j - i} = \prod_{i<j} \left(\frac{k_i - k_j + j - i}{j-i}\right)^2 = \prod_{1 \leq i < j \leq n} \left(\frac{k_i - k_j + j - i}{j - i}\right)^2$$

$$\cdot 1 \cdot \underbrace{\prod_{i=1}^{n} \prod_{j=n+1}^{\infty} \frac{(k_i - i + j)^2}{(-i+j)^2}}$$

$$\prod_{i=1}^{n} \frac{(n-i)!}{(k_i + n - i)!} \Longrightarrow \frac{1}{\prod \text{weights}} \text{ is finite}$$

(proven only for $N=1$ but valid in general)

$$\Rightarrow \prod_{i \neq j} \frac{k_i - k_j + j - i}{j - i} = \prod_{(i,j) \in \underset{\sim}{k}} \frac{1}{\text{hook}(i,j)^2} = \left(\frac{1}{|k|!} \dim R_{\underset{\sim}{k}}\right)^2$$

representation of $S_{|\underset{\sim}{k}|}$

$|\underset{\sim}{k}| = 0 \qquad 1$

$|\underset{\sim}{k}| = 1 \qquad \boxed{1} \qquad \Lambda^2/\hbar^2$

$|\underset{\sim}{k}| = 2 \qquad \boxed{2|1} \quad \boxed{\begin{smallmatrix}2\\1\end{smallmatrix}} \qquad 2 \cdot \frac{\Lambda^4}{4} = \frac{\Lambda^4}{2}/\hbar^4$

$|\underset{\sim}{k}| = 3 \qquad \boxed{3|2|1} \quad \boxed{\begin{smallmatrix}3&1\\1\end{smallmatrix}} \quad \boxed{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} \qquad \left(\frac{\Lambda}{\hbar}\right)^6 \left(\frac{1}{36} + \frac{1}{9} + \frac{1}{36}\right) = \frac{1}{6}\left(\frac{\Lambda}{\hbar}\right)^6$

$$\sum_{\underset{\sim}{k}} \frac{\Lambda^{2|\underset{\sim}{k}|}}{\hbar^{2|\underset{\sim}{k}|}} \prod \frac{1}{\text{hook}(i,j)^2} = e^{\frac{\Lambda^2}{\hbar^2}}$$

$N > 1$ $\boxed{\text{Nekrasov 4}}$

$$\underbrace{\Big(\sum_{\underset{\sim}{k}} \prod_{(l,i)\neq(n,j)} \left(\frac{a_l - a_n + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_l - a_n + \hbar(j-i)}\right) \Lambda^{2N|\underset{\sim}{k}|}\Big)}_{\text{pure instanton part}} Z^{\text{perturbative}}(a, \hbar, \Lambda)$$

contribution of pure gauge $A = 0$

$$\underbrace{}_{\parallel}$$

$$\prod_{(l,i)\neq(n,j)}^{\text{regularized}} (a_l - a_n + \hbar(i-j))$$

$$= \exp \sum_{l,n} \gamma_\hbar(a_l - a_n)$$

$$\gamma_\hbar(x) = \frac{d}{ds}\Big|_{s=0} \underbrace{\frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{4\sinh\left(\frac{\hbar t}{2}\right)}}$$

well defined for $\mathrm{Re}\, s > 2$, can be analytically continued to $\mathbb{C}$, well-defined derivative near 0

$$\left(\text{the integrand } \text{``}=\text{''} \sum_{i=1}^\infty \sum_{j=1}^\infty e^{-tx - t\hbar(i-j)} \right.$$

$$\left. \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma} \int_0^\infty \frac{dt}{t} t^s e^{-ty} = \frac{d}{dt} y^{-s} = \log \quad ?? \right)$$

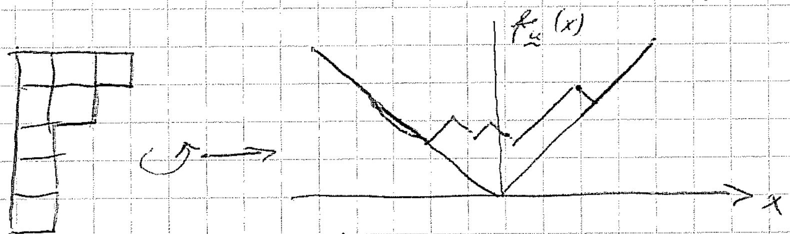
$$\gamma_\hbar(x+\hbar) + \gamma_\hbar(x-\hbar) - 2\gamma_\hbar(x) = -\log x$$

$$\gamma_\hbar(x) = \sum_{g=0}^\infty \gamma_g(x) \hbar^{2g-2} \qquad \gamma_0(x) = \frac{1}{2} x^2\left(\log x - \frac{3}{2}\right)$$

$$\gamma_1(x) = \frac{1}{12} \log x$$

$$\gamma_g(x) = \frac{B_g}{2g(2g-2)} x^{2g-2}$$

Profile of a partition



$f'_{\underset{\sim}{k}}(x) = \pm 1 \qquad f_{\underset{\sim}{k}}(x) \geq |x|, \quad f_{\underset{\sim}{k}}(x) = |x|$ for $|x| >> 0$

$|\underset{\sim}{k}| \to \infty \Rightarrow$ on suitable scale looks as differentiable smooth function

$f_{\underset{\sim}{k}}(x) \to f(x) : |f'(x)| \leq 1$

~~[illegible]~~ For coloured partition $\vec{\underset{\sim}{k}} = (\underset{\sim}{k}_1, \ldots, \underset{\sim}{k}_N)$ $\quad f_{\vec{\underset{\sim}{k}}}(x) = \sum_{l=1}^N f_{k_l}(x - a_l)$

a_l separated enough

$$f_{\vec{\underset{\sim}{k}}}(x) \to N|x| \text{ as } |x| >> 0$$

The claim is

$$Z \approx \exp -\frac{1}{4} \iint_{x<y} f''_{\vec{\underset{\sim}{k}}}(x) f''_{\vec{\underset{\sim}{k}}}(y) \gamma_\hbar(x-y)\, dx\, dy$$

Relation between 4-dim. $N=2$ SYM and equivariant cohomology

Twisting procedure

on $\mathbb{R}^4$ essentially another notation for the fields of SYM

advantages clear diff. geometric meaning

disadvantage symmetry $SU(2)_L \times SU(2)_R \times SU(2)_I$ (global symm. group) is not visible,

only $SU(2) \times SU(2)$

	$SU(2)_L$	$SU(2)_R$	$SU(2)_I$	$SU(2)_L$	$SU(2)_\Delta$	
A_μ	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	A_μ
$\lambda_{\alpha i}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\to$ vector ψ_μ
$\bar\lambda_{\dot\alpha i}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$1 \oplus 0$	$\chi^+_{\mu\nu}, \eta$ (self-dual 2-form)
ϕ	0	0	0	0	0	ϕ

the same set of fields in new notation

we may choose $SU(2)_L \times \underset{SU(2)_R \times SU(2)}{\underset{\uparrow}{SU(2)_\Delta}}$ as a "new Lorentz group"

Supercharges

$$Q_{\alpha i}, \bar{Q}_{\dot\alpha i} \longrightarrow Q, Q^+_{\mu\nu}, Q_\mu \qquad \{Q, Q_\mu\} = \partial_\mu \qquad Q^2 = 0 \text{ up to gauge transf.}$$

$Q^+_{\mu\nu}$: self-dual 2-form

$$\{Q^+_{\mu\nu}, Q_\lambda\} = (\varepsilon_{\mu\nu\lambda\sigma}\partial_\sigma)^+$$

self-dual combination (in $\mu\nu$)

$$QA_\mu = \underset{\underset{d_{\text{de Rham}} A_\mu}{\parallel}}{\psi_\mu} \qquad Q\psi_\mu = D_\mu \phi = \partial_\mu \phi + [A_\mu, \phi] \qquad Q\phi = 0$$

$\Rightarrow$ $Q = d_{\text{de Rham}} + i_{V(\phi)}$ acting on G-equivariant forms on space of all gauge fields $\mathcal{A}$

$\Rightarrow$ Q .. equivariant de Rham diff. on $\mathcal{A}$

The other supercharges:

$\mathcal{A}$ is hyperKähler ... I, J, K compatible complex structures compatible with the metric $\int \mathrm{Tr}\, \delta A \delta A$

(in fact connected to complex structures on $\mathbb{R}^4$)

$\Rightarrow$ we can define $\bar\partial_I, \bar\partial_J, \bar\partial_K \longleftrightarrow Q^+_{\mu\nu}$

translations of $\mathbb{R}^4$ act on $\mathcal{A}$ .. vector fields V_μ $\quad V_\mu A_\nu(x) = \partial_\mu A_\nu(x)$

$$i_{V_\mu} \longleftrightarrow Q_\mu \qquad \text{e.g. } Q_\mu \psi_\nu = F_{\mu\nu} = \partial_\mu A_\nu - D_\nu A_\mu$$

$\mathbb{R}^4$ in addition to translational symmetries also has rotational symmetries

$$\Omega_{\mu\nu} = \begin{pmatrix} \begin{smallmatrix} 0 & \varepsilon_1 \\ -\varepsilon_1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & \varepsilon_2 \\ -\varepsilon_2 & 0 \end{smallmatrix} \end{pmatrix} \qquad V_\Omega A_\mu = \Omega_{\tilde{\mu}\nu} x^\nu \partial_{\tilde{\mu}} A_\mu + \text{extra terms} \qquad (\mathcal{L}_{V_\Omega} A_\mu)$$

$\longrightarrow$ new supercharge $\tilde{Q}_\Omega = Q + V^\mu_\Omega Q_\mu$

$\tilde{Q}_\Omega{}^2 \sim$ gauge transf. + rotation $\Rightarrow$ O.K. on rotationally invariant operators

---

Path integral $\rightarrow$ integral over instanton moduli space

1) $\varepsilon = 0$ $\quad Q^2 = 0$ $\quad$ Consider $\mathcal{O}$

$\{Q, \mathcal{O}\} = 0$ up to $\mathcal{O} \rightarrow \mathcal{O} + \{Q, R\}$, doesn't depend on metric

$$S^{N=2}_{SYM} = \tau_0 \operatorname{Tr} F \wedge F + \{Q, R\} \qquad \tau_0 = \frac{\vartheta_0}{2\pi} + \frac{4\pi i}{g_0^2}$$

$$\Rightarrow \frac{\delta}{\delta \text{metric}} \langle \mathcal{O} \rangle = \langle \mathcal{O} \{Q, \frac{\delta R}{\delta \text{metric}}\} \rangle = \langle \underbrace{\{Q, \mathcal{O}\}}_{0} \frac{\delta R}{\delta \text{metric}} \rangle = 0$$

any G-invariant $P \Rightarrow \mathcal{O} = P(\phi)$ is Q-closed

e.g. R' : $\quad \{Q, R'\} = \bar{\tau}_0 \int \operatorname{Tr} \| F^+ \|^2 + \ldots$

lim $\bar{\tau}_0 \rightarrow \infty$, τ_0 fixed (i.e. suitable lim in ϑ_0, g_0 not preserving reality of them)

$\Rightarrow$ enforce $F^+ = 0$ $\Rightarrow$ path integral measure localizes sharply on the $F^+=0$ locus together with forcing ψ to be tangent vectors on X ($D_A^+ \psi = 0$, $D_A^* \psi = 0$)

$\Rightarrow \mathcal{O} \Longrightarrow$ closed diff. forms on X

2) $\varepsilon > 0$ $\quad \tilde{Q}_\Omega$ instead of Q $\Rightarrow$ any G-inv. polynomial $P \rightarrow \mathcal{O} = P(\phi(0))$ is $\tilde{Q}_\Omega$ closed etc.

Physical meaning of ε's

$\varepsilon = 0$ $\quad N=2$ 4-dim gauge theory can be obtained by dim reduction of $N=1$ 6-d SYM

A_M, $M = 1, 2, \ldots, 6$ $\qquad \operatorname{Tr} F_{MN} + \operatorname{Tr} \bar{\chi} \not{D} \chi$, everything x^5, x^6 independent

$\Rightarrow$ 4-d SYM $N=2$, $\phi = A_5 + i A_6$

instead of reducing on $T^2 \times \mathbb{R}^4$ (T^2 shrinks to zero size .. KK theory)

do generalised Scherk - Schwarz reduction $\quad \mathbb{R}^4$ bundle over T^2 $\quad \widetilde{\underset{(z,\bar{z})}{T^2} \times \underset{x^\mu}{\mathbb{R}^4}}$

$$G_{MN} dx^M dx^N = ds^2_G = A dz d\bar{z} + (dx^\mu + \Omega^\mu{}_\nu x^\nu dz + \bar{\Omega}^\mu{}_\nu x^\nu d\bar{z})^2$$

$$[\Omega, \bar{\Omega}] = 0$$

$\Rightarrow \mathcal{L} = \sqrt{G}\, G^{MM'} G^{NN'}\, \mathrm{Tr} F_{MN} F_{M'N'} + \mathrm{Tr}\, \bar\lambda \not{D} \lambda$

.. locally flat ~~but~~ globally nontrivial background

$A \to 0 \Rightarrow S^{N=2}_{SYM}(\varepsilon)$ + Wilson line in the $SU(2)_I$ $(..\Omega)$

instead of going $\underset{\widetilde{T^2 \times R^4}}{6 \to 4}$ one can also $\underset{S^1}{6 \to} \underset{\widetilde{S^1 \times R^4}}{5 \to 4}$ $\quad ds^2 = R^2 d\varphi^2 + (dx^\mu + \Omega^\mu{}_\nu x^\nu d\varphi)^2$

$\mathcal{N}=2$ SUSY in 4d as low-energy effective string theory

$M|_{CY\times S^1} \leftrightarrow IIA|_{CY} \xleftrightarrow{\text{mirror}} IIB|_{CY}$

R^4: $\mathcal{N}=2$ vector multiplets $(A, \lambda, \bar\lambda, \phi)$ .. abelian part $\leftarrow$ perturbative string spectrum, nonabelian part $\leftarrow D2|_{S^2}$

hyper multiplets

Ω-background $(\varepsilon \neq 0)$ .. M on $CY \times \widetilde{S^1 \times R^4}$ (in general requires $\varepsilon_1 = -\varepsilon_2$)

reduce on CY: 5d $\mathcal{N}=1$ SUSY gauge theory $\widetilde{S^1 \times R^4}$ $\to$ 4dim ε-deformed $\mathcal{N}=2$ SYM

reduce S^1: IIA strings on $CY \times R^4$ with nontrivial metric, nontrivial U(1) RR U(1) field strength, dilaton

use BCOV + GTNA procedure

topological string on CY

$\to \mathcal{F}_g$ = counts holomorphic curves of genus g on CY

Kähler moduli of ~~$\$$~~ CY

$\to (a_1, \ldots, a_N)$ of gauge theory

with the following differences

they had flat R^4 + constant RR-flux

~~F~~ $\sim \int dx^4 d^2\theta\, \mathcal{F}_0(a + ?\psi + \ldots)$

here RR-flux decays in $\infty \Rightarrow$ ~~$\not{E}$~~ everything normalizable, e.g. partition function, $F_{string} = -\frac{1}{\hbar^2}\mathcal{F}_0(a) + \mathcal{F}_1(a) + \ldots$

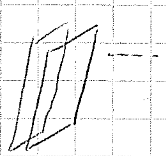
free energy

$= \sum \hbar^{2g-2} \mathcal{F}_g(a)$

topological string coupling

D-brane construction of $\mathcal{N}=2$ SYM

IIB string theory stack of N D3 branes



~~...~~ $\mathcal{N}=4$ U(N) gauge theory

$\mathcal{N}=2$ SYM + hypermultiplet in adjoint repr.

