Pairs of commuting quadratic elements in the universal enveloping algebra of Euclidean algebra and integrals of motion

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Introduction

When pairs of commuting integrals quadratic in the momenta for Hamiltonian systems in three spatial dimensions in Euclidean space are considered, the famous classical paper [Makarov, Smorodinsky, Valiev and Winternitz. Nuovo Cimento A Series 10, 52:1061–1084, 1967] which concerns the natural Hamiltonians

$$H = \frac{1}{2}\vec{p}^2 + V(\vec{x}) \tag{1}$$

concludes that the leading order terms of such integrals belong to the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}_3)$ of the Euclidean algebra and can be written in 11 possible forms, each corresponding to an orthogonal coordinate system in which the corresponding Hamilton–Jacobi or Schrödinger equation separates.

Introduction

In the presence of magnetic field we have observed in [A. Marchesiello and L. Šnobl, J. Phys. A 50 (24) (2017) 245202] that while the leading order terms must still belong to $\mathfrak{U}(\mathfrak{e}_3)$, a more general leading order structure of the pair of integrals can appear, at least in the particular case of integrals with leading order terms involving only linear momenta (a.k.a. Cartesian type).

Introduction

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Motivated by this result we study the algebraic structure corresponding to the leading order terms, that means classify three–dimensional Abelian subalgebras of quadratic elements in the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}_3)$ of the Euclidean algebra.

Formulation of the problem

Let us consider the Hamiltonian system in the three–dimensional Euclidean space of the form

$$H = \frac{1}{2}\vec{p}^2 + \vec{A}(\vec{x}) \cdot \vec{p} + V(\vec{x}) = \frac{1}{2}(\vec{p}^A)^2 + W(\vec{x}), \quad \vec{p}^A = \vec{p} + \vec{A}(\vec{x}) \tag{2}$$

and its integrals of motion polynomial in the momenta. The leading order terms of such an integral must belong to a representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}_3)$ of the Euclidean algebra $\mathfrak{e}_3 = \mathrm{span}\{p_1, p_2, p_3, l_1, l_2, l_3\}$ such that

$$\vec{p} \cdot \vec{l} = \sum_{j=1}^{3} p_j l_j = 0 \tag{3}$$

between the linear momenta $\vec{p} = (p_1, p_2, p_3)$ and the angular momenta $\vec{l} = (l_1, l_2, l_3)$, $l_j = \sum_{k,l} \epsilon_{jkl} x_k p_l$, holds. (I.e. the quadratic Casimir invariant $\vec{p} \cdot \vec{l}$ of $\mathfrak{e}_3$ vanishes in the representations relevant for our application.)

Formulation of the problem

Restricting ourself to the most tractable situation of quadratic integrals of motion we are looking for pairs of commuting quadratic elements in $\mathfrak{U}(\mathfrak{e}_3)$ which obviously also commute with the quadratic Casimir invariant $h = \vec{p}^2 = \sum_j p_j^2$ of $\mathfrak{e}_3$ and together with it may define leading order terms of a triple of commuting integrals of motion (including the Hamiltonian).

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To sum up, we are looking for the **classification of three–dimensional Abelian subalgebras of quadratic elements in $\mathfrak{U}(\mathfrak{e}_3)$ modulo equation $\vec{p} \cdot \vec{l} = 0$ and transformations from the Euclidean group.**

Classification: quadratic 3D Abelian subalgebras in $\mathfrak{U}(\mathfrak{e}_3)$ I

Any three–dimensional Abelian subalgebra $\mathrm{span}\{h = \vec{p}^2, X_1, X_2\}$ consisting of quadratic commuting elements in the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}_3)$ can be modulo equation $\vec{p} \cdot \vec{l} = 0$ and transformations from the Euclidean group written in terms of the following pairs of elements

(a

$$X_1 = l_1^2 + l_2^2 + l_3^2 + a l_3 p_3 + b p_3^2, \quad X_2 = l_3^2, \quad a, b \in \mathbb{R},$$

(b

$$\begin{aligned} X_1 &= l_1^2 + l_2^2 + l_3^2 + b(a p_2^2 + p_3^2), \\ X_2 &= a l_2^2 + l_3^2 - a b p_1^2, \, 0 < a \le \frac{1}{2}, b \in \mathbb{R}, \end{aligned}$$

