Superintegrability and time - dependent integrals

Ondřej Kubů and **Libor Šnobl**

Czech Technical University in Prague

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Abstract

While looking for additional integrals of motion of several minimally superintegrable systems in static electric and magnetic fields, we have realized that in some cases Lie point symmetries of Euler–Lagrange equations imply existence of explicitly time–dependent integrals of motion through Noether's theorem. These integrals allow a completely algebraic determination of the trajectories (including their time dependence) although the systems don't exhibit maximal superintegrability in the usual sense.

Report on work in progress, based on bachelor thesis of Ondřej Kubů.

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Integrability

A classical Hamiltonian system in n degrees of freedom is called **integrable** if it admits n functionally independent integrals of motion in involution.

Superintegrability

A classical Hamiltonian system in n degrees of freedom is **superintegrable** if it admits $n+k$ functionally independent integrals of motion (where $k \leq n-1$), out of which n are in involution.

Introduction, cont'd

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In these definitions, the Hamiltonian as well as the integrals are assumed to be functions on the phase space, i.e. time independent.

For time–dependent Hamiltonians also the integrals may naturally be explicitly time dependent. However, does it make any sense to consider time–dependent integrals also for time–independent Hamiltonians? Can they somehow naturally arise and can they be actually useful?

Introduction, cont'd

In particular: one may try to search for previously unknown integrals of the considered Hamiltonian system in the following way:

1. find point symmetries of the corresponding Euler–Lagrange equations (we need second or higher order equations to be able to determine symmetries algorithmically, thus Hamilton's equations are not suitable for this purpose),

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1. find point symmetries of the corresponding Euler–Lagrange equations (we need second or higher order equations to be able to determine symmetries algorithmically, thus Hamilton's equations are not suitable for this purpose),
2. among them find the ones which preserve the action, not only the Euler–Lagrange equations,

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1. find point symmetries of the corresponding Euler–Lagrange equations (we need second or higher order equations to be able to determine symmetries algorithmically, thus Hamilton's equations are not suitable for this purpose),
2. among them find the ones which preserve the action, not only the Euler–Lagrange equations,
3. associate to them integrals of motion via Noether's Theorem, first in Lagrangian formalism and next rewrite them in Hamiltonian mechanics.

Introduction, cont'd

We find that for some systems considered in A. Marchesiello & L.Š J. Phys. A: Math. Theor. 50, 245202 (2017) we construct in this way time–dependent integrals.

The main open question: **Is it just some pecularity of these systems or does it happen also in some more general situations?**

Point symmetries of ODEs

Let us first review what are the **point symmetries** of ODEs. The key concepts are the following:

1. We are given an ODE (or a set of them) of order p

$$F(x, y(x), y'(x), \dots, y^{(p)}(x)) = 0.$$

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Let us first review what are the **point symmetries** of ODEs. The key concepts are the following:

1. We are given an ODE (or a set of them) of order p

$$F(x, y(x), y'(x), \dots, y^{(p)}(x)) = 0. \tag{1}$$

2. Let $y(x)$ be a function on the domain $M \subset \mathbb{R}$. Its graph is the following subset of $M \times \mathbb{R}$

$$\Gamma_y = \{(x, y(x)) \,|\, x \in M\}. \tag{2}$$

We define also the k^{th} **prolonged graph** of the function y

$$\Gamma_y^{(k)} = \left\{ \left(x, y(x), y'(x), \dots, y^{(k)}(x) \right) \,|\, x \in M \right\} \subset M \times \mathbb{R}^{1+k}$$

and we denote the coordinates on the k^{th} **jet space** $\mathcal{J}^{(\mathbf{k})} \equiv M \times \mathbb{R}^{1+k}$, $k \geq 0$, by $x, u, u', \dots, u^{(k)}$.

Point symmetries of ODEs

3. We consider a 1–parameter (local) group of transformations of the space of dependent and independent variables $\mathcal{J}^{(0)}$, i.e. of u and x,

$$t \triangleright (x, u) = (\hat{x} = g_1(x, u; t), \hat{u} = g_2(x, u; t)), \qquad (3)$$

Such transformations are called **point transformation**.

