Symmetries and Invariant Solutions of the Supersymmetric Sine–Gordon Equation

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Assume that an open neighborhood $U \subset R^n$ with coordinates x^i is given. Consider the graph of a given smooth function $f : U \to \mathbb{R}$ as a section of the (trivial) fibre bundle $\mathcal{J}^{(0)} = U \times \mathbb{R}$, $\sigma_f(\vec{x}) = (\vec{x}, f(\vec{x}))$. It naturally induces a section of the jet bundle, e.g. for the 2nd order jet bundle $\mathcal{J}^{(2)} = U \times \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}^{n(n+1)/2}$

$$\sigma_f^{(2)}(\vec{x}) = (\, \vec{x}, f(\vec{x}), \partial_i f(\vec{x}), \partial_{ij} f(\vec{x}) \,)$$

(The interchangeability of mixed derivatives is assumed throughout.)

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(The interchangeability of mixed derivatives is assumed throughout.)

Let u be the coordinate on $\mathbb{R}$, together with u_i and u_{ij} defining the coordinates on the fibre of $\mathcal{J}^{(2)}$.

Let[1] $\mathbf{v} = \xi^i \partial_i + \mathcal{U}\partial_u$ be the generator of a one–parametric group of transformations of $\mathcal{J}^{(0)}$. Assume that the graph of f and consequently the section σ_f is transformed by the flow of $\mathbf{v}$, defining a new function $\hat{f}(\tau)$ for each value of the flow parameter τ provided $|\tau|$ is small enough. Consider the prolongations $\sigma^{(2)}_{\hat{f}(\tau)}$. Are they generated from $\sigma^{(2)}_f$ by flow of some vector field on $\mathcal{J}^{(2)}$?

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$$\partial_i \equiv \frac{\partial}{\partial x^i}, \qquad \partial_u \equiv \frac{\partial}{\partial u}$$

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Yes, the sought after vector field on $\mathcal{J}^{(2)}$ has the form of the 2nd prolongation

$$\mathrm{pr}^{(2)}(\mathbf{v}) = \xi^i \partial_i + \mathcal{U}\partial_u + \mathcal{U}^i \partial_{u_i} + \mathcal{U}^{ij}\partial_{u_{ij}},$$

where

$$\mathcal{U}^i = \mathcal{D}_i \mathcal{U} - \sum_j \mathcal{D}_i \xi^j u_j, \qquad \mathcal{U}^{ij} = \mathcal{D}_j \mathcal{U}^i - \sum_k \mathcal{D}_j \xi^k u_{ik}, \quad (1)$$

and $\mathcal{D}_i$ is the operator of the total derivative

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When does the vector field $\mathbf{v} = \xi^i \partial_i + \mathcal{U}\partial_u$ generate a one–parametric group of symmetries of a given 2nd order PDE

$$F(\vec{x}, f(\vec{x}), \partial_i f(\vec{x}), \partial_{ij} f(\vec{x})) = 0 \text{ ?} \tag{2}$$

In other words start with an arbitrary solution f of PDE (2). When do the functions $\hat{f}(\tau)$ solve the same PDE (2), for any choice of f?

Provided that $\operatorname{grad} F|_{F=0} \neq 0$ on $\mathcal{J}^{(2)}$ there is an equivalence

$\mathbf{v} = \xi^i \partial_i + \mathcal{U}\partial_u$ is a symmetry generator of PDE (2) if and only if

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Computation of symmetry generators of PDEs in practice

1. For given K–order PDE $F = 0$ find the prolongation of order K of an arbitrary vector field $\mathbf{v}$ on $\mathcal{J}^{(0)}$.
2. Solve $F(\vec{x}, u, u_i, u_{ij}, \ldots) = 0$ for suitable "derivative" $u_{AB\ldots}$ and substitute for it and all its differential consequences, e.g. $\mathcal{D}_i u_{AB\ldots}$, into
$$\left(\mathrm{pr}^{(K)}(\mathbf{v})F\right)\left(\vec{x}, u, u_i, u_{ij}, \ldots\right) = 0.$$
3. The resulting expression is an equation for unknown functions $\xi^i(x^j, u), \mathcal{U}(x^j, u)$ which must hold for any value of the remaining jet coordinates $u_i, u_{ij}, \ldots$. This gives an overdetermined system of linear PDEs for $\xi^i, \mathcal{U}$. If it can be solved we find all symmetry generators of the given PDE $F = 0$.