Classification: quadratic 3D Abelian subalgebras in $\mathfrak{U}(\mathfrak{e}_3)$ II

(c

$$
\begin{aligned}
X_1 &= l_1^2 + l_2^2 + l_3^2 + 2b(l_1 p_1 - (3a-1) l_2 p_2 - 2 l_3 p_3) + \\
&\quad + 3b^2((1-4a)p_1^2 - (3a^2 - 2a - 1)p_2^2 + 2(a-1)p_3^2), \\
X_2 &= a l_2^2 + l_3^2 + 6ab l_1 p_1 + 9ab^2(a p_3^2 + p_2^2), \quad 0 < a \leq \frac{1}{2}, b \in \mathbb{R} \backslash \{0\},
\end{aligned}
$$

(d

$$
X_1 = l_3^2, \quad X_2 = \frac{1}{2}(l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) + a l_3 p_3, \ a \geq 0,
$$

Classification: quadratic 3D Abelian subalgebras in $\mathfrak{U}(\mathfrak{e}_3)$ III

(e)

$$X_1 = l_3^2 + 2a(l_1 p_1 - l_2 p_2) + a^2 p_3^2,$$
$$X_2 = \frac{1}{2}(l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) - a p_1 p_2, \; a > 0,$$

(f)

$$X_1 = l_3^2 + a l_3 p_3 + b p_1^2 + c p_1 p_3 + d p_2 p_3,$$
$$X_2 = p_3^2, \quad a, b \in \mathbb{R}, \; c \geq 0, \; d \geq 0,$$

(g)

$$X_1 = l_3^2 + a p_3^2, \quad X_2 = l_3 p_3 + b p_3^2, \quad a, b \in \mathbb{R},$$

Classification: quadratic 3D Abelian subalgebras in $\mathfrak{U}(\mathfrak{e}_3)$ IV

(h

$$
\begin{aligned}
X_1 &= l_1 p_1 + a l_2 p_2 - (a+1) l_3 p_3 + b p_2^2, \\
X_2 &= p_1^2 + \frac{2a+1}{a+2} p_2^2, \quad -\frac{1}{2} < a \leq 0, b \in \mathcal{R},
\end{aligned}
$$

(i)

$$
X_1 = l_1 p_1 + a p_2^2 + b p_2 p_3, \quad X_2 = p_1^2, \quad a \in \mathbb{R}, \; b \geq 0,
$$

Classification: quadratic 3D Abelian subalgebras in $\mathfrak{U}(\mathfrak{e}_3)$ V

(j)

$$\begin{aligned} X_1 &= l_1p_1 + al_2p_2 - (a+1)l_3p_3 + \frac{\omega}{2}\left(l_1p_3 + p_3l_1 - l_3p_1 - p_1l_3\right) \\ &\quad +2bp_1p_2 + c\left(p_2^2 - p_3^2\right), \\ X_2 &= p_1^2 + \frac{6\omega}{4a-1}p_1p_3 + \frac{a+2}{4a-1}p_2^2 - \frac{5a+1}{4a-1}p_3^2, \end{aligned}$$

where $\omega = \sqrt{1+a-2a^2}$, $-\frac{1}{2} < a \leq 0$, $b \geq 0$, $c \in \mathbb{R}$,

(k)

$$\begin{aligned} X_1 &= p_1^2 + ap_2^2, \\ X_2 &= p_2^2 + bp_1p_2 + cp_1p_3 + dp_2p_3, \quad 0 \leq a \leq \frac{1}{2},\ b \geq 0,\ c \geq 0,\ d \in \mathbb{R}. \end{aligned}$$

Comparison with integrals for natural Hamiltonians I

Let us compare our classification with the leading order terms of the pairs of commuting quadratic integrals of the systems classified in Makarov et al. There, the following possibilities appeared, each related to an orthogonal coordinate system:

I Cartesian $X_1 = p_1^2, \quad X_2 = p_2^2.$

II Cylindrical $X_1 = l_3^2, \quad X_2 = p_3^2.$

III Elliptic cylindrical
$X_1 = l_3^2 + Ap_1^2, \quad X_2 = p_3^2, \quad A > 0.$

IV Parabolic cylindrical $X_1 = l_3 p_1 + p_1 l_3, \quad X_2 = p_3^2.$

V Spherical $X_1 = l_1^2 + l_2^2 + l_3^2, \quad X_2 = l_3^2.$

VI Prolate spheroidal
$X_1 = l_1^2 + l_2^2 + l_3^2 - A(p_1^2 + p_2^2), \quad X_2 = l_3^2, \quad A > 0.$

Comparison with integrals for natural Hamiltonians II

VII Oblate spheroidal
$X_1 = l_1^2 + l_2^2 + l_3^2 + A(p_1^2 + p_2^2), \quad X_2 = l_3^2, \quad A > 0.$

VIII Parabolic rotational (also known as circular parabolic) $X_1 = l_3^2, \quad X_2 = l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2,$

IX Conical
$X_1 = l_1^2 + l_2^2 + l_3^2, \quad X_2 = B^2 l_2^2 + C^2 l_3^2, \quad C > B > 0.$

X Ellipsoidal
$X_1 = l_1^2 + l_2^2 + l_3^2 + (A^2 + B^2) p_1^2 + A^2 p_2^2 + B^2 p_3^2,$
$X_2 = B^2 l_2^2 + A^2 l_3^2 + A^2 B^2 p_1^2, A > B > 0.$

XI Paraboloidal
$X_1 = l_3^2 + A\,(l_1 p_2 + p_2 l_1) - B\,(l_2 p_1 + p_1 l_2) - ABp_3^2,$
$X_2 = (l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) - A(p_2^2 + p_3^2) - B(p_1^2 + p_3^2), \quad B > A > 0.$

Comparison with integrals for natural Hamiltonians III

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The Cartesian case **I** is included in the class (k), which however contains four parameters, whereas the Cartesian case **I** has none.

The cylindrical **II** and elliptic cylindrical **III** cases are included in the class (f) (with $A = b$); however, three additional terms of the form $l_3 p_3$, $p_1 p_3$ and $p_2 p_3$ in X_1 are allowed by the algebraic structure of $\mathfrak{U}(\mathfrak{e}_3)$.

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The parabolic cylindrical case **IV** is contained in the class (j) upon a transformation, with the parameters chosen as $a = 0$, $\omega = 1$, $b = c = 0$.

Comparison with integrals for natural Hamiltonians IV

We observe that the spherical, prolate and oblate spheroidal cases, i.e. **V, VI, VII**, all belong to the class (a) with the identification $A = b \in \mathbb{R}$ which, however, allows an additional term of the form $l_3 p_3$ in X_1.

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The parabolic rotational (also known as circular parabolic) case **VIII** is contained in the class (d) which again allows extra term $l_3 p_3$, this time in X_2.

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The conical and ellipsoidal cases **IX** and **X** correspond to the case (b), namely, the conical becomes (b) with $a = \frac{B^2}{C^2}$ and $b = 0$, the ellipsoidal becomes (b) with $a = B^2/A^2$ and $b = -A^2$.

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The paraboloidal case **XI** can be matched precisely with the class (e).

Comparison with integrals for natural Hamiltonians V

The classes (c), (g), (h) and (i) do not have their counterparts in the classification of quadratically integrable natural Hamiltonian systems.

Example 1

To illustrate that these generalizations are indeed realized in some integrable systems, let us consider the class (i) with vanishing parameter b and nonvanishing a. We first rotate our reference frame to write the algebra in an equivalent form

$$X_1 = l_3 p_3 + a p_1^2, \quad X_2 = p_3^2, \quad a \neq 0. \tag{4}$$

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Solving the equations coming from various powers of the momenta in the commutativity conditions

$$\{H, X_1\}_{P.B.} = \{H, X_2\}_{P.B.} = \{X_1, X_2\}_{P.B.} = 0, \tag{5}$$

we find a five parameter family of integrable systems with magnetic field.