+ 6 scalars

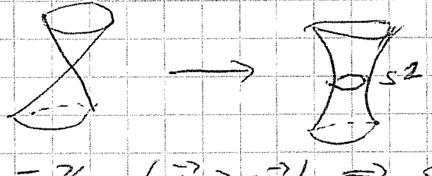
we want to kill fluctuations in 4 of 6 transverse directions

$\to$ do the orbifold $\quad R^{3,1} \to R^{3,1} \times R^{4}/\Gamma \qquad \Gamma \subset SU(2)$

add N D3 branes $\Rightarrow$ at the fixed point $\mathcal{N}=2$ SYM
fractional $\quad$ correspon. to irreducible reps.

Fractional branes

resolve the singularity



E.g. the simplest $\Gamma = \mathbb{Z}_2 \ (\vec{x} \to -\vec{x}) \Rightarrow S^2 \to$ manifold $= T^*S^2 \times R^2 \times R^{3,1}$

wrap N D5 branes on $S^2 \to \mathcal{N}=2 \quad 4d\,U(N)\,SYM$

worldvolume of D5 ... $S^2 \times R^{3,1}$

S^2 small $\longrightarrow \mathcal{N}=2 \quad 3+1 \quad U(N)$ SYM

One option: $\quad$ replace $\mathbb{R}^2$ by $\mathbb{R}^1 \times S^1$, replace $\underbrace{R^1 \times \widetilde{S^1 \times R^4}}_{\parallel}$ in euclidean, instead of $R^{3,1}$

precisely Ω-background setup

instantons in the $U(N)$ gauge theory $\Longleftrightarrow k \times D1$ wrapping S^2

partition function $\quad Z[a_1, \dots, a_N, \hbar, \Lambda]$

$$\Lambda^2 = e^{A}, A = \text{area}(S^2)$$

the simplest case $N=1 \qquad Z[a, \hbar, \Lambda] \qquad$ "$U(1)$ SYM"

On the gauge theory side

$$Z[a, \hbar, \Lambda] = \left(\sum_{\underset{\sim}{k}} \left(\frac{\Lambda}{\hbar}\right)^{2\Sigma k_i} \prod \left(\frac{k_i - k_j + j - i}{j - i} \right) \right) \cdot \underbrace{\Lambda^{\frac{a^2}{2\hbar^2}}}_{\text{classical piece}}$$

$$= \Lambda^{\frac{a^2}{2\hbar^2}} e^{\frac{\Lambda^2}{\hbar^2}}$$

$$\Rightarrow F_0^{U(1)} = \log \Lambda \cdot \frac{a^2}{2} + \Lambda^2$$

σ-models usually $A \equiv t \Rightarrow F_0^{U(1)} = \frac{t a^2}{2} + e^{-t}$

... type A topological string on $\mathbb{C}P^1$

add couplings

$$Z[a; t_1, t_2, \dots; \hbar] = \int \mathcal{D}A \dots \exp \int d^4x\, d^4\theta \left(\sum_{k=1}^{\infty} t_k \, \mathrm{Tr} \frac{\Phi^{k+1}}{k+1} \right) + \dots$$

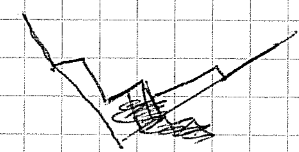
on the σ-model side $\qquad$ "gravitational descendants"

$$= \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \left\langle\!\left\langle \exp \int_\Sigma a \cdot 1 + t_k \, \sigma_{k-1}(\omega) \right\rangle\!\right\rangle_{g, \mathbb{C}P^1}$$

Back to $O(N)$ SYM

$Z[a_1,\dots,a_N,\hbar,\Lambda]$ partition function in theory with N colours $= \sum_{N\text{-coloured partitions}} \exp\left(-\frac{1}{4}\iint_{x\neq y} f''(x) f''(y)\, \gamma_\hbar(x-y)\,dx\,dy\right)$

Reminder: Profile f: e.g. $(5,3,1,1,1) \longrightarrow$



$\underset{\sim}{k} = (k_1 \geq k_2 \geq k_3 \geq \dots)$ f: $\frac{1}{2} f''(x) = \delta(x) + \sum_{i=1}^{m} \big(\delta(x+\hbar(k_i - i + 1))$
$- \delta(x+\hbar(k_i - i)) + \delta(x+\hbar(-i))$
$- \delta(x+\hbar(1-i)\big)$

$$\sum_i k_i = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{2} f''_{\underset{\sim}{k}}(x)\,dx$$

$\vec{\underset{\sim}{k}} = (\underset{\sim}{k}_1,\dots,\underset{\sim}{k}_N) \Rightarrow f_{\vec{\underset{\sim}{k}}} = \sum_{l=1}^{N} f_{\underset{\sim}{k}_l}(x - a_l)$ $\quad f_{\vec{\underset{\sim}{k}}}(x) = N|x|$ for $|x| >> 0$

and $\gamma_\hbar(\xi) = \frac{\xi^2}{2\hbar^2}\log\Lambda + \frac{\xi^2(\log\xi - \frac{3}{2})}{2\hbar^2} + \frac{1}{12}\log\xi + \sum_{g=2}^{\infty} \frac{\hbar^{2g-2}}{\xi^{2g-2}} \frac{B_{2g}}{2g(2g-2)}$

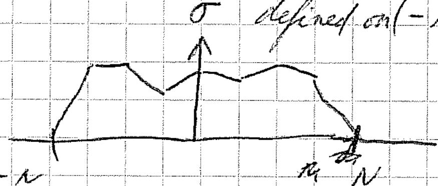
$\hbar \to 0$ $\quad \sum_i k_i \sim \frac{1}{\hbar^2}$ the most important contribution to $Z \Rightarrow$ "almost smooth f"

$\Rightarrow$ what's the saddle point of $S(f) = \iint_{x\neq y} f''(x) f''(y)\,(x-y)^2\left(\log\frac{|x-y|}{\Lambda} - \frac{3}{2}\right)$

$\Rightarrow$? how to extract a_l's from f .. add Lagrange multipliers to S .. $\sum_l \xi_l a_l$

$\sum_l \xi_l a_l = \int dx\, \sigma(f')$

σ defined on $(-N,N)$ $\quad$ (assume $\sum_l \xi_l = 0$)



$\sigma'(y) = \xi_l \quad -N+(2l-1) < y < -N+2l$

$\Rightarrow$ we find extremal of $S(f,\xi_l)$

$f'(x) \sim$ eigenvalue density in matrix model

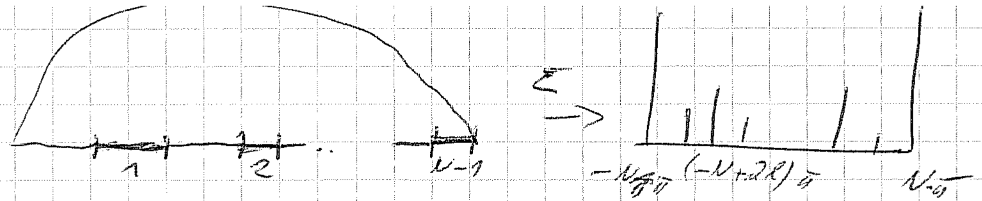
$$\chi(x) = \int_{y\neq x} f''(y)\,(x-y)\left(\log\left|\frac{x-y}{\Lambda}\right| - 1\right)dy$$

$f = f_*$ is minimiser of $S \Leftrightarrow \begin{cases} \chi_*(x) = \xi_l & -N+2(l-1) < f'_*(x) < -N+2l \\ & l = 1,\dots,N \\ \chi_*(x) \in (\xi_l, \xi_{l+1}) & f'_*(x) = -N+2l,\ l = 1,\dots,N-1 \end{cases}$

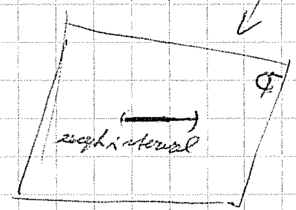
$$\pi f'_*(x) + i\chi'_*(x) = \Upsilon(x)$$

where $\Upsilon(x) = \Sigma(x+i0)$ $\quad$ $\Sigma(z)$ is analytic in the upper half plane

$\boxed{\text{Nekrasov } 7}$



$$e^{\frac{i\Sigma}{2}} + e^{-\frac{i\Sigma}{2}} = \frac{1}{\Lambda^N} P_N(z)$$



$$z \to \infty \Rightarrow e^{i\frac{\Sigma}{2}} \sim \left(\frac{z}{\Lambda}\right)^N$$

$$w = e^{i\Sigma(z)/2} \quad \Rightarrow \quad w + \frac{1}{w} = \frac{1}{\Lambda^N}\left(z^N + u_1 z^{N-1} + \dots + u_N\right)$$

$$dS_w = z\,\frac{dw}{w} = z\,\Sigma'(z)\,dz = ?$$

$$\Sigma(z) = i\int dy \; f''(y)\log\left|\frac{z-y}{\Lambda}\right| + N\pi \quad \Rightarrow \quad \Sigma'(z) = i\int dy\,\frac{f''(y)}{z-y}$$

$$= 2i\sum_l \frac{1}{z-a_l} + \sum_{k=1}^{\infty} \frac{1}{z-a_l+\hbar(k_{l_i}-i+1)} - \frac{1}{z-a_l+\hbar(k_{l_i}-i+1)} + \frac{1}{z-a_l+\hbar(-i)} - \frac{1}{z-a_l+\hbar(1-i)}$$

$$\Rightarrow \oint_{A_l} dS_w = \oint \frac{z\,dz}{z-a_l} = 2\pi i\, a_l \quad \dots \text{really } \underline{\text{a period}}$$

$\Rightarrow$ partition function saturated by certain instanton configuration

in a limit $\quad Z = \exp\left(V F_0 + F_1 + V^{-1} F_2 + \dots\right)$

Legendre transform of F $\to$ fix instanton number K $\to \tilde{F}_0(K) = K \hat{F}_0$

$\sim$ probability of instanton in unit volume

$\sim e^{-4\pi^2/g^2} / (a^2)^{N-1}$

Quantum gauge theories

diff. classical × quantum properties

E.g. massless QCD $\quad SU(N) \quad F_{\mu\nu}^2 + \sum_{i=1}^{N_f} q \, \not{D} q^i$

IR → strongly coupled (unable to prove results)

1) confinement → $SU(N)$ invisible

2) chiral symmetry breaking

$$q_\alpha \to U_L q_\alpha \qquad \tilde{q}_{\dot\alpha} \to U_R \tilde{q}_{\dot\alpha}$$

$$(U_L, U_R) \in U(N_f)_L \times U(N_f)_R \xrightarrow{\text{broken}} U(N_f)_{diag}$$

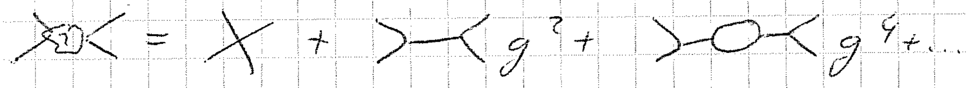
condensate $\langle q\tilde{q}\rangle \neq 0$ (but in perturbation theory $\langle q\tilde{q}\rangle = 0$)

→ problem to understand the vacuum

3) mass gap $\quad$ spectrum of $\hat{H}$ has a gap $\quad$ excit. / vacuum

→ spontaneous generation of a mass scale Λ_{QCD}

1), 2), 3) assumed to be <u>nonperturbative</u> !



$\sum a_n g^{2n}$, $\; a_n \sim$ # graphs at n-loop $\sim n!$

⇒ asymptotic expansion (doesn't converge)

+ nonperturbative corrections $\sim e^{-\frac{1}{g^2}}$

large N $\quad SU(N) \quad N \to \infty$

⇒ only the planar diagrams contribute in perturbative computation

$\sum b_n g^{2n} \qquad b_n \sim C^n \qquad$ ⇒ might be analytic !