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The effect of such a transformation on any function $y : M \to \mathbb{R}$ is defined using the transformation of the graph of the function $y(x)$,

$$\Gamma_{t \triangleright y} = t \triangleright \Gamma_y \equiv \{(g_1(x, y(x); t), g_2(x, y(x); t)) \,|\, x \in M\}. \tag{4}$$

We shall assume that the 1-parameter group (3) is generated by its generator X, i.e. the vector field

$$X = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}, \quad \text{s.t.} \quad t \triangleright (x, u) \equiv \Phi_X^t(x, u).$$

Point symmetries of ODEs

4. Both the original graph and the transformed graph can be extended to their k^{th} prolongation. We may ask ourselves whether for a given generator X there exists a vector field $\mathrm{pr}^{(k)}X$ on $\mathcal{J}^{(k)}$ such that for every function $y : M \to \mathbb{R}$ we have

$$\Gamma^{(k)}_{t \triangleright y} = \Phi^{t}_{\mathrm{pr}^{(k)}X}\left(\Gamma^{(k)}_{y}\right). \tag{5}$$

That means that the vector field $\mathrm{pr}^{(k)}X$ should encode in itself the fact that the derivatives $u'(x), \ldots, u^{(n)}(x)$ in the differential equation (1) transform in a unique way once a point transformation (3) is chosen.

Point symmetries of ODEs - prolongation of the generator

Such $\mathrm{pr}^{(k)}X$ on $\mathcal{J}^{(k)}$, called the k^{th} **prolongation** of the vector field X, indeed exists and is given by the formula

$$\mathrm{pr}^{(k)}X = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \sum_{j=1}^{k} \eta^{(j)}(x,u,u',\ldots,u^{(j)})\frac{\partial}{\partial u^{(j)}} \tag{6}$$

where the components $\eta^{(j)}(x,u,u',\ldots,u^{(j)})$ are constructed recursively

$$\eta^{(j)}(x,u,u',\ldots,u^{(j)}) = \mathcal{D}_x\eta^{(j-1)} - u^{(j)}\mathcal{D}_x\xi \tag{7}$$

using the operator of total derivative

$$\mathcal{D}_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + \sum_{j=1}^{k-1} u^{(j+1)}\frac{\partial}{\partial u^{(j)}}.$$

Point symmetries of ODEs - definition

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Actually, this "definition" needs to be taken with a grain of salt – not all functions can be mapped by a generic point transformation. Thus we consider the (local) 1–parameter group of point transformations and consider only functions on which it can act when the group parameter t is sufficiently close to 0.

Point symmetries of ODEs - theorem

Theorem (On generators of symmetries of ODEs)

Let $M \subset \mathbb{R}$ *and let* $F : \mathcal{J}^{(k)} \to \mathbb{R}$ *define a differential equation*

$$F(x, u(x), u'(x), \dots, u^{(k)}(x)) = 0. \tag{8}$$

Let

$$\Sigma_F = \{(x, u, u', \dots, u^{(k)}) \in \mathcal{J}^{(k)} | F(x, u, u', \dots, u^{(k)}) = 0\}$$

and $\mathrm{d}F(v) \neq 0, \quad \forall v \in \Sigma_F$. *Then a* **vector field** $X \in \mathfrak{X}(\mathcal{J}^{(0)})$ **generates a local 1–parameter group of point symmetries** *of the differential equation* (8) *if and only if*

$$\mathrm{pr}^{(k)} F(v) = 0, \quad \forall v \in \Sigma_F. \tag{9}$$

Point symmetries of ODEs - theorem

N.B.: For more dependent variables and thus systems of ODEs everything works almost the same, just indices a, α labelling u_a and F_α appear and summations over the index a in the definitions of $\mathcal{D}_x$ and $\mathrm{pr}^{(k)}X$ show up.