Example 1

Its magnetic field, the vector potential and the electrostatic potential are

$$\vec{B}(\vec{x}) = (-b_\varphi x_2, b_\varphi x_1, b_Z), \ \vec{A}(\vec{x}) = \left(-\frac{b_Z}{2}x_2, \frac{b_Z}{2}x_1, -\frac{b_\varphi}{2}(x_1^2 + x_2^2)\right), \quad (6)$$

$$W(\vec{x}) = -\frac{b_\varphi^2(x_1^2+x_2^2)^2}{8} - \frac{ab_\varphi b_Z(x_1^2 - x_2^2)}{2} + w_3(x_1^2+x_2^2) + w_1 x_1 + w_2 x_2.$$

We have the integral

$$\begin{aligned} X_1 \quad = \quad & l_3 p_3 + a p_1^2 + \frac{2x_2 w_3 + w_2}{b_\varphi} p_1 + \frac{ab_Z b_\varphi x_1 - 2x_1 w_3 - w_1}{b_\varphi} p_2 - b_\varphi a x_1^2 p_3 - \\ & -a(ab_Z b_\varphi - 2w_3)x_1^2 + \frac{w_1(4ab_\varphi + b_Z)}{2b_\varphi} x_1 + \frac{ab_Z^2}{4} x_2^2 + \frac{b_Z w_2}{2b_\varphi} x_2 \quad (7) \end{aligned}$$

and the second integral reduces to a first order one,

$$X_2 = p_3. \quad (8)$$

Example 1 – limits

The system described by (6) does not possess any other integrals linear or quadratic in the momenta for generic nonvanishing values of its parameters determining the magnetic field b_φ and b_Z. Thus its integrals cannot be expressed in the form of any of the classes **I**,...,**XI** of Makarov et al.

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Example 1 – limits

Thus we have a quadratically minimally superintegrable system with

$$\vec{B}(\vec{x}) = (-b_\varphi x_2, b_\varphi x_1, 0)\,,\ W(\vec{x}) = -\frac{b_\varphi^2}{8}(x_1^2+x_2^2)^2+w_3(x_1^2+x_2^2)+w_1 x_1+w_2 x_2, \tag{9}$$

and the integrals

$$\begin{aligned} X_1' &= l_3 p_3 - \frac{2w_3}{b_\varphi} l_3 + \frac{w_2}{b_\varphi} p_1 - \frac{w_1}{b_\varphi} p_2, \\ X_1'' &= p_1^2 - b_\varphi x_1^2 p_3 + 2w_3 x_1^2 + 2w_1 x_1, \\ X_2 &= p_3. \end{aligned} \tag{10}$$

Among them we can choose a commuting pair of integrals X_1'' and X_2 corresponding to Cartesian separation, i.e. class **I**.

Example 1 – limits

Similarly, we may consider the limit $b_\varphi \to 0$ in which, however, the integral X_1 becomes singular. The system is now characterized by the constant magnetic field and the potential of the form of a (shifted) isotropic harmonic oscillator in the x_1x_2 plane

$$\vec{B}(\vec{x}) = (0, 0, b_Z), \quad W(\vec{x}) = w_3(x_1^2 + x_2^2) + w_1 x_1 + w_2 x_2. \quad (11)$$

Assuming that the frequency of the oscillator is nonvanishing, i.e. $w_3 \neq 0$, we find the first order integral

$$\tilde{Y}_1 = l_3 - \frac{w_2}{4w_3}(2p_1 + b_Z y) + \frac{w_1}{4w_3}(2p_2 - b_Z x_1) \qquad (12)$$

by multiplying X_1 by b_φ and next taking the limit $b_\varphi \to 0$. We also observe that provided $w_3 \neq 0$ we can now set $w_1 = w_2 = 0$ without loss of generality by a shift of the coordinate system accompanied by a gauge change.

Example 1 – limits

The system (11) and its integrals obtain a particularly simple form

$$\vec{B}(\vec{x}) = (0, 0, b_Z), \quad W(\vec{x}) = w_3(x_1^2 + x_2^2), \quad \tilde{Y}_1 = l_3, \quad Y_2 = p_3, \tag{13}$$

belonging to the cylindrical class **II**. The system (13) does not possess any other independent linear or quadratic integrals (except when $w_3 = -\frac{1}{8} b_Z^2$ holds).