Note: the existence of quantum gauge theory still ~~not~~ mathematically proven

<u>SUSY gauge theories</u>

I. phenomenology : needed in GUTs

II. 1st step towards M-theory

III. laboratory

early 1990s [Seiberg] analytic (holomorphic) expressions ⇒ might have exact solutions

[Seiberg – Witten 1994] $N=2$

... $SU(N)$ $\mathcal{N}=2$ becomes abelian $U(1)^{N-1}$ at low energies .. dual weakly coupled theory

effective geometry ... Riemann surface Σ of genus $N-1$

coupling matrix τ: $F^i_{\mu\nu}$ $i=1,..,N-1$ .. $U(1)^{N-1}$

$$\tau_{ij} F^i_+ F^j_+ + \text{c.c.} \quad \Rightarrow \quad \tau_{ij} = \text{period matrix of } \Sigma$$

$\Rightarrow$ Plan of lectures:

$$\mathcal{N}=1 \text{ SUSY} \longrightarrow \text{perturbation theory} \xrightarrow{U(N)} \text{planar 0-dim diagrams} = \text{matrix model}$$

$\downarrow$ large $\tilde{N}$

geometry $\xrightarrow{\text{nonperturbative}}$ $\mathcal{N}=1$ SUSY

$\mathcal{N}=1$ SYM, gauge group G simple

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} \bar{\lambda} \not{D} \lambda\right) \qquad (A_\mu, \lambda_\alpha, \lambda_{\dot\alpha}) \text{ adjoint valued}$$

$$+ \frac{\theta}{8\pi^2} \int \mathrm{Tr}\, F \wedge F \qquad \text{gauginos}$$

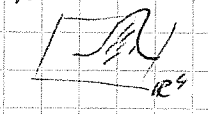
$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

$U(1)_R$ - symmetry $\lambda_\alpha \to e^{i\varphi}\lambda_\alpha$, $\lambda_{\dot\alpha} \to e^{-i\varphi}\lambda_{\dot\alpha}$ semi-classically broken by instantons

$$J^5_\mu = \mathrm{Tr}(\bar\lambda \gamma_5 \gamma_\mu \lambda)$$

$$\Rightarrow \partial_\mu J^\mu_5 = \frac{1}{32\pi^2} \mathrm{Tr}_{adj} (F \wedge F)$$

instanton



topological charge $\Rightarrow \frac{1}{32\pi^2} \int_{\mathbb{R}^4} \mathrm{Tr}_{adj} F \wedge F = n h$ $\quad n \in \mathbb{Z}$ the instanton number

h = dual Coxeter #
$= c_2(Adj) = \mathrm{Tr}_{adj} T_A T_A$
$= \begin{cases} N & SU(N) \\ N \mp 2 & SO(N)/Sp(N) \end{cases}$

$\Rightarrow$ index of $\not{D} = 2hn$

E.g. $n=1$ $\Rightarrow$ fermions are created $2h \times \lambda_\alpha$, i.e. $\langle \lambda^{2h} \rangle \sim e^{-8\pi^2/g^2}$

$\Rightarrow U(1)_R$ broken by instantons to $\mathbb{Z}_{2h}$ (e.g. $\mathbb{Z}_{2N}$ for $SU(N)$)

Strong coupling (only strong indirect evidence, not proved!)

confinement $G \to$ nothing, mass gap, chiral sym. breaking $\mathbb{Z}_{2h} \to \mathbb{Z}_2$ $(\lambda \to -\lambda)$

$$\langle \mathrm{Tr}\,\lambda^2 \rangle_k = \frac{1}{2} \langle \mathrm{Tr}\, \lambda_\alpha \lambda_\beta \varepsilon^{\alpha\beta} \rangle_k = \langle \mathrm{Tr}\, \lambda_1 \lambda_2 \rangle_k \neq 0$$

$$\langle (\mathrm{Tr}\,\lambda^2)^h \rangle \sim e^{-8\pi^2/g^2} \qquad e^{2\pi i k/h}\, e^{2\pi i \tau/h} \qquad k = 0,..,h-1$$

h vacua $|0\rangle, |1\rangle, .., |h-1\rangle$ $\qquad \langle \mathrm{Tr}\,\lambda^2 \rangle_k = \langle k | \mathrm{Tr}\,\lambda^2 | k \rangle$

Witten index
$\mathrm{Tr}\,(-1)^F = h$

$\mathbb{Z}_{2h}$ ~ permutations of vacua

$\Rightarrow$ ~~...~~ $\sim \frac{1}{k}$ instantons = "fractional instantons" $\boxed{\text{Dijkgraaf 2}}$

Questions : - why ?

- can be generalised to YM with matter ?

$\mathcal{N}=1$ SUSY

$$Q_\alpha, \overline{Q_{\dot\alpha}} \qquad \{Q_\alpha, \overline{Q_{\dot\alpha}}\} = \gamma^{M}_{\alpha\dot\alpha} P_M \equiv P_{\alpha\dot\alpha}$$

superspace $(x^M, \theta^\alpha, \theta^{\dot\alpha})$

$$Q_\alpha = i\underbrace{\partial_\alpha}_{=\frac{\partial}{\partial\theta^\alpha}} - \frac{1}{2}\bar\theta^{\dot\beta}\underbrace{\partial_{\alpha\dot\beta}}_{\sim\frac{\partial}{\partial x^M}} \qquad \overline{Q}_{\dot\alpha} = \dots$$

$$D_\alpha = \partial_\alpha + \frac{1}{2}\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta} = e^{-U}\partial_\alpha e^{U} \quad \text{where } U = \frac{1}{2}\theta^\alpha\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}$$

$\overline{Q}^2 = 0 \Rightarrow$ cohomology

chiral operators $\mathcal{O}$: $\overline{Q_{\dot\alpha}}\,\mathcal{O} = 0$, $\mathcal{O} \cong \mathcal{O} + \overline{Q_\alpha}\,\mathcal{O}'_\alpha \Rightarrow$ chiral ring

chiral superfield ϕ $\quad \overline{D}_{\dot\alpha}\phi = 0 \Rightarrow e^{U}\hat\phi(x,\theta) = \phi$

assume we have SUSY vacuum $\quad Q|0\rangle = \overline{Q}|0\rangle = 0$

$\langle 0|\mathcal{O}_1(x)\dots\mathcal{O}_n(x_n)|0\rangle$ $\quad$ i) independent of x_i's

chiral ops.

ii) depends only on the cohomology class $[\mathcal{O}_i]$

e.g. if $[\mathcal{O}_1]=0 \Rightarrow \langle 0|\underbrace{[\overline{Q},\mathcal{O}']}_{\overline{Q}\mathcal{O}'-\mathcal{O}'\overline{Q}}\mathcal{O}_2\dots\mathcal{O}_n|0\rangle = 0$

(commute to act on $\langle 0|$, resp. $|0\rangle$ $\Rightarrow 0$)

$$\partial_\mu\mathcal{O} = [P_\mu, \mathcal{O}] = \{\overline{Q}_{\dot\alpha}, [Q_\alpha, \mathcal{O}]\} = 0 \bmod \overline{Q}$$

~~...~~ $|0\rangle$ ~~...~~ satisfies <u>cluster decomposition property</u> $\quad \langle\mathcal{O}_1\dots\mathcal{O}_n\rangle = \langle\mathcal{O}_1\rangle\dots\langle\mathcal{O}_n\rangle$

for x_i far apart

$\because$ diagonalization of chiral ring $\qquad$ (thus holds $\to$ because of i) decomposition $\forall x_i$)

chiral field $\phi(x,\theta)$, $\overline{\Phi}(x,\bar\theta)$ $\Rightarrow$ F-terms $\int d^4x\, d^2\theta\, W(\phi)$

holomorphic

D-terms $\int d^4x\, d^4\theta\, K(\phi,\overline{\Phi})$

e.g. $\mathcal{L} = \int d^4x\, d^4\theta\, \phi\overline{\Phi} + \int d^4x\, d\theta\left(\frac{1}{2}m\phi^2 + \frac{1}{6}g\phi^3\right) + \text{c.c.}$

$W(\phi)$ superpotential

$$\phi = \underbrace{\varphi(x)}_{\text{scalar}} + \underbrace{\psi_\alpha(x)}_{\text{fermion}}\theta^\alpha + F\theta^2$$

$$\Rightarrow \mathcal{L} = \int d^4x\left[(\partial_\mu\varphi)^2 + \bar\psi\not{D}\psi\right) + \left|\frac{\partial W}{\partial\varphi}\right|^2 + W''\psi\psi + \text{c.c.}\Big]$$

Note: D-term $\int d^4x\, d^2\theta\, d^2\bar\theta\, K(\phi,\overline{\Phi}) = \int d^4x\, d^2\theta\,(\overline{D}^2K)$ $\qquad$ ($\Leftarrow \int d\theta\,\theta = 1$)

vanishes in the chiral ring $\Rightarrow$ only F-terms important in correlation functions

$\Rightarrow \langle\mathcal{O}\rangle$ independent of D-terms

Equations of motion $\bar{D}^2\phi = W'(\phi) = 0 \mod \bar{Q}$

E.g. $W(\phi) = \phi^n + \dots \Rightarrow$ chiral ring generated by $1, \phi, \phi^2, \dots, \phi^{n-2}$,

$\phi^{n-1} = \bar{Q}(\dots)$

$\int d^4x\, d^4\theta\ \bar{\Phi}\phi \to \int d^4x\, d^4\theta\ \bar{\Phi} e^V \phi$ $\quad V(x,\theta,\bar\theta)$ adjoint valued SUSY connection

field strength $W_\alpha = -i \bar{D}^2(e^{-V} \underset{\frac{\partial}{\partial\theta^\alpha}}{D_\alpha} e^V)$

~ "$\frac{1}{2}$ form" $\quad$ chiral superfield

$W_\alpha = \lambda_\alpha(x) + \theta^\alpha F_{\alpha\beta}(x) + \dots$

$F_{\alpha\beta} = F_{\beta\alpha} = \gamma^{\mu\nu}_{\alpha\beta} \underset{\text{self-dual part}}{F^+_{\alpha\beta}}$

$\Rightarrow$ SYM action

$S = \frac{1}{32\pi^2} \mathrm{Tr}_{adj}(W_\alpha W^\alpha)$ $\quad$ "glueball field" $\quad S \sim \mathrm{Tr}\,\lambda^2 + \dots + \theta^2\, \mathrm{Tr}(F^+)^2$

$S^{N=1}_{SYM} = \int d^4x\, d^2\theta\, (2\pi i \tau S) + \text{c.c.}$

Chiral ring of this theory (for $G = U(N)$) is generated by $1, \mathrm{Tr}\, W_\alpha, S \sim \mathrm{Tr}\, W^2$

(would be missing in simple G)

operators like $\mathrm{Tr}\, W^3$ are trivial in cohomology $\Big([W_\alpha, \mathcal{O}] = [\bar{Q}_{\dot\alpha}, \partial_{\alpha\dot\alpha}\mathcal{O}] = 0 \mod \bar{Q}$

e.g. $W_\alpha W_\beta + W_\beta W_\alpha = 0 \mod \bar{Q}$

$\Rightarrow W_1, W_2, W_1 W_2$ and everything else is $0 \mod \bar{Q}$

$\Rightarrow \mathrm{Tr}\, W^n = 0 \mod \bar{Q}, \quad n \geq 3 \Big)$

$SU(N)$ .. the chiral ring $[1, S, S^2, S^3, \dots]_{\mathbb{Z}}$

<u>Gaugino condensation</u>

$\langle \mathrm{Tr}\,\lambda^2 \rangle_k \neq 0$ $\quad$ in the chiral ring $\langle S \rangle_k = e^{\frac{2\pi i k}{N}} \Lambda^3$

$\Lambda^3 = e^{-\frac{8\pi^2}{g^2 N}} \Lambda_0^3$

Note: running of coupling constant $e^{-\frac{8\pi^2}{g^2(\mu)}} (\mu)^{3N} = \Lambda^{3N}_{\text{scale}}$

$\Rightarrow$ idea $\quad$ effective superpotential for the glueball field S .. Veneziano, Jankielowicz

$W_{eff}(S) = N S [1 - \log(S/\Lambda^3)]$

$W_{eff}' = 0 \Rightarrow \log(S/\Lambda^3)^N = 0 \Rightarrow S = e^{2\pi i k/N} \Lambda^3$

N critical points reproducing the 1-point functions

Note:

SYM $\quad SU(N)$ $\quad W_\alpha$ chiral field $\quad \{W_\alpha, W_\beta\} = \bar{D}\Theta \approx 0$

$S = \mathrm{Tr}\, W^2 \Rightarrow 1, S, S^2 \dots$ commute

W_α can be simultaneously diagonalised $(w^i_\alpha) \Rightarrow w^i_\alpha w^j_\beta = w^j_\beta w^i_\alpha \Rightarrow$ classically $S^N \approx 0$

Dijkgraaf 3

surely $S^{N^2} = 0$ always

quantum theory $\langle S^N \rangle \sim e^{-\frac{8\pi^2}{g^2}} \Lambda^{3N}$

$\Rightarrow$ in quantum ~~theory~~ chiral ring $S^N \approx \Lambda^{3N}$

$$W_{eff}(S) = NS\left[1 - \log\left(S/\Lambda^3\right)\right]$$

?Describe the behaviour with general matter?