Noether's Theorem in Lagrangian mechanics

We use geometrical formulation of Lagrangian dynamics on the evolution space $TM \times \mathbb{R}$ (where M is the configuration space of our system). Assuming regularity of the given Lagrangian L we encode the dynamics in the **dynamical vector field**

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \Lambda^i(q^i, \dot{q}^i, t) \frac{\partial}{\partial \dot{q}^i} \tag{10}$$

where

$$\Lambda^i(q^i, \dot{q}^i, t) = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)^{-1} \left(- \frac{\partial^2 L}{\partial \dot{q}^j \partial q^k} \dot{q}^k - \frac{\partial^2 L}{\partial \dot{q}^j \partial t} + \frac{\partial L}{\partial q^j} \right). \tag{11}$$

Its integral curves after projection to the extended configuration space $M \times \mathbb{R}$ give us solutions of the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0. \tag{12}$$

Cartan's forms

We define the **Cartan's 1–form**

$$\theta = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^i}(\mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t) \in \Omega^1(TM \times \mathbb{R}), \qquad (13)$$

and **Cartan's 2–form** $\mathrm{d}\theta$. The dynamical vector field Γ can then be characterized equivalently by the conditions

$$i_\Gamma \mathrm{d}\theta = 0, \quad \langle \Gamma, dt \rangle = 1, \qquad (14)$$

i.e. Γ is a suitably normalized characteristic vector field of the Cartan's 2–form $\mathrm{d}\theta$ (which by definition means $i_\Gamma \mathrm{d}\theta = 0\,\&\, i_\Gamma \mathrm{d}(\mathrm{d}\theta) = 0$).

Dynamical and $\mathrm{d}\theta$–symmetries

A vector field $Y \in \mathfrak{X}(TM \times \mathbb{R})$ is a (generator of a) **dynamical symmetry** of the dynamical vector field $\Gamma \in \mathfrak{X}(TM \times \mathbb{R})$ if and only if a function $g \in C^\infty(TM \times \mathbb{R})$ exists such that

$$[Y, \Gamma] = g \cdot \Gamma. \tag{15}$$

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A particular subclass of dynamical symmetries are $\mathrm{d}\theta$–symmetries. A vector field $Y \in \mathfrak{X}(TM \times \mathbb{R})$ is a **$\mathrm{d}\theta$–symmetry** of the dynamical vector field $\Gamma \in \mathfrak{X}(TM \times \mathbb{R})$ if and only if it satisfies

$$\mathscr{L}_Y \mathrm{d}\theta = 0. \tag{16}$$

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1. to every $\mathrm{d}\theta$–symmetry Y is associated an integral of motion F of the form

$$F = f - i_Y\theta, \quad \text{where} \quad \mathrm{d}f = \mathscr{L}_Y\theta. \tag{17}$$

   F is defined locally (use of Poincaré lemma in its construction) and is determined up to a constant.

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3. To every integral of motion F exists unique $X \in \mathfrak{X}(TM \times \mathbb{R})$ such that $\langle X, \mathrm{d}t\rangle = 0$. As a consequence $[X, \Gamma] = 0$.

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3. To every integral of motion F exists unique $X \in \mathfrak{X}(TM \times \mathbb{R})$ such that $\langle X, \mathrm{d}t \rangle = 0$. As a consequence $[X, \Gamma] = 0$.
4. Integral of motion F is an invariant of the $\mathrm{d}\theta$–symmetry Y, i.e. $Y(F) = 0$.

(Cf. W. Sarlet, F. Cantrijn, SIAM Rev. 23(4) 467-494 (1981).)

Example of system with time–dependent integrals

Now let us apply these ideas to the chosen system. We proceed as follows:

1. rewrite the given Hamiltonian system in its Lagrangian formulation,
2. find generators of point symmetries of its Euler–Lagrange equations,
3. extend them from $\mathcal{J}^{(0)}$ to $\mathcal{J}^{(1)}$ through their first prolongation to get the corresponding dynamical symmetries and establish which of them are $\mathrm{d}\theta$–symmetries,
4. associate to $\mathrm{d}\theta$–symmetries corresponding integrals of motion via Noether's Theorem above.