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belonging to the cylindrical class **II**. The system (13) does not possess any other independent linear or quadratic integrals (except when $w_3 = -\frac{1}{8} b_Z^2$ holds). Thus we see that in the limit $b_\varphi \to 0$ the integrable system (6) goes into the integrable system (13); however, the class the commuting integrals belong to changes in this limit, from (i) to (f). In other words the limit $b_\varphi \to 0$ forces also the parameter a in the class (i) to vanish despite the fact that the Hamiltonian goes into the form characterised by (13) irrespectively of the value of a.

Example 2 – mathematical model of helical undulator in a solenoid

As a second illustration of the relevance of our classification, let us consider the generalized cylindrical class (f) with the parameter choice $b = 0$. As reported in our recent preprint O. Kubů, A. Marchesiello and L. Šnobl, arXiv:2206.15305, in this class several integrable systems can be found.

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$$\vec{B}(x,y,z) = \left(-b_3 \cos\left(\frac{2z}{a}\right), b_3 \sin\left(\frac{2z}{a}\right), b_1\right), \quad W(x,y,z) = 0,$$

$$\vec{A}(x,y,z) = \left(-\frac{b_3 a}{2} \cos\left(\frac{2z}{a}\right), b_1 x + \frac{b_3 a}{2} \sin\left(\frac{2z}{a}\right), 0\right). \quad (14)$$

Example 2 – mathematical model of helical undulator in a solenoid

If $b_3 = 0$, the system reduces to the well–known superintegrable system with constant magnetic field. The superintegrability of the system with $b_1 = 0$ is known as well, see [A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A 48 (39) (2015) 395206.]. As observed in [T. Heinzl, A. Ilderton, J. Phys. A 50 (34) (2017) 345204], it describes motion of electrons in a nonrelativistic limit of a helical undulator.

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Remark: Undulators are devices for generation of powerful coherent radiation using beams of charged high energy particles, typically electrons. Electrons are traversing suitable magnetic field, forced to undergo oscillations and thus to radiate energy.

Example 2 – mathematical model of helical undulator in a solenoid

When $b_1 b_3 \neq 0$ we study the motion of (nonrelativistic) electrons in the field of a helical undulator placed in an infinite solenoid. Relativistic version of such a system was recently proposed and experimentally realized as a simple and efficient source of coherent spontaneous THz undulator radiation in [N. Balal, I. V. Bandurkin, V. L. Bratman, A. E. Fedotov, Phys. Rev.: Accelerators and Beams 20 (2017) 122401.].

Example 2 – mathematical model of helical undulator in a solenoid

The system (14) admits three first order integrals reading

$$
\begin{aligned}
Y_1 &= p_x^A + b_1 y + \frac{b_3 a}{2} \cos\left(\frac{2z}{a}\right), \\
Y_2 &= p_y^A - b_1 x - \frac{b_3 a}{2} \sin\left(\frac{2z}{a}\right), \\
Y_3 &= l_z^A - \frac{a}{2} p_z^A - \frac{1}{2}\left[b_1 r^2 + b_3 a \sin\left(\frac{2z}{a}\right) x + b_3 a \cos\left(\frac{2z}{a}\right) y\right].
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Y_3 &= l_z^A - \frac{a}{2} p_z^A - \frac{1}{2}\left[b_1 r^2 + b_3 a \sin\left(\frac{2z}{a}\right) x + b_3 a \cos\left(\frac{2z}{a}\right) y\right].
\end{aligned}
\tag{15}
$$

They do not commute, as their Poisson brackets read

$$
\{Y_1, Y_2\} = b_1 \cdot 1, \quad \{Y_1, Y_3\} = -Y_2, \quad \{Y_2, Y_3\} = Y_1. \tag{16}
$$

Thus together with the constant identity function 1 they constitute a solvable Lie algebra.

Example 2 – mathematical model of helical undulator in a solenoid

Its Casimir invariants are $H, 1$ and

$$K = Y_1^2 + Y_2^2 + 2b_1 Y_3 \tag{17}$$

(see e.g. [L. Šnobl and P. Winternitz, Classification and Identification of Lie Algebras, AMS 2014]). By an explicit calculation we find that all second order integrals are functions of integrals H, Y_i.