$G = U(N)$ deformed $\mathcal{N}=2 \Rightarrow \mathcal{N}=1$ SYM + 1adj. chiral scalar ϕ

$$\int \mathrm{Tr}\, W(\phi)\, d^4x\, d^4\theta$$

polynomial $W(\phi) = \sum_{k=0}^{n+1} t_k \phi^k$ $\Rightarrow$ broken $\mathcal{N}=2$ to $\mathcal{N}=1$, critical points (we assume isolated)

$\Rightarrow$ n critical points a_i

$$W'(\phi) = \prod_{i=1}^{n} (\phi - a_i)$$

we may assume $t_{n+1} = 1$, $t_0 = 0$ $\Rightarrow$ moduli $t_1, \ldots, t_n$

We don't care about renormalizability $\rightarrow$ effective QFT

ϕ hermitean matrix $\rightarrow$ can be diagonalized $\phi = U^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_N \end{pmatrix} U$

$$\Rightarrow \mathrm{Tr}\, W(\phi) = \sum_{i=1}^{N} W(\lambda_i)$$

$W'(\phi) = 0 \Rightarrow \lambda_i = a_k \Rightarrow$ vacua labelled by partitions $N = N_1 + \ldots + N_n$

$N_k = \#\{\lambda_i = a_k\}$

$$\Rightarrow \phi = \begin{pmatrix} a_1 \mathbb{1}_{N_1\times N_1} & & \\ & \ddots & \\ & & a_n \mathbb{1}_{N_n\times N_n} \end{pmatrix} \Rightarrow U(N) \longrightarrow U(N_1)\times\ldots\times U(N_k)$$

in this vacuum

$U(N_i) \cong U(1)\times SU(N_i) \xrightarrow{Q.M.} U(1)$, $\langle S_i \rangle$ condensate

$\subset$ glue in the i-th block

$\Rightarrow$ abelian $U(1)^n$, gauge fields $F^i_{\mu\nu}$ $i = 1, .., n$ & condensates S_i

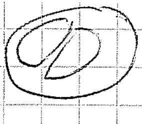
We want to compute (i) effective superpotential $W_{eff}(S_1, .., S_n; t_1, .., t_n)$

(ii) $\tau_{ij}(S_1, .., S_n; t_1, .., t_n)$

(i) $W_{eff} = \sum_{i=1}^{n}\left(N_i \frac{\partial F_0}{\partial S_i} + 2\pi i \tau_0 S_i\right)$

$$F_0(S_i, t_i) = \sum -\tfrac{1}{2} S_i^2 \log\left(S_i/\Lambda_0^3\right) + \overbrace{\sum a_n S^n}^{\text{matter field calculations}}$$

$$\Rightarrow W_{eff} = \sum N_i \underbrace{S_i \log\left(S_i/\Lambda^3\right)}_{\text{VY-term}} + \Sigma \ldots$$

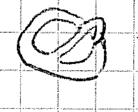
$a_n = \sum\limits_{\substack{\text{planar} \\ \text{Feynman diagrams} \\ \text{at the level } n}}$  $\tilde\tau^{eff}_{ij} = \dfrac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j}$

Feynman diagrams ... 0-dim. theory with action $= \mathrm{Tr}\, W(\phi)$, ϕ. matrix

... F. diags of a matrix model

$$\int\limits_{\tilde N \times \tilde N} d\phi \; e^{-\frac{1}{g_s}\mathrm{Tr}\, W(\phi)}$$ g_s . string coupling $(=\hbar)$

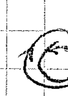
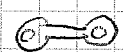
E.g.: $W = \frac{1}{2} m\phi^2 + \frac{1}{3} g \phi^3$ $\phi \cong 0$

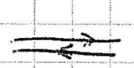
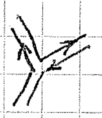
 planar   (e.g.  is non-planar)

factor $\tilde N g_s$ $\forall$ hole ← better boundary - holes on S^2

We want to have S_i $\forall$ hole ⇒ we identify $S_i = \tilde N_i g_s$

$U(N) \to U(1)$, S

$\mathcal{F}_0 = -\frac{1}{2} S^2 \log S/\Lambda^3 +$  $\frac{1}{2} \log m\, S^2 +$   $\frac{1}{3}\frac{1}{m^3} g^2 S^3 + \sum a_n S^n$

 $\frac{1}{m}$  g

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Matrix models

$$Z = \int\limits_{\tilde N \times \tilde N} d\phi \; e^{-\frac{1}{g_s}\mathrm{Tr}\, W(\phi)} \frac{1}{\mathrm{vol}\, U(\tilde N)} \qquad \left(\phi \to U^{-1}\phi U \Rightarrow \frac{1}{\mathrm{vol}\, U(\tilde N)}\right)$$

$$\phi = U^{-1}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{\tilde N}\end{pmatrix} U \qquad d\phi = d\lambda + [\xi, \lambda] \qquad U = e^{\xi}$$

⇒ measure
$$d\phi = dU \cdot \prod_{i=1}^{\tilde N} d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)$$

$$Z = \int d\lambda_i \underbrace{\prod_{i<j}(\lambda_i - \lambda_j)}_{\text{1-loop effect}} e^{-\frac{1}{g_s}\sum_i W(\lambda_i)}$$

1-loop effect   ~ eff. action for λ_i after integrating out the off-diagonal elements

$$= \int d\lambda_i \exp -\frac{1}{g_s} S_{eff}(\lambda_1, \ldots, \lambda_{\tilde N}), \quad S_{eff} = \underbrace{\sum W(\lambda_i) + 2g_s \sum \log(\lambda_i - \lambda_j)}_{\text{"2dim Coulomb gas}}$$

Saddle point $W'(\lambda_i) + 2g_s \sum\limits_{i \neq j} \frac{1}{\lambda_i - \lambda_j} = 0$

⇒ 1. ϕ . complex scalar → λ_i complex

Finally we want $\tilde N \to \infty$, $g_s \to 0$, $g_s \tilde N$ finite add probe to the system $S_{eff}(\lambda_1,\dots,\lambda_{\tilde N}, x)$ ~ effect of the gas on x $\boxed{\text{Dijkgraaf 4}}$

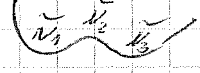
$$y(x) = \frac{\partial S_{eff}}{\partial x} = W'(x) + 2g_s\sum_i \frac{1}{x-\lambda_i}$$

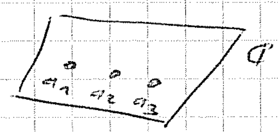
$y\,dx$ 1-form on y-plane

Resolvent : $\omega(x) = \frac{1}{\tilde N}\mathrm{Tr}\frac{1}{x-\phi} = \frac{1}{\tilde N}\sum_i\frac{1}{x-\lambda_i}$

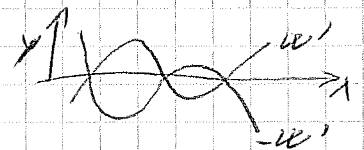
$$\Rightarrow y(x) = W'(x) + 2(g_s\tilde N)\omega(x)$$

Classically $g_s = 0 \Rightarrow W'(\lambda_i) = 0$

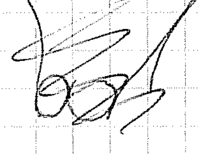


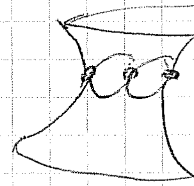


$$\Rightarrow y^2 = (W'(x))^2$$



We reinterpret $y^2 = (W'(x))^2$ as algebraic curve





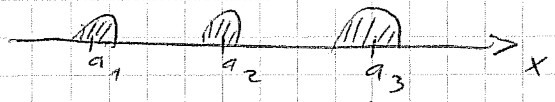
← double points
singular surface

Quantum $g_s = \hbar \to 0$, $\tilde N_i \to \infty$ s.t. $g_s \tilde N_i = S_i$

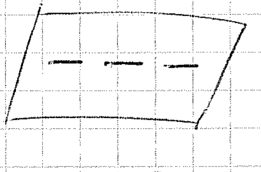
$\Rightarrow$ eigenvalues ~ Fermi liquid

eigenvalue density $\rho_{clas}(x) = \frac{1}{\tilde N}\sum\delta(x-\lambda_i)$ becomes smooth

$\Rightarrow$



$\Rightarrow$ eigenvalue cuts in x-plane



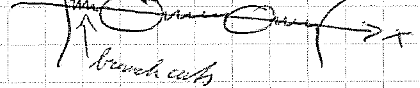
Wigner: $\int d\phi\, e^{-\frac{1}{g_s}\mathrm{Tr}\,\phi^2}$, $W = x^2$

Gaussian ensemble $\Rightarrow$ Semi-circle law $\rho(x) = \frac{\sqrt{S-x^2}}{\pi S}$

$$y^2 = W'(x)^2 \xrightarrow[\text{def.}]{\text{quantum}} y^2 = W'(x)^2 + f(x)$$

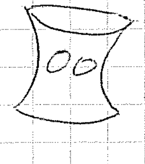
$$f(x) = \sum_{k=0}^{n-1} b_k x^k, \quad b_k \leftrightarrow S_i$$

$\Rightarrow$ single zeros for generic f



$$(\omega(x))^2 = \left(\frac{1}{\tilde N}\sum\frac{1}{x-\lambda_i}\right)^2$$

$\Rightarrow$



a smooth Riemann surface of genus $g = n-1$

$$= \frac{1}{\tilde N^2}\sum_{i\neq j}\frac{1}{(x-\lambda_i)(x-\lambda_j)} + \underbrace{\frac{1}{\tilde N^2}\sum_i\frac{1}{(x-x_i)^2}}$$

double poles $= \frac{1}{\tilde N}\,\omega'(x) \to 0$ as $\tilde N \to \infty$

$$= \tilde{N}^2 \sum_{i \neq j} \frac{\kappa}{(x-\lambda_i)(\lambda_i-\lambda_j)} + O\left(\frac{1}{\tilde{N}^2}\right) \qquad \frac{1}{(x-a)(x-b)} = \frac{1}{(x-a)(a-b)} + \dots$$

$$= \frac{1}{\tilde{N}\tilde{N} g_s} \sum \frac{-w'(\lambda_i) + w'(x) - w'(x)}{x-\lambda_i} \qquad w'(\lambda_i) = -2g_s \sum \frac{1}{\lambda_i - \lambda_j}$$

$$= \frac{1}{\underset{\sum_i S_i}{\underset{\|}{S}}} \frac{1}{\tilde{N}} \sum_i \underbrace{\frac{w'(x) - w'(\lambda_i)}{x-\lambda_i}}_{} + \frac{1}{S}\,\omega(x)\, W'(x)$$

is again a polynomial ($x = \lambda_i \Rightarrow w'() - w'() = 0$) of degree $n-2$

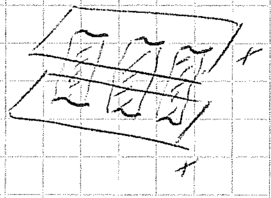
$$\Rightarrow \boxed{y^2 = (W' + 2S\omega)^2 = W'^2_{(x)} + f(x)}$$

terms containing ω — Haven't we dropped $O(\frac{1}{\tilde{N}^2})$ term we would have a nontrivial diff. eqn for ω

Riemann surface (hyperelliptic curve $y^n = \prod_{i=1}^{n} (x-\alpha_i)(x-\beta_i)$)

$$y(x)dx = dS_{\text{eff}}(x)$$

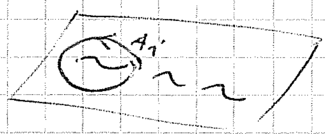
→ 2 sheeted cover of x-plane



$$y(x)dx = dW + 2g_s \sum \frac{1}{x-\lambda_i}$$

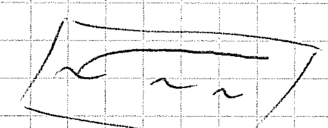
$\Rightarrow$ simple poles at λ_i

contour integral around branch cuts $\frac{1}{2\pi} \oint_{A_i} y\,dx = g_s \tilde{N}_i = S_i$



$\Rightarrow \quad S_i \leftrightarrow b_i \quad \sum b_i x^i = f(x)$

a dual cycle



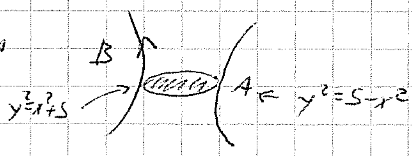
$$A_i \cap B_j = \delta_{ij}$$

$$\frac{1}{2\pi} \int y\,dx = \frac{\partial \mathcal{F}_0}{\partial S_i}$$

$$\Rightarrow \boxed{\begin{array}{l} \text{matrix model} \rightarrow \text{algebraic curve} + \text{1-form } y\,dx \\ \text{periods} \oint_{A_i} y\,dx = S_i \qquad \oint_{B_i} y\,dx = \frac{\partial \mathcal{F}_0}{\partial S_i} \quad \dots \text{special geometry} \end{array}}$$

$y(x) \sim \frac{1}{x-\lambda_i} \quad \Rightarrow \rho(x) = $ discontinuity in $y(x)$

$W = \frac{1}{2} m \phi^2 \quad \Rightarrow y^2 = x^2 + S \quad \rightarrow$ hyperbola, 2 cycles



$$\Rightarrow \frac{\partial \mathcal{F}}{\partial S} = \int^{B} y\,dx \sim S \log S$$

$$\frac{1}{\mathrm{vol}\, U(\tilde{N})} \int_{\tilde{N}\times\tilde{N}} d\phi\, e^{-\frac{m\phi^2}{g_s}} \Rightarrow \mathcal{F}_0 = -\frac{1}{2}\tilde{N}^2 \log\left(\frac{g_s \Lambda}{m}\right) + (-g_s^2) \log \underbrace{\mathrm{vol}\, U(\tilde{N})}_{\sim (\tilde{N})^{\tilde{N}^2}} = -\frac{1}{2} S^2 \log (S/\Lambda^2 m)$$

$\Rightarrow \frac{\partial \mathcal{F}}{\partial S} \sim S \log S$ the same as above

Yesterday:

$\boxed{\text{Dijkgraaf 5}}$

Canonical example: deformed $\mathcal{N}=2 \Longrightarrow \mathcal{N}=1$ SYM + ϕ_{adj}

at tree level $\qquad \int d^2\theta\, d^4x \quad \mathrm{Tr}\, W_{tree}(\phi) \qquad W = \sum_{k=1}^{n+1} t_k \phi^k$

$$U(N) \to U(N_1)\times \ldots \times U(N_n) \to \underset{S_1}{U(1)}\times \ldots \times \underset{S_n}{U(1)}$$

$$W_{eff}(S_i, t_i) = \sum_i [N_i \frac{\partial F_0}{\partial S_i} + 2\pi i\, \tau_0 S_i]$$

$$F_0 = F_0^{gauge} + F_0^{matter}$$

$$F_0^{gauge} = \sum -\tfrac{1}{2} S_i^2 \log(S_i/\Lambda_0^3) \qquad \Lambda_0 \text{ cutoff}$$

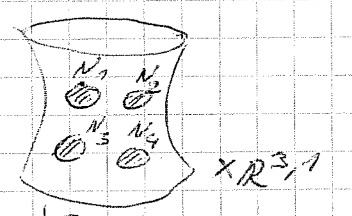
$$F_0^{matter} = \sum_{\text{planar diagrams}} S^{\ell} \quad \ell \leftarrow \#\text{ of loops}$$

matrix model $\to$ curve $\quad y^2 = W'(x)^2 + f(x)$

$$S_i = \oint_{A_i} y\,dx, \qquad \frac{\partial F_0}{\partial S_i} = \int_{B_i} y\,dx$$