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(N=1) Supersymmetric sine–Gordon equation

A prototype of nonlinear supersymmetric field equation. The dependent variable is a real bosonic superfield

$$\Phi(x, t, \theta_1, \theta_2) = \frac{1}{2}u(x, t) + \theta_1\phi(x, t) + \theta_2\psi(x, t) + \theta_1\theta_2 F(x, t).$$

The N=1 supersymmetric sine–Gordon equation (SSG) reads

$$D_x D_t \Phi = \sin\Phi \tag{3}$$

where the covariant derivative operators are

$$D_x = \partial_{\theta_1} + \theta_1\partial_x \qquad \text{and} \qquad D_t = \partial_{\theta_2} + \theta_2\partial_t.$$

The quantities x, t, ϕ, F have bosonic (even, commuting) character, $\theta_1, \theta_2, \phi, \psi$ are fermionic (odd, anticommuting).

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Supersymmetry transformations

It is invariant under the supersymmetry transformations

$$x \to x - \underline{\eta}_1\theta_1,\ \theta_1 \to \theta_1 + \underline{\eta}_1,\ t \to t - \underline{\eta}_2\theta_2,\ \theta_2 \to \theta_2 + \underline{\eta}_2,$$

where $\underline{\eta_1}$ and $\underline{\eta_2}$ are arbitrary constant fermionic parameters.

These transformations are generated by the infinitesimal supersymmetry generators

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Component equations

SSG (3) can be equivalently written in terms of the component fields

$$\begin{aligned} &\text{(i) } u_{xt} = -\sin u + 2\phi\psi \sin\left(\frac{u}{2}\right), \\ &\text{(ii) } \phi_t = -\psi \cos\left(\frac{u}{2}\right), \\ &\text{(iii) } \psi_x = \phi \cos\left(\frac{u}{2}\right) \end{aligned} \tag{5}$$

and

$$F = -\sin\left(\frac{u}{2}\right).$$

Which formulation allows to find the supersymmetry generators by an analogue of the usual method of computation of symmetry generators for PDEs?

Symmetries in the component approach

Based on the methods developed in
M. A. Ayari, V. Hussin and P. Winternitz, *J. Math. Phys.* **40**, 1951 (1999),
V. Hussin, *Mathematics Newsletter (India)* **10**, 47–57 (2000),
N. Alvarez–Moraga and V. Hussin, *J. Phys. A: Math. Gen.* **36**, 9479–9506 (2003).

Take the vector field

$$\begin{aligned}\mathbf{v} =&\xi(x,t,u,\phi,\psi)\partial_x + \tau(x,t,u,\phi,\psi)\partial_t + \mathcal{U}(x,t,u,\phi,\psi)\partial_u \\ &+ \Sigma(x,t,u,\phi,\psi)\partial_\phi + \Psi(x,t,u,\phi,\psi)\partial_\psi,\end{aligned} \tag{6}$$

where ξ, τ and $\mathcal{U}$ are bosonic functions while Σ and Ψ are fermionic.

Application of the algorithm is then a straightforward generalization of the one for bosonic variables. Note that the anticommuting quantities are present only in the dependent variables and corresponding jet space coordinates, so that in each term in the prolongation formulae (1) is at most one anticommuting object. Consequently, (almost) no ordering ambiguities arise.

BUT by an explicit computation we find only the Poincaré group in $1+1$ dimensions

$$P_x = \partial_x,\ P_t = \partial_t,\ D = 2x\partial_x - 2t\partial_t - \phi\partial_\phi + \psi\partial_\psi.$$

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Where did the supersymmetry go?

The problem is that the supersymmetry doesn't act on the component fields u, ψ, ϕ, F as a point transformation, transforming one graph into another. It involves derivatives since under supersymmetry transformation (for simplicity taking $\eta_2 = 0$) we have

$$\delta u \sim \eta_1 \phi, \qquad \delta \phi \sim \frac{1}{2} \eta_1 u_x,$$

$$\delta \psi \sim -\eta_1 F, \qquad \delta F \sim \eta_1 \psi_x,$$

which due to the presence of u_x obviously cannot be rewritten as point transformation (even if we use the equations of motion). Consequently, the supersymmetry acts more like a contact transformation and doesn't show up in the computation of point ones.