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4. associate to $\mathrm{d}\theta$–symmetries corresponding integrals of motion via Noether's Theorem above.

Example of system with time–dependent integrals

The system under consideration is taken from A. Marchesiello & L.Š., J. Phys. A: Math. Theor. 50, 245202 (2017). Its Hamiltonian reads

$$
\begin{aligned}
H &= \frac{1}{2}\left(\vec{p}+\vec{A}(\vec{x})\right)^2 + W(\vec{x}) \qquad (18)\\
&= \frac{1}{2}\left(p_1^2+p_2^2+(p_3-\Omega_1 y-\Omega_2 x)^2\right)+\frac{\Omega_1\Omega_2}{2S}\left(Sx-y\right)^2,
\end{aligned}
$$

i.e. describes a particle in a constant magnetic field with

$$
\begin{aligned}
\vec{B}(\vec{x}) &= (-\Omega_1,\Omega_2,0), \quad \vec{A}=(0,0,-\Omega_2 x-\Omega_1 y),\\
W(\vec{x}) &= \frac{\Omega_1\Omega_2}{2S}\left(Sx-y\right)^2 \qquad (19)
\end{aligned}
$$

and $e=-1$, $m=1$.

Example of system with time–dependent integrals

It is known to be minimally superintegrable (A. Marchesiello & L.Š J. Phys. A: Math. Theor. 50, 245202 (2017)) and for

$$S = \frac{\Omega_1}{\Omega_2}\kappa^2 \qquad (20)$$

with $\kappa = \frac{m}{n} \in \mathbb{Q}$ even maximally superintegrable, with an additional integral of order $m + n - 1$ (cf. A. Marchesiello & L.Š., SIGMA 14 (2018), 092).

Point symmetries of the corresponding E.-L. equations

The corresponding Euler–Lagrange equations read

$$\begin{aligned} \ddot{x} &= \Omega_2 \dot{z} - \Omega_1 \Omega_2 \left(\frac{\Omega_1 \kappa^2 x}{\Omega_2} - y \right), \\ \ddot{y} &= \dot{z} \Omega_1 + \frac{\Omega_2^2}{\kappa^2} \left(\frac{\Omega_1 \kappa^2 x}{\Omega_2} - y \right), \\ \ddot{z} &= -\Omega_1 \dot{y} - \Omega_2 \dot{x}. \end{aligned} \tag{21}$$

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They possess generically the following 8–dimensional algebra of point symmetry generators

Point symmetries of the corresponding E.-L. equations

$$
\begin{gathered}
Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = \frac{\partial}{\partial z}, \\
Y_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \qquad Y_4 = \frac{\partial}{\partial x} + \frac{\Omega_1}{\Omega_2}\kappa^2\frac{\partial}{\partial y}, \\
Y_5 = \sin(\omega t)\frac{\partial}{\partial x} + \frac{\Omega_2}{\omega}\cos(\omega t)\frac{\partial}{\partial z}, \\
Y_6 = \cos(\omega t)\frac{\partial}{\partial x} - \frac{\Omega_2}{\omega}\sin(\omega t)\frac{\partial}{\partial z}, \\
Y_7 = \sin\left(\frac{\omega}{\kappa}t\right)\frac{\partial}{\partial y} + \frac{\Omega_1\kappa}{\omega}\cos\left(\frac{\omega}{\kappa}t\right)\frac{\partial}{\partial z}, \\
Y_8 = \cos\left(\frac{\omega}{\kappa}t\right)\frac{\partial}{\partial y} - \frac{\Omega_1\kappa}{\omega}\sin\left(\frac{\omega}{\kappa}t\right)\frac{\partial}{\partial z}
\end{gathered}
\tag{22}
$$