From string theory: wrapping D5 branes on CY_3

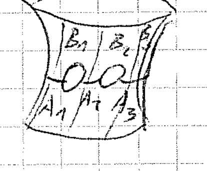
worldsheet $\mathbb{R}^4 \times \mathbb{P}^1$



holography

$n\ \mathbb{P}^1$'s $\to$ n wrapped D5 branes on $\mathbb{P}^1$

$$S_i = \mathrm{Tr}_{U(N_i)} W^2$$



A_i smooth spheres

$\{A_i, B_j\}$ basis of $H_3(CY)$

3-form field $H = H_{RR} + \tau_0 H_{NS}$ $\quad$ fluxes through the branches between spheres

$$\oint_{A_i} H = N_i \qquad \int_{B_i} H = \tau_0\, 2\pi i$$

Ω .. holomorphic (3,0) form $\qquad \oint_{A_i}\Omega = S_i \qquad \oint_{B_i}\Omega = \frac{\partial F_0}{\partial S_i}$

$\Rightarrow$

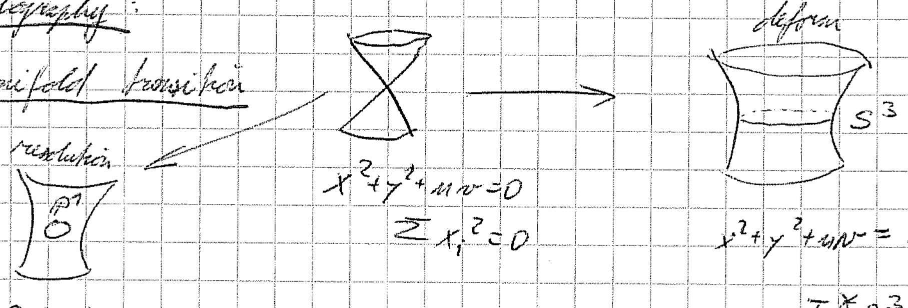
S_i moduli in closed string theory

$$\boxed{W_{eff} = \int_{CY} H\wedge\Omega = \sum_i \Big[\oint_{A_i} H \oint_{B_i}\Omega - \oint_{B_i} H \oint_{A_i}\Omega\Big]} = \sum_i \Big[N_i \frac{\partial F}{\partial S_i} - \tau_0 S_i\, 2\pi i\Big]$$

$\Rightarrow$ the formula above was motivated by string theory

3d holography:

conifold transition



$x^2+y^2+uv=0$

$\sum x_i^2=0$

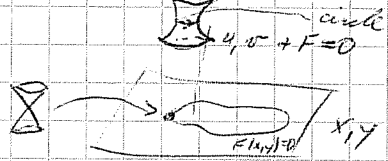
$x^2+y^2+uv=S$

$\ldots\ T^*S^3$

$\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to P^1$

CY: $F(x,y)+uv=0$ in $\mathbb{C}^4$ (affine hypersurface)

$\Rightarrow$ assoc. curve C $\quad F(x,y)=0$



(3,0) form $\Omega = \dfrac{du\wedge dx\wedge dy}{u}$

$\Rightarrow$ periods of Ω $\quad \int_{A_i}\Omega = \int\frac{du\wedge dx\wedge dy}{u} = \int_{F\geq 0} dx\,dy = \int_{F=0} y\,dx$

$\quad //$

$\Rightarrow$ periods of $y\,dx$ on $F=0$

Computation without strings, in field theory only

$\mathcal{N}=1$ SYM, gauge group G, matter fields ϕ in repr. R

$$\int d^4\theta d^4x\,\bar{\Phi}e^V\phi + \int d^2\theta d^4x\, \underset{\substack{\uparrow \\ G\text{-invariant}}}{W(\phi)} + \int d^2\bar{\theta} d^4x\,\overline{W}(\bar{\Phi})$$

} may not be renormalizable (effective low-energy theory)

background gauge field

$$\nabla_\alpha = e^{-V}D_\alpha e^V \qquad D_\alpha = \frac{\partial}{\partial\theta_\alpha}+\ldots$$

$$W_\alpha = -i\bar{D}^2\nabla_\alpha = \lambda_\alpha + \theta^\alpha F_{\alpha\beta}+\ldots$$

chiral ring $\quad S = \mathrm{Tr}\, W_\alpha^2$

claim: $\quad S^h \approx 0 \ (\mathrm{mod}\ \bar{D})$ $\quad$ classically

$\quad S^h \approx \Lambda^{3h}$ $\quad$ quantum

Nk terms ; S^n, $n\geq h$ ambiguous

Compute $W_{eff}(S)$: $\quad \int D\phi D\bar{\phi}\, e^{\int d^4x d^4\theta\,\bar{\phi}e^V\phi + \int d^4x d^2\theta\, W(\phi) + c.c.} = \exp\left[\int d^4x d^2\theta\right.$

$\cdot W_{eff}(S) + \int d^4x d^4\theta \ldots$

$W_{tree} = \sum t_k \phi^k$ $\qquad$ $\overline{W}_{tree} = \sum \bar t_k \bar\phi$ $\qquad$ $\boxed{\text{Dijkgraaf 6}}$

$\Rightarrow W_{eff}(S_i, t_k)$

Proof: e.g. promote constants t_k to chiral fields $\to$ $\bar t_k$ antichiral fields

$\Rightarrow \int W_{eff}(S, t_k)\, d^2\theta$ doesn't care about antichiral fields $\Rightarrow$

One may make $\bar t_k$ indep. of $(t_k)^*$ $\Rightarrow$ $\overline{W} = \frac{1}{2}\bar m \bar\phi^2$, (put $\bar m = 0$?) dangerous

$\bar\Phi$ enters into ~~path integral~~ action

$$\int d^4\theta\, \bar\Phi e^V \phi + \int d^2\bar\theta\, \frac{\bar m}{2}\bar\phi^2$$

we put $\tilde\phi = \bar\phi e^V$ and use $\int d^2\theta K = D^2 K$ $\qquad \to \int d^4\theta\, \frac{\bar m}{2}\tilde\phi \frac{1}{\nabla^2}\tilde\phi = \int d^4\theta\, \frac{\bar m}{2}\tilde\phi \frac{\bar\nabla^2}{\Box_+}\tilde\phi$

$$\Box_+ = \bar\nabla^2\nabla^2 = \nabla_\mu\nabla^\mu - iW^\alpha D_\alpha - \frac{i}{2}\nabla^\alpha W_\alpha$$

eliminate $\tilde\phi$: $\quad \tilde\phi = D^2\phi$

$\Rightarrow$ action just for ϕ $\quad \int d^4x \int d^2\theta \left[W(\phi) + \frac{1}{2\bar m}\phi\Box_+\phi \right]$ $\quad$ where $\Box_+ = \nabla_\mu\nabla^\mu - iW^\alpha D_\alpha$

where we assume $[\nabla_\mu, W^\alpha] = 0$ and substitute $\phi \to e^V\phi$, $\nabla_\alpha \to D_\alpha$

rescale $\quad x \leftarrow \sqrt{\bar m}\, x \qquad \theta \leftarrow \bar m\, \theta \quad \Rightarrow d^4x\, d^2\theta \to \bar m^2 \frac{1}{\bar m^2} d^4x\, d^2\theta$

$\int d\theta \cdot \theta \equiv 1$

$$\Rightarrow \phi(\nabla_\mu\nabla^\mu - iW^\alpha D_\alpha)\phi = \phi\left[\frac{\partial^2}{\partial x_\mu \partial x^\mu} - iW^\alpha\frac{\partial}{\partial\theta^\alpha}\right]\phi$$

$\longrightarrow \bar m\, \phi(\cdots)\phi \qquad \Rightarrow$ indep. of $\bar m$

? $\bar m \overset{!}{=} 0 \Rightarrow$ constraint $\Box_+\phi = 0$ $\xrightarrow[\text{on euclidean space time}]{}$ $\phi \sim$ constant $\Rightarrow$ localize to constant fields ?

anomaly $\Rightarrow$ we rather put $\bar m = 0$

$W = \frac{1}{2}m\phi^2 + g\phi^3 + \dots$

$\Rightarrow$ quadratic pieces: $\quad \frac{1}{2}m\phi^2 + \phi\Box\phi + i\phi W^\alpha D_\alpha\phi$

acts in repr. R

Feynman rules:

1st quantised picture $\quad \underset{\text{coords}}{(x^\mu, \theta^\alpha)} \longleftrightarrow \underset{\text{momenta}}{(p^\mu, \pi^\alpha)} \qquad \pi^\alpha = \frac{\partial}{\partial\theta^\alpha}$

$\xrightarrow[p^\mu, \pi^\alpha]{}$ $\quad$ we assume the length S $\Rightarrow$ propagator $e^{-S[p_\mu^2 + W^\alpha\pi_\alpha + m]}$

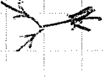
$m\bar m = |m|^2$, $\bar m = 1$

Schwinger param $\Rightarrow \int_0^\infty ds\, e^{-S[\cdots]} = \frac{1}{p^2 + W^\alpha\pi_\alpha + m}$ = propagator

General diagram: give length $\underbrace{S_1, S_2, \dots S_E}_{\#\text{edges}}$ to every line

moduli space of metrics on graph

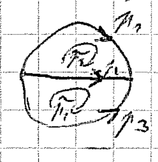
$\xrightarrow[p^\mu, \pi^\alpha]{S_i}$ $\Leftrightarrow$ i^{th} propagator



$\left.\begin{array}{l} p_1 + p_2 + p_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 0 \end{array}\right\} \Rightarrow$ momenta labelled by $\tilde p_a$ ... loop momenta $\quad p_i = \sum_{a=1}^{l} L_{ia}\tilde p_a$ $\quad$ (#loops l)

$\pi_i = \sum L_{ia}\tilde\pi_a$

e.g.



$$p_1 = \tilde{p}_1$$
$$p_2 = -\tilde{p}_1 + \tilde{p}_2$$
$$p_3 = -\tilde{p}_2$$

We shall compute

$$\int ds \int d\tilde{p}\, d\tilde{\pi} \prod_{i=1}^{\#\text{edges}} e^{-s_i [p_i^2 + W\pi_i + m]}$$

$$\sum_i s_i p_i^2 = \sum_i s_i \sum_a \sum_b (L_{ia}\tilde{p}_a)(L_{ib}\tilde{p}_b) = \sum_{a,b} M_{ab}\, \tilde{p}_a \tilde{p}_b$$

$$M_{ab} = \sum_i s_i L_{ia} L_{ib}$$

(in string theory ~ Im(period matrix))

$$\int (d^4\tilde{p})^l\, e^{-M_{ab}\tilde{p}_a\tilde{p}_b} = \frac{1}{(\det M)^2} \leftarrow \frac{\dim \text{ of } \mathbb{R}^{3,1}}{2}$$

e.g. $\ominus$ $\quad s_1(\tilde{p}_1)^2 + s_2(-\tilde{p}_1+\tilde{p}_2)^2 + s_3(-\tilde{p}_2)^2 = (s_1+s_2)\tilde{p}_1^2 - 2s_2\tilde{p}_1\tilde{p}_2 + (s_2+s_3)\tilde{p}_2^2$

$$\Rightarrow M = \begin{pmatrix} s_1+s_2 & -s_2 \\ -s_2 & s_2+s_3 \end{pmatrix} \qquad \det M = s_1 s_2 + s_1 s_3 + s_2 s_3$$

$$\Rightarrow \quad \ominus_{\text{bosonic}} = \frac{1}{(s_1s_2 + s_2s_3 + s_1s_3)^2}$$

$$\int d\tilde{\pi}^\alpha_a\, e^{-s_i W_\alpha L_{ia}\tilde{\pi}_a} \sim s^{2l}\, W^{-2l} \quad \Rightarrow \text{ the whole } \ominus \text{ (bosonic + fermionic) is conformally invariant} \sim s^0$$

$2l$ $\tilde{\pi}$'s

$$\int d\pi\, e^{-sW\pi} = sW$$

$$\Rightarrow \quad \int ds \underbrace{\left[\frac{P_{2l}(s)}{(\det M)^2}\right]}_{\text{homogeneous of degree } 0} e^{\sum s_i m}$$

claim: 1) this diagram is <u>finite</u>

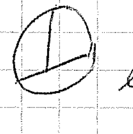
2) recall: in F-terms $\operatorname{Tr} W^{n\geq 3} \approx 0 \Rightarrow W^{2l} \sim [\operatorname{Tr} W^2]^l \sim S^l$

$$\Rightarrow W_{eff} = \sum_{l\text{-loop}} \overset{\text{finite}}{a_l}\, S^l$$

<u>Review of above:</u> F-terms chiral path integral ... functional of $\phi(x,\theta)$ only

involves $\int d^4x\, d^2\theta \left\{ \frac{1}{2}\phi[\Box - iW^\alpha D_\alpha + m]\phi + W_{int}(\phi) \right\}$

$$\frac{}{s} \quad e^{-sH}, \quad H = p_\mu^2 + W^\alpha \pi_\alpha + m$$

 l loops $\quad$ (group factor) $\cdot\, (\operatorname{Tr} W^2)^l \int \prod ds_i \cdot 1 \cdot e^{-(\sum s_i)m}$

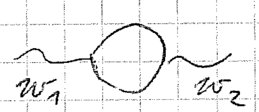
$\Rightarrow$ propagator $\quad — \quad \frac{1}{m}$

vertex $\quad \succ \quad g$

$\mathcal{L} = \mathrm{Tr}\, W^2 = S$

$\Big\}$ 0-dim Feynman rules

E.g. $\quad l = 1$



$$\int d^4p\, d^2\pi \int_\Lambda^\infty \frac{dS}{6}\, e^{-S(p^2 + W^\alpha \pi_\alpha + m)} =$$

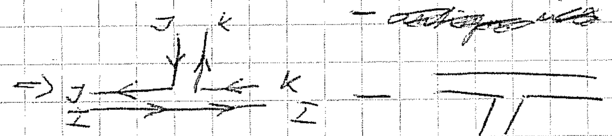
$$= \int_\Lambda^\infty \frac{dS}{6}\, \frac{S^2\, \mathrm{Tr}\, W^2}{S^2}\, e^{-Sm} = \mathrm{Tr}_{adj}\, W^2 \log\left(\frac{m}{\Lambda}\right)$$

$\qquad \cong h$

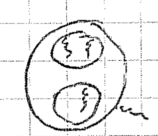
E.g. $\quad G = U(N)$, ϕ in adj. repr.