Can this be improved in the superfield formalism?

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Superfield approach – the form of the generator

Explicitly, the SSG (3) reads

$$\theta_1\theta_2\Phi_{xt} - \theta_2\Phi_{t\theta_1} + \theta_1\Phi_{x\theta_2} - \Phi_{\theta_1\theta_2} = \sin\Phi, \tag{7}$$

where each successive subscript (from left to right) indicates a successive left partial derivative.

We generalize the method of prolongations so as to include also the independent fermionic variables. We write

$$\begin{aligned}\mathbf{v} =&\xi(x,t,\theta_1,\theta_2,\Phi)\partial_x + \tau(x,t,\theta_1,\theta_2,\Phi)\partial_t + \rho(x,t,\theta_1,\theta_2,\Phi)\partial_{\theta_1}\\ &+ \sigma(x,t,\theta_1,\theta_2,\Phi)\partial_{\theta_2} + \Lambda(x,t,\theta_1,\theta_2,\Phi)\partial_\Phi,\end{aligned} \tag{8}$$

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We need the fermionic analogues $\mathcal{D}_{\theta_1}, \mathcal{D}_{\theta_2}$ of the bosonic total derivatives $\mathcal{D}_x, \mathcal{D}_t$, e.g.

$$\begin{aligned}\mathcal{D}_{\theta_1} =& \partial_{\theta_1} + \Phi_{\theta_1}\partial_{\Phi} + \Phi_{x\theta_1}\partial_{\Phi_x} + \Phi_{t\theta_1}\partial_{\Phi_t} + \Phi_{\theta_2\theta_1}\partial_{\Phi_{\theta_2}} + \\ &+ \Phi_{xx\theta_1}\partial_{\Phi_{xx}} + \Phi_{xt\theta_1}\partial_{\Phi_{xt}} + \Phi_{x\theta_2\theta_1}\partial_{\Phi_{x\theta_2}} + \\ &+ \Phi_{tt\theta_1}\partial_{\Phi_{tt}} + \Phi_{t\theta_2\theta_1}\partial_{\Phi_{t\theta_2}},\end{aligned} \tag{9}$$

We note that due to the use of left derivatives the chain rule for a Grassmann–valued function $f(g(x))$ is

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The 2nd prolongation

Now we can write the superspace version of the prolongation formulae. With proper respect for ordering they read

$$\begin{aligned}\mathrm{pr}^{(2)}\mathbf{v} = & \xi\partial_x + \tau\partial_t + \rho\partial_{\theta_1} + \sigma\partial_{\theta_2} + \Lambda\partial_\Phi + \Lambda^x\partial_{\Phi_x} + \\ & + \Lambda^t\partial_{\Phi_t} + \Lambda^{\theta_1}\partial_{\Phi_{\theta_1}} + \Lambda^{\theta_2}\partial_{\Phi_{\theta_2}} + \Lambda^{xx}\partial_{\Phi_{xx}} + \\ & + \Lambda^{xt}\partial_{\Phi_{xt}} + \Lambda^{x\theta_1}\partial_{\Phi_{x\theta_1}} + \Lambda^{x\theta_2}\partial_{\Phi_{x\theta_2}} + \Lambda^{tt}\partial_{\Phi_{tt}} + \\ & + \Lambda^{t\theta_1}\partial_{\Phi_{t\theta_1}} + \Lambda^{t\theta_2}\partial_{\Phi_{t\theta_2}} + \Lambda^{\theta_1\theta_2}\partial_{\Phi_{\theta_1\theta_2}}\end{aligned} \tag{10}$$

where the components are

$$\Lambda^A = \mathcal{D}_A\Lambda - \sum_B \mathcal{D}_A\zeta^B\Phi_B, \qquad \Lambda^{AB} = \mathcal{D}_B\Lambda^A - \sum_C \mathcal{D}_B\zeta^C\Phi_{AC}, \tag{11}$$

and $A, B, C \in \{x, t, \theta_1, \theta_2\}$, $\zeta^A = (\xi, \tau, \rho, \sigma)$.