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The Casimir invariant K commutes with all integrals of order at most 2, as well as the Hamiltonian. Thus, instead of K we may equivalently consider our integral $X_2 = 2H - K$:

$$X_2 = \left(p_z^A\right)^2 - b_3 a \cos\left(\frac{2z}{a}\right) p_x^A + b_3 a \sin\left(\frac{2z}{a}\right) p_y^A + a b_1 p_z^A - \frac{b_3^2 a^2}{4}. \tag{18}$$

Example 2 – mathematical model of helical undulator in a solenoid

Every commuting triple of quadratic integrals can then be written as a linear span of H, X_2 and another second order integral constructed out of Y_1, Y_2, Y_3. Its role can be played by, e.g., the generalized cylindrical integral

$$X_1 = Y_3^2 - \frac{a^2}{4}X_2 = Y_3^2 + \frac{a^2}{4}(Y_1^2 + Y_2^2 + 2b_1 Y_3 - 2H), \qquad (19)$$

or by the integral Y_1^2; thus, we see that the system (14) lies at the intersection of classes (f) and (k) and does not belong to any other class. It can be shown that its Hamilton–Jacobi equation does not separate in any orthogonal coordinate system in $\mathbb{R}^3$.

Example 2 – mathematical model of helical undulator in a solenoid

In the absence of the electrostatic potential, $W(\vec{x}) = 0$, the relativistic Hamiltonian expressed in the instant form

$$H_{\mathrm{rel}} = \sqrt{1 + (p_j^A)^2} + W(\vec{x}),$$

is a function of the nonrelativistic Hamiltonian (2).

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is a function of the nonrelativistic Hamiltonian (2). Thus the same nonabelian algebra of integrals of motion (15) is also present when motion of a relativistic electron in the helical undulator placed in an infinite solenoid is considered, as long as the radiation emitted by the electron is neglected. This observation may be helpful in theoretical analysis of undulators of the type proposed by Balal et al.

Conclusions

We have classified all possible three–dimensional Abelian algebras of quadratic elements in any representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}_3)$ such that the quadratic Casimir element $\vec{p} \cdot \vec{l}$ vanishes.

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We have classified all possible three–dimensional Abelian algebras of quadratic elements in any representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}_3)$ such that the quadratic Casimir element $\vec{p} \cdot \vec{l}$ vanishes. The motivation and need for this classification come from the theory of integrable systems, since such subalgebras encode possible leading order terms of commuting integrals of motion quadratic in the momenta. We have seen that this classification in principle allows for more general forms of integrals of motion than the well known result of Makarov et al. which was derived for systems involving only the scalar potential.

Conclusions

Next, we have demonstrated an example of an integrable system involving magnetic field with such more general structure of its integrals, cf. (6). We have seen that in two limits simplifying the structure of the magnetic field we have different behaviour.

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Conclusions

Next, we have demonstrated an example of an integrable system involving magnetic field with such more general structure of its integrals, cf. (6). We have seen that in two limits simplifying the structure of the magnetic field we have different behaviour. Namely, in one limit one of the original integrals splits into two independent integrals, i.e. the system becomes minimally superintegrable, and a simpler choice of the pair of commuting integrals is available. In another limit, the system stays only quadratically integrable but the leading order structure of the integrals changes. One original integral becomes singular in the limit and a new integral arises, effectively forcing a parameter present in the leading order term, which is not directly involved in the limiting process, to vanish.

Conclusions

The systems and their integrals arising from the more general leading order structure presented here may demonstrate unusual singular properties in the limit of vanishing magnetic field, as in Example 1, and find applications in the fields of physics where explicitly computable stable trajectories of particles in magnetic fields are of relevance and interest, e.g. in plasma physics.

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The systems and their integrals arising from the more general leading order structure presented here may demonstrate unusual singular properties in the limit of vanishing magnetic field, as in Example 1, and find applications in the fields of physics where explicitly computable stable trajectories of particles in magnetic fields are of relevance and interest, e.g. in plasma physics.

For physical applications, the results of Example 2 are of particular interest. Our observation that the electron in helical undulator placed in an infinite solenoid possesses a nontrivial algebra of integrals of motion may help in theoretical study of its properties, e.g., in allowing more efficient numerical simulations.

Acknowledgement

Inspiration and need for this research came from numerous discussions with Pavel Winternitz over several years. Unfortunately, while he was alive we always found other more urgent research directions to follow together. Thus we dedicate our results to his memory.

Thank you for your attention!

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