$\mathrm{Tr}\{\phi [W^\alpha, D_\alpha \phi]\}$ $\quad$ if $\begin{smallmatrix} I \\ J \end{smallmatrix} \rightleftarrows \Leftrightarrow \phi_{IJ} \Rightarrow \phi_{IJ} W^\alpha_{JK} D_\alpha \phi_{KI}$

$-$ ~~antisym...~~ ...

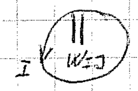


if we assume $W_{int} = \mathrm{Tr}(\phi^3) \Rightarrow$

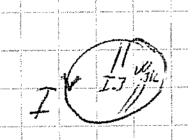


$\overbrace{\#\text{ index loop}}^{\text{number of holes}} = h \qquad$ quantum loops $= l$

$\sum_{I=1}^{N} \circlearrowleft I = N = \mathrm{Tr}(1)$



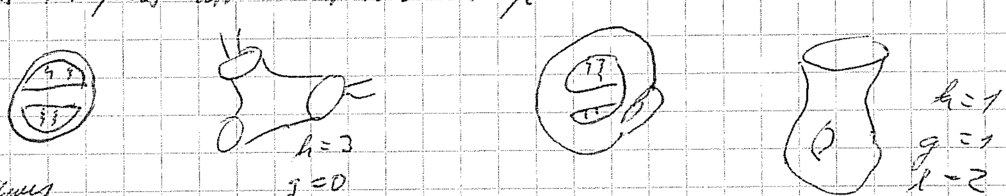
$\delta_{IJ} W^\alpha_{IJ} = \mathrm{Tr}\, W^\alpha$



$\delta_{II} W^\alpha_{IJ} W^\beta_{JI} = \mathrm{Tr}(W^\alpha W^\beta)$

$(\mathrm{Tr}\, W^2)^l \rightarrow$ W's in pairs attached to index loop

e.g.



$h = 3$, $g = 0$, $l = 2$ $\qquad$ $h = 1$, $g = 1$, $l = 2$

h, g genus

$l = 2g + h - 1$

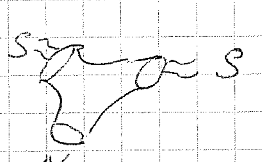
$\Rightarrow h = l + 1 - 2g$

For $\boxed{h \geq l} \Rightarrow g = 0 \Rightarrow$ only planar diagrams contribute

$\Rightarrow h = l + 1$

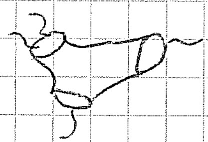
$\Rightarrow$ factor $(l+1) S^l N$

$\qquad\quad \overset{\shortparallel}{N} \frac{\partial}{\partial S}(S^{l+1})$



$S \quad S \quad N$

$$\Rightarrow \quad W_{eff} = \sum a_\ell (\ell+1) S^\ell N = N \frac{\partial}{\partial S}\Big[\underbrace{\sum a_\ell S^{\ell+1}}_{\mathcal{F}_0^{matter}(S)}\Big]$$

Another option  $\sim \binom{\ell+1}{2} S^{\ell-1} (\mathrm{Tr}\, W)(\mathrm{Tr}\, W) = \tilde{\tau}_{eff}(S)\, F_{\mu\nu}^2$ effective coupling of $U(N) \to U(1)$

$F \equiv \mathrm{Tr}\, F \qquad F \equiv \mathrm{Tr}\, F$

$$\tau_{eff} = \frac{\partial^2}{\partial S^2}\Big[\sum a_\ell S^{\ell+1}\Big] = \frac{\partial^2}{\partial S^2} \mathcal{F}_0$$

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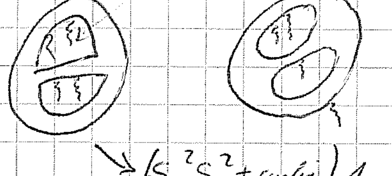
shaded $\} \sim e^{-S_i W \pi_i}$



$$\frac{(S_1^2 S_2^2 + \text{cyclic})\, 1}{(S_1 S_2 + S_2 S_3 + S_1 S_3)^2} \qquad \frac{(S_1^2 S_2 S_3 + \text{cyclic})\, 2}{(S_1 S_2 + S_2 S_3 + S_1 S_3)^2}$$

Note $S_i \to 0$ isn't sufficient to create singularity $\Rightarrow$ we must put e.g. $S_1 \cong S_2 \cong \varepsilon \to 0$

$$\Rightarrow \quad \frac{\varepsilon^2 S_3^2 + \varepsilon^3 (\ldots)}{(2\varepsilon \cdot S_3)^2} \longrightarrow \frac{S_3^2}{4 S_3^2} \quad \text{finite}$$

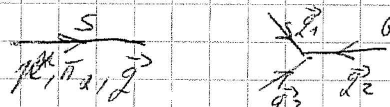
Similarly for higher loops

Abelian gauge field $\quad W_\alpha W_\beta = - W_\beta W_\alpha$ (exactly, not mod $\bar{D}$)

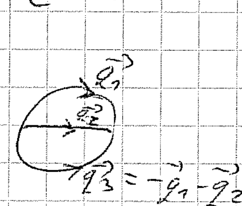
$$\Rightarrow W_\alpha^{n} = 0 \qquad n > M$$

$$G \to U(1)^M$$

$\phi_{rep\text{-}R} \to$ charged fields $\phi \quad \vec{q} = (q_1, \ldots, q_n)$

$\Rightarrow$  $\quad$ G-invariant $\sum \vec{q}_i = 0 \qquad \vec{W}_\alpha = (W_\alpha^I,\ I = 1, \ldots, M)$

$$e^{-S(p^2 + \vec{W}_\alpha \cdot \vec{q}\, \pi_\alpha + m)}$$

 $\quad \Rightarrow$ "loop charges" $\quad \vec{q}_i = \sum_a L_{ia} \vec{q}_a$

introduced before for p^μ, π^α

$$\Rightarrow \quad e^{-\sum_i S_i (L_{ia} p_a)^2 + \vec{W}_\alpha (L_{ia} \vec{q}_a)(L_{ib} \pi_b^\alpha)}$$

$$= e^{-(M_{ab} p_a p_b + \vec{W}_\alpha \cdot M_{ab} \vec{q}_a \pi_b^\alpha)}$$

$$\Rightarrow \int d^4 p\, d^2 \pi\, (\ldots) = \int d^4 p\, d^2 \pi\, e^{-(p_a p_a + \vec{W}_\alpha \cdot \vec{q}_a \pi_a^\alpha)}\, e^{-\sum S_i m}$$

$p \to M^{-\frac{1}{2}} p$

$\pi \to \ldots \pi \Rightarrow d^4 p\, d^2 \pi$ invariant

$\Rightarrow$ (group factor) $\cdot \prod_{\alpha, a} (\vec{v}_\alpha \cdot \vec{q}_a) \int \prod dS_i \, e^{-\sum S_i \epsilon_m}$

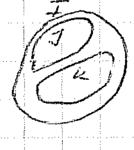
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \left(\frac{1}{m}\right)^{\#\text{ of prop}}$

Dijkgraaf P

$(\mathrm{Tr}\, W_1 W_2)^l \det_{l\times l} (\vec{q}_a \cdot \vec{q}_b)$

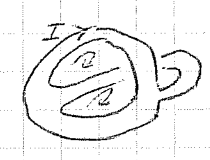
only $\neq 0$ if $\vec{q}_1, \dots, \vec{q}_l$ linearly independent

$\Rightarrow$ defines (up to n loops) analogue of planar diagrams !



ϕ_{IJ} $\quad q_1 = \vec{e}_I - \vec{e}_J \quad \vec{e}_1, \dots, \vec{e}_N$ basis of $\mathbb{R}^N$

$\quad\quad q_2 = \vec{e}_J - \vec{e}_K$



1 index loop $\quad \phi_{II} \quad e_I - e_I = 0 \Rightarrow$ chargeless field

$\rightarrow$ invisible in abelian theory $\rightarrow$ zero

$\#$ of indep. group theory indices $= h \geq l$

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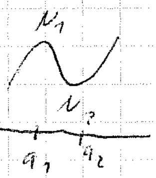
single S $\quad U(N) \rightarrow U(1)$ ... perturbation around $\Phi = 0$

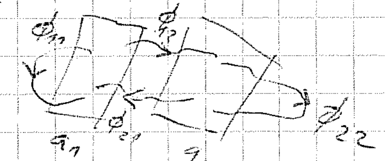


$U(N) \rightarrow U(N_1) \times U(N_2)$ $\quad \Phi = \begin{pmatrix} a_1 \mathbb{1}_{N_1\times N_1} & 0 \\ 0 & a_2 \mathbb{1}_{N_2 \times N_2} \end{pmatrix} + \underbrace{\Phi^{qu}}_{\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}}$

$\Rightarrow$ spontaneous symmetry breaking

$W = \frac{1}{2}(a_1 - a_2)\phi^2 + \frac{1}{3}\phi^3$ $\quad\quad \approx$

branes



i.e. $W = \frac{1}{2}(a_1 - a_2)(\phi_{11}^2 - \phi_{22}^2) + O(\phi^3)$

$\Rightarrow \phi_{12}, \phi_{21}$ zeromodes

... vacuum manifold $\frac{U(N)}{U(N_1)\times U(N_2)}$ $\quad \dim(\dots) = 2N_1 N_2$

Gauge fixing $\quad \phi_{12} = \phi_{21} = 0 \quad \delta\phi_{12} = [c, \phi]_{12} \Rightarrow \int d^4x \, d^2\theta \, \mathrm{Tr}(b[\phi, c])$

$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

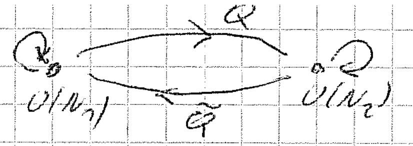
$\Rightarrow$

$W = (a_1 - a_2)\, \mathrm{Tr}(b_{12} c_{21} + c_{12} b_{21})$

$\Rightarrow$ non-interacting branes but ghosts stretching between them

Def. $\phi_{11} = \phi_+, \; \phi_{22} = \phi_-, \; (b_{12}, c_{12}) = Q_i, \; (b_{21}, c_{21}) = \tilde{Q}_i$

$\Rightarrow W \sim m\phi_+^2 - m\phi_-^2 + Q_i \phi_+ \tilde{Q}_i + \tilde{Q}_i \phi_- Q_i + O(\phi_\pm^3)$



Prescription: replace $U(N)$ by supergauge group $U(N_1|N_2)$ (unitary transf. on $\mathbb{C}^{N_1|N_2}$)

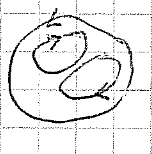
in adjoint rep. $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ $\qquad \phi_{11}, \phi_{22}$ bosons

$\qquad \phi_{21}, \phi_{12}$ fermions

$STr(\phi) = Tr\,\phi_{11} - Tr\,\phi_{22}$

$Tr\,F_{\mu\nu}^2 \Rightarrow$ non-unitary theory but well-behaved

Claim: in many respects (F-terms) $U(N_1|N_2) \simeq U(\underbrace{N_1-N_2}_{N^{eff}})$ $\quad$ might be <0 !



index loop $\quad Tr\,\mathbb{1} = N$

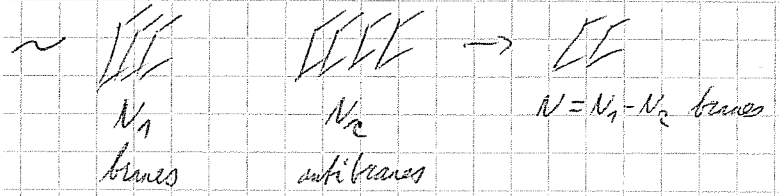
$STr\,\mathbb{1} = N_1 - N_2$

$U(N_1|N_2) \to U(N_1) \times U(N_2)$ $\quad$ assume $\phi = \begin{pmatrix} \phi_{11} & 0 \\ 0 & \phi_{22} \end{pmatrix}$ $\quad$ i.e. $\phi_{12} = \phi_{21} = 0 \Rightarrow$ ghosts which are bosons !