Applying the second prolongation (10) to the equation (7), we obtain the following condition

$$\begin{aligned}\rho\left(\theta_2\Phi_{xt}+\Phi_{x\theta_2}\right)-\sigma\left(\theta_1\Phi_{xt}+\Phi_{t\theta_1}\right)-\Lambda\left(\cos\Phi\right)\\+\Lambda^{xt}\left(\theta_1\theta_2\right)+\Lambda^{t\theta_1}\left(\theta_2\right)-\Lambda^{x\theta_2}\left(\theta_1\right)-\Lambda^{\theta_1\theta_2}=0.\end{aligned} \tag{12}$$

Next, we substitute the SSG equation (12), i.e. eliminate $\Phi_{\theta_1\theta_2}$, and proceed as before, carefully keeping track of the ordering.

As was anticipated, we have found the full super–Poincaré algebra in $(1+1)$ dimensions, spanned by the generators

$$\begin{aligned}L=-2x\partial_x+2t\partial_t-\theta_1\partial_{\theta_1}+\theta_2\partial_{\theta_2},\qquad P_x=\partial_x,\qquad P_t=\partial_t,\\ Q_x=-\theta_1\partial_x+\partial_{\theta_1},\qquad Q_t=-\theta_2\partial_t+\partial_{\theta_2}.\end{aligned} \tag{13}$$

Applying the second prolongation (10) to the equation (7), we obtain the following condition

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Construction of invariant solutions of PDEs

The knowledge of a 1–parametric group of symmetries of given PDE (such that its orbits have dimension one in the space of independent variables) allows a reduction of the number of independent variables.

It works as follows: one finds the invariants I_k, $k = 1, .., n$ of the action of the group on $\mathcal{J}^{(0)}$, and constructs the coordinates on $\mathcal{J}^{(0)}$ out of them and one additional function of original independent variables, say ω. One of the invariants is chosen as the new dependent variable $\tilde{u} \equiv I_n$.

Once the PDE is expressed in these new dependent and independent variables, one assumes that the sought solution is invariant with respect to the action of the group, i.e. $\tilde{u}$ depends only on $I_1, \ldots, I_{n-1}$.

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Construction of invariant solutions of PDEs

The general theory guarantees that such reduction is consistent and we obtain a PDE with one less independent variables. Repeating this procedure one is able to reduce PDE to ODE provided suitable symmetry group is present at each step.

Of course, this procedure allows to find only special solutions of the original PDE, namely those invariant with respect to some 1–parametric symmetry group. But for nonlinear PDEs it is often the only known method giving at least some nontrivial solutions.

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Invariant solutions of SSG

- It is possible to reduce SSG without any difficulty to a system of ODEs when the 1–parametric subgroup is constructed out of bosonic generators L, P_x, P_t. Whether or not at least particular nontrivial solutions of these ODEs and the corresponding invariant solutions of SSG can be found explicitly depends on the chosen subgroup.
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Example

Consider the transformations generated by $\underline{\mu} Q_x$. The invariants are t, θ_2, Φ and any quantity of the form

$$\tau = \underline{\mu} f\left(x, t, \theta_1, \theta_2, \Phi\right).$$

It is an open question as to whether or not these can lead to a reduced system of equations expressible in terms of the invariants. It is clearly not possible for every function f. E.g., taking $\tau = \underline{\mu} x \theta_1$ we get

$$\underline{\mu} x \theta_2 \mathcal{A}_{t\tau} + \underline{\mu} x \mathcal{A}_{\tau\theta_2} + \sin \mathcal{A} = 0, \qquad (14)$$

for the field

$$\Phi = \mathcal{A}\left(t, \tau, \theta_2\right).$$

The presence of the variable x in equation (14) demonstrates that we do not obtain a reduced equation expressible in terms of the invariants only.

Similar problem arises for the transformations generated by $P_x + \underline{\mu} Q_x + \underline{\nu} Q_t$ or $P_x + \varepsilon P_t + \underline{\mu} Q_x + \underline{\nu} Q_t$ and others.

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Conclusions

- With proper care it is possible to extend the conventional approach to the search for symmetries of PDEs to the superspace.
- It seems to be the only approach in which the supersymmetry demonstrates itself as point symmetry.
- In the case of the super–sine–Gordon equation no hidden, unexpected symmetries were found.
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Thank you for your attention