where $\omega = \sqrt{\Omega_1^2\kappa^2 + \Omega_2^2}$,

Point symmetries of the corresponding E.-L. equations

which is enhanced to 12–dimensional one when $\kappa = 1$

$$\begin{aligned}
Y_9 &= z\frac{\partial}{\partial x} + z\frac{\Omega_1}{\Omega_2}\frac{\partial}{\partial y} - \left(\frac{\Omega_1}{\Omega_2}y + x\right)\frac{\partial}{\partial z}, \\
Y_{10} &= y\frac{\partial}{\partial x} + \left(\frac{\Omega_1^2 - \Omega_2^2}{\Omega_1\Omega_2}y + x\right)\frac{\partial}{\partial y} + \frac{\Omega_1}{\Omega_2}z\frac{\partial}{\partial z}, \\
Y_{11} &= \left[\left(\frac{\Omega_1}{\Omega_2}y + x\right)\sin(\omega t) + \frac{\omega}{\Omega_2}z\cos(\omega t)\right]\frac{\partial}{\partial x} \\
&+ \frac{\Omega_1}{\Omega_2^2}\left[(\Omega_1 y + \Omega_2 x)\sin(\omega t) + \omega z\cos(\omega t)\right]\frac{\partial}{\partial y} \\
&+ \frac{\omega}{\Omega_2^2}\left[(\Omega_1 y + \Omega_2 x)\cos(\omega t) - \omega z\sin(\omega t)\right]\frac{\partial}{\partial z}, \\
Y_{12} &= \left[\left(\frac{\Omega_1}{\Omega_2}y + x\right)\cos(\omega t) - \frac{\omega}{\Omega_2}z\sin(\omega t)\right]\frac{\partial}{\partial x} \\
&+ \frac{\Omega_1}{\Omega_2^2}\left[(\Omega_1 y + \Omega_2 x)\cos(\omega t) - \omega z\sin(\omega t)\right]\frac{\partial}{\partial y} \\
&- \frac{\omega}{\Omega_2^2}\left[(\Omega_1 y + \Omega_2 x)\sin(\omega t) + \omega z\cos(\omega t)\right]\frac{\partial}{\partial z}.
\end{aligned} \tag{23}$$

Associated integrals of motion

Among the generators $Y_1, \ldots, Y_8$ all except Y_3 give rise to $\mathrm{d}\theta$–symmetries through their prolongation. The corresponding integrals read

$$
\begin{aligned}
E &= \frac{1}{2}\dot{\vec{x}}^2 + \frac{\Omega_2^2}{2\kappa^2}\left(\frac{\Omega_1}{\Omega_2}\kappa^2 x - y\right)^2, \quad p_z = \dot{z} + \Omega_1 y + \Omega_2 x, \\
F_4 &= -\frac{\Omega_1}{\Omega_2}\kappa^2 \dot{y} - \dot{x} + \frac{\omega^2}{\Omega_2} z, \\
F_5 &= -\frac{\Omega_2}{\omega} p_z \cos(\omega t) - \dot{x}\sin(\omega t) + \omega x \cos(\omega t), \\
F_6 &= \frac{\Omega_2}{\omega} p_z \sin(\omega t) - \dot{x}\cos(\omega t) - \omega x \sin(\omega t), \qquad (24) \\
F_7 &= -\frac{\Omega_1 \kappa}{\omega} p_z \cos\left(\frac{\omega}{\kappa} t\right) - \dot{y}\sin\left(\frac{\omega}{\kappa} t\right) + \frac{\omega}{\kappa} y \cos\left(\frac{\omega}{\kappa} t\right), \\
F_8 &= \frac{\Omega_1 \kappa}{\omega} p_z \sin\left(\frac{\omega}{\kappa} t\right) - \dot{y}\cos\left(\frac{\omega}{\kappa} t\right) - \frac{\omega}{\kappa} y \sin\left(\frac{\omega}{\kappa} t\right).
\end{aligned}
$$

Corresponding integrals

We can consider F_5 and F_6 (F_7 and F_8, respectively) as real and imaginary parts of complex integrals

$$J_5 = \left(\omega x - \frac{\Omega_2}{\omega} p_z + \mathrm{i}\dot{x}\right) \mathrm{e}^{\mathrm{i}\omega t}, \quad J_7 = \left(\frac{\omega}{\kappa} y - \frac{\Omega_1 \kappa}{\omega} p_z + \mathrm{i}\dot{y}\right) \mathrm{e}^{\mathrm{i}\frac{\omega}{\kappa} t}.$$