$\Rightarrow Q_i, \tilde{Q}_i$ are bosons (bifundamentals of $U(N_1) \times U(N_2)$)

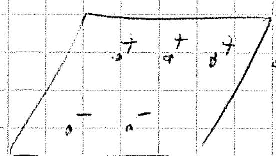
$\quad$ – quiver gauge theory

The model $\; m\phi_+^2 - m\phi_-^2 + Q_i\,\phi_+\,\tilde{Q}_i + \tilde{Q}_i\,\phi_-\,Q_i$ was proposed as holographic dual to conifold theory



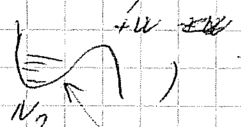
$N_1, N_2 \to \infty, \quad N_1 - N_2 = N$

Claim: matrix model for $U(N_1|N_2)$ is <u>equivalent</u> to $U(N_1-N_2)$ model

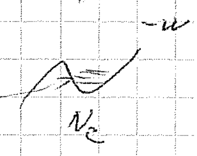


← eigenvalues for the supergroup have both positive & negative charge

$\Rightarrow$ + eigenvalues fill



, −eigenvalues



Dirac sea

"Stabilization" in K-theory ("study of $\underbrace{V_1 - V_2}_{\mathbb{C}\text{ vector spaces}}$")

$N_1, N_2 \gg 1 \Rightarrow$ perturbative computation $\Rightarrow S^\ell \neq 0$ for arbitrary ℓ, no problem with nilpotency of S

$\Rightarrow$ stabilization $\cong$ matrix model $\cong$ anomaly $\cong$ string theory

Klemm 1

Topological string theory on Calabi-Yau geometries

Literature: Clay School on Mirror Symmetry AMS 2003

Birmingham, Blau, Thomson, Phys-Rep. 91

I. Principals of topological theories

Motivation: basic object we want to calculate $Z[X,J] = \int \mathcal{D}X \, e^{iS[X,J]}$

$$S[X,J] = \int \mathcal{L}(X) + XJ \quad \text{sources}$$

$\Rightarrow$ correlation functions

$$\langle 0 | X(x_1) \dots X(x_n) | 0 \rangle = \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J}\right) \dots \left(-i\frac{\delta}{\delta J}\right) Z[J] \Big|_{J=0}$$

knowledge of all correlation functions $\cong$ solving the theory

However $\mathcal{D}X \begin{cases} \text{contains all classical sol}^{\text{ns}} \\ \text{contains all quantum fluctuations} \end{cases}$

One may try semiclassical approximation $S[X] = S[X_{cl}] + \frac{(\delta X)^2}{2}\frac{\delta^2}{\delta X^2} S[X_{cl}] + O(\delta X^3)$

$$\Rightarrow Z[J=0] = \int \mathcal{D}X \, e^{iS[X]} \approx \sum\!\!\!\!\!\!\int_{\substack{\text{classical}\\ \text{sol}^{\text{ns}}}} e^{iS[X_{cl}]} \int (\delta X)^2 \underbrace{e^{-i\frac{(\delta X)^2}{2}\frac{\delta^2}{\delta X^2} S[X_{cl}]}}_{\text{Gaussian}}$$

$$\cong \sum\!\!\!\!\!\!\int_{\text{classical sol}^{\text{ns}}} e^{iS[X_{cl}]} \det\left(\frac{\delta^2 S}{\delta^2 X}\right)$$

Note: $\sum\!\!\!\!\!\!\int$ .. integral over moduli of instantons (= classical sol$^{\text{ns}}$)

How good is this approximation? Is there a class of operators so that $Z[X,J]_{\text{semiclass}}$ is a good generator for the correlations?

Supersymmetry and localization

0-dim QFT $\qquad S = \frac{1}{2}(\partial h)^2 - (\partial^2 h)\psi_1 \psi_2$

$h(x)$, X boson $\qquad \psi_i$ fermions . Grassman variables

$$Z = \int dX \, d\psi_1 d\psi_2 \, e^{-S}$$

$\int \psi_1 \psi_2 \, d\psi_2 \, d\psi_1 = -1 \qquad$ zero otherwise

SUSY transformation

$X \longrightarrow X + \delta X, \; \delta X = \varepsilon(\psi_1 + \psi_2) \qquad \varepsilon$ .. Grassmann param.

$\psi_i \to \psi_i + \delta\psi_i, \; \delta\psi_1 = \varepsilon \, \partial h, \; \delta\psi_2 = -\varepsilon \, \partial h$

$\Rightarrow \left.\begin{array}{l} \delta S = (\partial h)\partial^2 h \, \varepsilon(\psi_1 + \psi_2) - \partial^2 h(\varepsilon \partial h \, \psi_2 - \psi_1 \varepsilon \, \partial h) = 0 \\ \delta(dX d\psi_1 d\psi_2) = 0 \end{array}\right\} \Rightarrow$ all correlation functions invariant

For $\partial h \neq 0 \qquad \hat{X} = X - \frac{\psi_1 \psi_2}{\partial h} \qquad \hat{\psi}_1 = \psi_1 \qquad \hat{\psi}_2 = \psi_1 + \psi_2$

$\Rightarrow \quad \delta \hat{X} = 0 = \delta \hat{\psi}_2, \quad \delta \hat{\psi}_1 = \varepsilon \, \partial h$, we put $\varepsilon = -\frac{\hat{\psi}_1}{\partial h}$

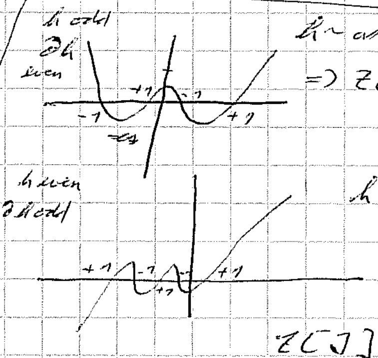
$\Rightarrow \quad S(\hat{X}, 0, \hat{\psi}_2) \Rightarrow Z[0] = 0$ outside points $\partial h = 0$

Critical points of h: $h = h(x_c) + \frac{\alpha_c}{2}(x-x_c)^2 + \dots$

locally $h \sim \frac{\alpha}{2}(x-x_c)^2 \Rightarrow \int \frac{dX}{\sqrt{2\pi}}\, d\psi_1\, d\psi_2\, e^{-\frac{1}{2}\alpha X^2 + \alpha \psi_1 \psi_2} = \int \frac{dX}{\sqrt{2\pi}}\, \alpha\, e^{-\frac{1}{2}\alpha X^2}$

$$= \frac{\alpha}{|\alpha|} = \mathrm{sign}(\alpha)$$

Sum over critical points $\quad Z = \sum_{\partial h(x_c)=0} \mathrm{sign}\left(\partial^2 h\big|_{x_c}\right)$



$h \sim a x^n$, n odd $\Rightarrow Z[J] = 0$

$h \sim a x^n$, $a > 0$, n even

$Z[J] = 1$

$a < 0$ $\quad Z[J] = -1$

Moral of toy example

- contributions to Z localize due to SUSY
- we calculate an index which is independent of deformations keeping boundary conditions in infinity

1-dimensional supersymmetric field theory (SUSYQM)

$X(t),\ \psi_1(t),\ \psi_2(t) \qquad L = \frac{1}{2}\dot{X}^2 - \frac{1}{2}(h'(x))^2 + i(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''\bar{\psi}\psi$

$$\bar{\psi} = \psi_1 - i\psi_2, \quad \psi = \psi_1 + i\psi_2$$

momenta $\quad \Pi = \frac{\partial L}{\partial \psi} = i\bar{\psi} \qquad p = \frac{\partial L}{\partial \dot{x}} = \dot{x} \qquad \{\Pi, \psi\} = i \qquad \{X, P\} = i$

Supersymmetry transformation $\quad \delta_\varepsilon X = \varepsilon\bar{\psi} - \bar{\varepsilon}\psi \qquad \delta_\varepsilon \psi = \varepsilon(i\dot{X} + h'(x)) \qquad \delta_\varepsilon \bar{\psi} = \bar{\varepsilon}(-i\dot{X} + h$

$\varepsilon = \varepsilon_1 + i\varepsilon_2$

$$\Rightarrow \delta_\varepsilon S = \int dt\, \delta L = \int dt \frac{d}{dt}(\dots) = 0$$

Hilbert space operators corresponding to δ_ε .. $\quad Q = \bar{\psi}(i\dot{X} + h'(x)) \qquad \bar{Q} = \psi(-i\dot{X} + h'(x))$

$\delta_\varepsilon X = \varepsilon[Q, X] \qquad \delta_\varepsilon \psi_{1,2} = \varepsilon\{Q, \psi_{1,2}\} \qquad$ 2 real SUSY charges

Hamiltonian $\quad H = p\dot{x} + \Pi\dot{\psi} - L \Rightarrow H$ is expressible in terms of Q: $H = \frac{1}{2}\{Q, \bar{Q}\}, \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0$

$\Rightarrow$ BRST operator $\quad \mathcal{H}_F \xrightarrow{Q} \mathcal{H}_B \xrightarrow{Q} \mathcal{H}_F \xrightarrow{Q} \mathcal{H}_B \dots \qquad \mathbb{Z}_2$ graded complex

grading $(-1)^F$: $\quad \{Q, (-1)^F\} = \{\bar{Q}, (-1)^F\} = 0 \qquad F$ .. operator for counting # of fermi

In partial $\quad Q_1 = Q + Q^\dagger \qquad Q_1^2 = 2E \Rightarrow$ invertible if $E > 0$

$$\Rightarrow Q: \mathcal{H}_B \cong \mathcal{H}_F \quad \text{for } E > 0$$

$Q^2 = 0 \Rightarrow$ cohomology $\quad H_B(Q) = \frac{\mathrm{Ker}\ \mathcal{H}_B \to \mathcal{H}_F}{\mathrm{Im}\ \mathcal{H}_F \to \mathcal{H}_B} = H_B^{(E=0)}(Q) = \bigoplus_{\text{even}} \mathcal{H}_p$

$$H_F(Q) = \frac{\mathrm{Ker}\ \mathcal{H}_F \to \mathcal{H}_B}{\mathrm{Im}\ \mathcal{H}_B \to \mathcal{H}_F} = H_F^{(0)}(Q) = \bigoplus_{p\ \text{odd}} \mathcal{H}_p$$

The ground state of theory is characterized by cohomology of Q

$$\mathcal{H} = \bigoplus_{p=0} \mathcal{H}_p \qquad (F\mathcal{H}_p = p\mathcal{H}_p)$$

$$\boxed{\mathrm{Tr}_{\mathcal{H}}(-1)^F = \sum_p (-1)^p \dim \mathcal{H}_p \quad \text{Witten index}}$$

Klemm 2

Dictionary $\psi_j \psi_k \cong d\theta_j \wedge d\theta_k = -d\theta_k \wedge d\theta_j \cong -\psi_k \psi_j$

F measures degree of the form

Path integral formulation of e.g. Witten index

SQM on S^1

$$\mathrm{Tr}\, e^{-\beta H} = \int \mathcal{D}X\, d\psi_2\, d\bar\psi_2\, e^{-S}\qquad S = \int_0^\beta \mathcal{L}_E dt$$

$X(0) = X(\beta)$

$\left.\begin{matrix}\psi(0) = -\psi(\beta)\\ \bar\psi(0) = -\bar\psi(\beta)\end{matrix}\right\}$ antiperiodic

can't be SUSY because of antiperiodic boundary conditions $\sim \epsilon$ cannot be antisymmetric

$\Rightarrow$ add $(-1)^F$

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int \mathcal{D}X\, d\psi\, d\bar\psi\, e^{-S_E}$$

$X(0) = X(\beta)$

$\left.\begin{matrix}\psi(0) = -\psi(\beta)\\ \bar\psi(0) = -\bar\psi(\beta)\end{matrix}\right\}$ periodic

... this object defines Witten index $\to \mathrm{Tr}(-1)^F$ $E>0$ boson $\xleftrightarrow[\text{pairing}]{}$ fermion

$\mathrm{Tr}(-1)^F e^{-\beta H}$ can be proved doesn't depend on β

$$\partial_\beta \mathrm{Tr}(-)^F e^{-\beta H} = \int_{\text{periodic}} \mathcal{D}X\, d\psi\, d\bar\psi\ H\, e^{-S_E} = \frac{1}{2}\int \mathcal{D}X d\psi d\bar\psi\ \underbrace{\{Q,\bar Q\}}_{\delta \bar Q} e^{-S_E}$$

$$\overset{SUSY}{\cong} 0$$

$\Rightarrow$ we can take $\beta \to 0 \Rightarrow$ only time independent modes contribute $\to$

0-dim SUSY FT ... "secretly lower dimensional quantities"

1-dimensional σ-model on the target space M

$$L = \frac{1}{2} g_{IJ}\, \dot x^I \dot x^J + \frac{1}{2} g_{IJ} (\bar\psi^I D_t \psi^J - D_t \bar\psi^I \psi^J) \qquad M\text{ - Riemannian manifold}$$

$$-\frac{1}{2} R_{IJKL}\, \psi^I \bar\psi^J \psi^K \bar\psi^L \qquad D_t \psi^I = \partial_t \psi^I + \Gamma^I_{KL}\, \partial_t X^K \psi^L$$

SUSY invariant $\quad \delta X^I = \epsilon \bar\psi^I - \bar\epsilon \psi^I$

$$\delta \psi^I = \epsilon(i \dot X^I - \Gamma^I_{KL}\, \bar\psi^K \psi^L)$$

$$\delta \bar\psi^I = \bar\epsilon(-i\dot X^I - \Gamma^I_{KL}\, \bar\psi^K \psi^L)$$

Algebra $\quad [X^I, P_J = \frac{\partial L}{\partial \dot x_J}] = \delta^I_J \qquad \{\psi^I, \psi^J\} = g^{IJ}$

Quantization $\quad X^I \to x^I\cdot \qquad \bar\psi^I \to dx^I \wedge$

$P_I \to i\nabla_I \qquad \psi^I \to g^{IJ}\frac{\partial}{\partial x^J}$

States $\quad |0\rangle = 1$

$\bar\psi^I|0\rangle = dx^I$ ... $\bar\psi^1 \ldots \bar\psi^n |0\rangle = dx^1 \wedge \ldots \wedge dx^n$

$\mathcal{H}^{(E>0)} = H^*(Q) \cong H^*_{dR}(M)$

$Q = i\psi^I P_I \cong d \qquad \bar Q = -i\bar\psi^I P_I \cong d^+ \qquad H = \frac{1}{2}\{Q, \bar Q\} = \frac{1}{2}\Delta$

$\mathrm{Tr}(-1)^F = \chi(M)$ Euler characteristic

1-dim σ-models on Kähler manifold M $\underbrace{\qquad\qquad}_{SQM}$

Kähler ... complex manifold with a hermitian metric g s.t. $\exists$ 2-form $\omega(X,Y) = g(JX,Y)$ (complex structure J),

$$\omega = i g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}, \quad d\omega = 0$$

$\Rightarrow$ locally $\exists$ Kähler potential K with $g_{i\bar{j}} = i\, \partial_i \partial_{\bar{j}} K$ $\qquad \partial_i g_{\bar{j}k} = \partial_{\bar{j}} g_{ik}$

$\Gamma^i_{jk} = g^{i\bar{l}}\, \partial_j g_{k\bar{l}}$ pure in the indices

$$L = g_{i\bar{j}} \dot{X}^i \dot{X}^{\bar{j}} + g_{i\bar{j}} \bar{\psi}^{\bar{j}} D_t \psi^i + i g_{i\bar{j}} \bar{\psi}^i D_t \psi^{\bar{j}} + R_{i\bar{j}k\bar{l}}\, \bar{\psi}^i \psi^k \psi^{\bar{j}} \psi^{\bar{l}}$$

$\delta X^i = \varepsilon_+ \bar{\psi}^i - \varepsilon_- \psi^i \qquad \delta \bar{X}^i = -\bar{\varepsilon}_+ \psi^i + \bar{\varepsilon}_- \psi^i$