Two time-independent integrals can be constructed as squares of their norms. After simplification, they read

$$\tilde{F}_5 = \dot{x}^2 + \frac{\left[\Omega_2(\dot{z} + \Omega_1 y) - \Omega_1^2 \kappa^2 x\right]^2}{\omega^2}, \; \tilde{F}_7 = \dot{y}^2 + \left(\frac{\Omega_1 \kappa^2 (\dot{z} + \Omega_2 x) - \Omega_2^2 y}{\kappa\omega}\right)^2,$$

out of which only one is independent of E, p_z and F_4 since they are related to the energy through $\tilde{F}_5 + \tilde{F}_7 = 2E$.

Corresponding integrals

In addition if $\kappa = \frac{m}{n} \in \mathbb{Q}$, we can combine J_5 and J_7 to get

$$J_{57} = J_5^n \bar{J}_7^m = \left(\omega x - \frac{\Omega_2}{\omega} p_z + \mathrm{i}\dot{x}\right)^n \left(\frac{n\omega}{m} y - \frac{m\Omega_1}{n\omega} p_z - \mathrm{i}\dot{y}\right)^m, \tag{25}$$

where bar means complex conjugation. Its real or imaginary part is an additional independent integral, the other four being E, p_z, F_4 and $\tilde{F}_5$ (or $\tilde{F}_7$). Explicit formulas for the real or imaginary part can be obtained in terms of Chebyshev polynomials.

Thus, we have recovered through the point symmetry approach the four known time-independent integrals which imply minimal superintegrability of our system, as well as the fifth for rational κ.

Corresponding integrals of motion

Thus we have recovered through the point symmetry approach the four known time–independent integrals which were found in A. Marchesiello & L.Š J. Phys. A: Math. Theor. 50, 245202 (2017) and imply minimal superintegrability of our system.

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We have also found for rational κ an additional integral, which makes the system maximally superintegrable, in accordance with A. Marchesiello & L.Š SIGMA 14, 092 (2018).

For $\kappa = 1$ out of additional integrals only Y_9 leads to a $\mathrm{d}\theta$–symmetry. The corresponding integral reads

$$F_9 = \frac{1}{2\Omega_2}\left((2\Omega_1 xy + 2x\dot{z} - 2\dot{x}z)\Omega_2 \right.$$
$$\left. + ((y^2+z^2)\Omega_1 + 2y\dot{z} - 2\dot{y}z)\Omega_1 + (x^2+z^2)\Omega_2^2\right).$$

Corresponding integrals of motion

The integrals (24) involve six independent functions on the 7–dimensional manifold $TM \times \mathbb{R}$, i.e. their values determined by the initial data restrict the dynamics to a curve in $TM \times \mathbb{R}$ and allow to find the trajectories algebraically.

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We may also observe that the time–dependent integrals $F_5, \ldots, F_8$ are actually integrated Euler–Lagrange equations for x and y with a suitable integrating factor, e.g.

$$\frac{\mathrm{d}}{\mathrm{dt}} F_5 = \omega \dot{x} \cos(\omega t) - \omega^2 x \sin(\omega t) + \Omega_2 p_z \sin(\omega t) - \ddot{x} \sin(\omega t) - \omega \dot{x} \cos(\omega t)$$
$$= -\sin(\omega t) \left(\ddot{x} + \omega^2 x - \Omega_2 p_z \right).$$

Conclusions - open questions

The example just presented (and a few others) lead us to two essential open questions

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  Namely, we have so far found in this way only integrals involving trigonometric functions sin and cos of the time argument t, leading to some oscillatory behavior, and consequently, to some harmonic oscillator behind the considered system. Do less trivial examples exist?

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  Also, maximal superintegrability in the usual sense implies that bounded trajectories are closed. Here the integrals restrict the trajectory to a curve in the extended phase space whose projection on the phase space may be bounded but not closed.

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Thank you for your attention!