$\delta \psi^i = i \bar{\varepsilon}_- \dot{X}^i - \varepsilon_+ \Gamma^i_{jk} \bar{\psi}^j \psi^k$

$\delta \bar{\psi}^{\bar{i}} = -i \varepsilon_- \dot{X}^{\bar{i}} - \bar{\varepsilon}_+ \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}^{\bar{j}} \psi^{\bar{k}}$

$\delta \bar{\psi}^i = -i \varepsilon_+ \dot{X}^i - \varepsilon_- \Gamma^i_{jk} \psi^k \psi^j - \varepsilon_+ \Gamma^i_{jk} \bar{\psi}^j \psi^k \qquad (??)$

$\delta \psi^{\bar{i}} = -i \bar{\varepsilon}_+ \dot{X}^i - \bar{\varepsilon}_- \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}^{\bar{j}} \psi^{\bar{k}}$

SUSY generators $\qquad Q_+ = g_{i\bar{j}} \psi^i \dot{X}^{\bar{j}} \qquad Q_- = g_{i\bar{j}} \bar{\psi}^i \dot{X}^{\bar{j}}$

$\qquad \bar{Q}_+ = g_{i\bar{j}} \bar{\psi}^{\bar{j}} \dot{X}^i \qquad \bar{Q}_- = g_{i\bar{j}} \psi^{\bar{j}} \dot{X}^i$

$Q = i(Q_- + \bar{Q}_+) \qquad \bar{Q} = -i(\bar{Q}_- + Q_+)$

axial & vector symmetry generators ν, α

$\bar{\psi}^i \longrightarrow e^{i(-\nu+\alpha)} \bar{\psi}^i \qquad \bar{\psi}^{\bar{i}} \to e^{i(\nu+\alpha)} \bar{\psi}^{\bar{i}}$

$\psi^i \longrightarrow e^{i(\nu-\alpha)} \psi^i \qquad \psi^{\bar{i}} \to e^{-i(\nu+\alpha)} \psi^{\bar{i}}$

$\Rightarrow$ Noether currents

$F_A = g_{i\bar{j}} (\bar{\psi}^{\bar{j}} \psi^i + \bar{\psi}^i \psi^{\bar{j}}) = g_{IK} \psi^I \psi^K$

$F_V = g_{i\bar{j}} (\psi \psi + \psi \psi) = -i g_{IK} \psi^I J^K_L \psi^L$

$[x_j, p_k] = i \delta^j_k \qquad [x^{\bar{i}}, p_{\bar{j}}] = i \delta^{\bar{i}}_{\bar{j}} \qquad \{\psi^i, \bar{\psi}^{\bar{j}}\} = g^{\bar{j}i}, \qquad \{\psi^{\bar{i}}, \bar{\psi}^j\} = g^{\bar{i}j}$

$\bar{\psi}^i \leftrightarrow dz^i \wedge, \qquad \bar{\psi}^{\bar{i}} \leftrightarrow d\bar{z}^{\bar{i}} \wedge, \qquad \psi^i \leftrightarrow g^{i\bar{j}} \frac{\partial}{\partial \bar{z}^{\bar{j}}}, \qquad \psi^{\bar{i}} \leftrightarrow g^{\bar{i}j} \frac{\partial}{\partial z^j}$

$\eta \in \Omega^{p,q}(M) \qquad \eta = \eta_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q}\, dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{i}_1} \wedge \ldots \wedge d\bar{z}^{\bar{i}_q}$

$$\sim \bar{\psi}^{i_1} \ldots \bar{\psi}^{i_p} \bar{\psi}^{\bar{i}_1} \ldots \bar{\psi}^{\bar{i}_q} |0\rangle = |\eta\rangle$$

$F_A |\eta\rangle = (q-p) |\eta\rangle \qquad F_V |\eta\rangle = (q+p) |\eta\rangle$ ... measure Dolbeault decomposition of form

States have natural decomposition $\bigoplus_{p,q=1}^{M} \Omega^{p,q}(M)$

Cohomology of SQM and M

$Q_- = \bar{\psi}^j p_j \leftrightarrow i\partial = dz^j \wedge i \frac{\partial}{\partial z^j}, \quad \bar{Q}_+ = \bar{\psi}^{\bar{j}} p_{\bar{j}} \leftrightarrow i\bar{\partial} = d\bar{z}^{\bar{j}} \wedge i \frac{\partial}{\partial \bar{z}^{\bar{j}}}$

$$\Omega^{p,q} \xrightarrow{\partial} \Omega^{p+1,q} \qquad \Omega^{p,q} \xrightarrow[\bar\partial]{} \Omega^{p,q+1}$$

Klinn 3

$$\bar{Q}_- = Q_-^{+} \leftrightarrow i\partial^+ \qquad Q_+ = \bar{Q}_+^{+} \leftrightarrow i\bar\partial^+$$

$$H = \underset{\Delta_\partial}{\{Q_+, \bar{Q}_+^*\}} = \underset{\Delta_{\bar\partial}}{\{Q_-, \bar{Q}_-^*\}} = \underbrace{\tfrac{1}{2}\{Q, Q^+\}}_{=\frac{1}{2}\Delta}$$

$$\mathcal{H}^{p,q}(M) = H_\partial^{p,q}(M) = H_{\bar\partial}^{p,q}(M) \quad \text{Q.K.} \qquad \mathcal{H} = \bigoplus_{p,q} \mathcal{H}^{p,q}(M)$$

Some remarks on cohomological theories

homology $\leftrightarrow$ cohomology ? are there quantities which do not depend on the representative in $[\alpha]$, $\alpha \sim \alpha + d\gamma$, $\alpha \in H^p$
or in $[A]$, $A \sim A + \partial C$

Invariant quantities are period integrals $\int_A \alpha + d\gamma = \int_A \alpha + \underbrace{\int_{\partial A} \gamma}_{=0}$ if $\boxed{\partial A = 0}$

and similarly $\int_{A+\partial C} \alpha = \int_A \alpha + \int_{\partial C} \alpha = \int_A \alpha + \underbrace{\int_C d\alpha}_{=0}$ if $\boxed{d\alpha = 0}$

SQM on M of any topological theory defined by Q with $Q^2 = 0$

$$\langle 0|\mathcal{O}_1 \dots \mathcal{O}_n|0\rangle \qquad [\mathcal{O}_i] \quad \mathcal{O}_i \sim \mathcal{O}_i + \{Q, \gamma\}, \quad \{\mathcal{O}_i, Q\} = 0$$

$$\langle 0|\mathcal{O}_1 \dots \mathcal{O}_i + \{Q, \gamma\} \dots \mathcal{O}_n|0\rangle = 0 \qquad Q|0\rangle = 0$$

We make a choice to restrict ourselves to topological subsector. ? Does it contain interesting correlation functions?

A and B topological models

(Witten hep-th/9112056)

Consider 2-d σ-model on a Calabi-Yau 3-fold M: $X: \Sigma_g \longrightarrow M$
Σ_g: worldsheet, oriented Riemann surface

$$S = t\int_{\Sigma_g} dz \Big(\tfrac{1}{2} g_{IJ} \partial_z X^I \partial_{\bar z} X^J + i\psi_-^{\bar i} D_z \psi_-^i g_{i\bar i} + \psi_+^{\bar i} D_{\bar z} \psi_+^i g_{i\bar i} + R_{i\bar i j \bar j} \psi_+^i \psi_+^{\bar i} \psi_-^j \psi_-^{\bar j} \Big)$$

$$\psi_+^i \in \Gamma(K^{1/2} \oplus X^* T^{10}M) \qquad \psi_+^{\bar i} \in \Gamma(K^{1/2} \oplus X^* T^{01}M)$$

$$\psi_-^i \in \Gamma(\bar{K}^{1/2} \oplus X^* T^{10}M) \qquad \psi_-^{\bar i} \in \Gamma(\bar{K}^{1/2} \oplus X^* T^{01}M)$$

where K is the canonical bundle on Σ_g

$$D_{\bar z} \psi^i = \frac{\partial}{\partial \bar z} \psi^i + \partial_{\bar z} X^j \Gamma^i_{jk} \psi^k$$

$N=(2,2)$ supersymmetry $\quad \varepsilon_+, \bar{\varepsilon}_+ \in K^{-1/2} \quad \varepsilon_-, \bar{\varepsilon}_- \in K^{1/2}$ susy params

$\delta X^i = i\varepsilon_- \psi_+^i + i\psi_-^i \qquad \delta X^{\bar{i}} = i\varepsilon_- \psi_+^{\bar{i}} + i\bar{\varepsilon}_+ \psi_-^{\bar{i}}$

$\delta \psi_+^i = -\bar{\varepsilon}_- \partial_z X^i - i\varepsilon_+ \bar{\psi}^j \Gamma^i_{jm} \psi_+^m \qquad \delta\psi_+^{\bar{i}} = -\varepsilon_- \partial_{\bar{z}} X^{\bar{i}} - i\bar{\varepsilon}_+ \psi_-^j \Gamma^{\bar{i}}_{\bar{j}\bar{m}} \psi_+^{\bar{m}}$

$\delta\psi_-^i = -\bar{\varepsilon}_- \partial_{\bar{z}} \cancel{X}^i - i\varepsilon_+ \psi_+^j \Gamma^i_{jk} \psi_+^k \qquad \delta\psi_-^{\bar{i}} = \varepsilon_+ \partial_{\bar{z}} X^{\bar{i}} - i\varepsilon_- \psi_+^j \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \psi_-^{\bar{k}}$

$\delta_\varepsilon W = i\varepsilon [Q, W] \qquad Q^2 = 0 \qquad \{Q_\pm, \bar{Q}_\pm\} = H \pm P$ (probably some mistakes)

$Q_+, Q_-, \bar{Q}_+, \bar{Q}_-$

also axial & vector symmetry $F(x, \theta^\pm, \bar{\theta}^\pm)$ $\quad$ axial: $e^{i\alpha F_A} : (\theta^\pm, \bar{\theta}^\pm) \to (e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\bar{\theta}^\pm)$

vector: $e^{i\nu F_V} : (\theta^\pm, \bar{\theta}^\pm) \to (e^{-i\nu}\theta^\pm, e^{i\nu}\bar{\theta}^\pm)$

For arbitrary Riemann surface susy is broken by the antisymmetric boundary conditions for fermions

for susy one needs $\nabla\varepsilon = 0$ & we don't have them on Σ_g $\to$ modify transf. properties of spinors

$\delta_\varepsilon S = \int \nabla_\mu \varepsilon G^\mu$ (G^μ: supercurrent) $\quad$ and keep the Q generators

<u>topological twisting</u> $\quad$ we take the ψ's be sections of different bundles

on + fields (right fields) $\quad$ <u>+ twist</u> $\quad \psi_+^i, \psi_+^{\bar{i}} \in \Gamma(K^{1/2} \otimes X^* T^{10} M), \Gamma(K^{1/2} X^* T^{01} M)$

$\downarrow$

$\psi_+^i, \psi_+^{\bar{i}} \in \Gamma(X^* T^{10} M), \Gamma(K \otimes X^* T^{01} M)$

<u>− twist</u> $\quad \psi_+^i, \psi_+^{\bar{i}} \longrightarrow \psi_+^i, \psi_+^{\bar{i}} \in \Gamma(K \otimes X^* T^{10} M), \Gamma(X^* T^{01} M)$

similarly one may ± twist the − fields $\psi_-^i, \psi_-^{\bar{i}} \Rightarrow 4$ possibilities

interesting:

A-twist $\quad$ (+right, −left) $\Longrightarrow \begin{cases} \psi_+^i \in \Gamma(X^* T^{10} M) \\ \psi_+^{\bar{i}} \in \Gamma(K \otimes X^* T^{01} M) \\ \psi_-^{\bar{i}} \in \Gamma(X^* T^{01} M) \\ \psi_-^i \in \Gamma(K \otimes X^* T^{10} M) \end{cases}$

B-twist $\quad$ (−right, −left) $= \begin{cases} \psi_+^i \in \Gamma(K \otimes X^* T^{10} M) \\ \psi_+^{\bar{i}} \in \Gamma(X^* T^{01} M) \\ \psi_-^i \in \Gamma(\bar{K} \otimes X^* T^{10} M) \\ \psi_-^{\bar{i}} \in \Gamma(X^* T^{01} M) \end{cases}$

Symmetries for model on $\mathbb{R}^2$ $\qquad SO(2) \times \underbrace{U_V(1) \times U_A(1)}_{R\text{-symmetries}}$

operators $\qquad S \qquad F_V \qquad F_A$

twisting — new $S' = S + R \qquad$ A-twist $\; S' = S + F_V \qquad$ B-twist $\; S' = S + F_A$

$\Rightarrow$

	$U(1)_V$	$U(1)_A$	$U(1)_S$	K	$U(1)_{S'}$ (A)	K	$U(1)_{S'}$ (B)	K
X	0	0	0	1	0	1	0	1
ψ_-	−1	1	1	$K^{1/2}$	0	1	2	K
$\bar{\psi}_+$	1	1	−1	$\bar{K}^{1/2}$	0	1	0	1
$\bar{\psi}_-$	1	−1	1	$K^{1/2}$	2	K	0	1
ψ_+	−1	−1	−1	$\bar{K}^{1/2}$	−2	$\bar{K}$	−2	$\bar{K}$

Illinum 9

						(A)		(B)
Q_-	-1	1	1	$K^{\frac{1}{2}}$	0	1	2	K
$\bar{Q}_+$	1	1	-1	$\bar{K}^{\frac{1}{2}}$	0	1	0	1
$\bar{Q}_-$	1	-1	1	$K^{\frac{1}{2}}$	2	K	0	1
Q_+	-1	-1	-1	$\bar{K}^{\frac{1}{2}}$	-2	$\bar{K}$	-2	$\bar{K}$

A-model $\quad Q_A = Q_- + \bar{Q}_+$ $\qquad$ B-model $\quad Q_B = \bar{Q}_- + \bar{Q}_+$

$\Rightarrow$ susy operators on curved manifolds $\qquad Q_B^2 = Q_A^2 = 0$

BRST operators