Integrable and superintegrable systems in static electromagnetic fields

Antonella Marchesiello, **Libor Šnobl** and Pavel Winternitz

Department of Physics
Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague

Centre de recherches mathématiques &
Département de mathématiques et de statistique,
Université de Montréal

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Contents

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Introduction

We consider superintegrable systems, i.e. Hamiltonian systems that have more globally defined integrals of motion than degrees of freedom, in three spatial dimensions. Such Hamiltonian systems in $\mathbb{R}^3$ were considered and under some restrictions classified in detail for the case when the Hamiltonian is the sum of the kinetic energy and the scalar potential.

In J. Bérubé, P. Winternitz. J. Math. Phys. 45 (2004), no. 5, 1959-1973 the structure of the gauge–invariant integrable and superintegrable systems involving vector potentials was considered in two spatial dimensions. Among other results it was shown there that under the chosen assumptions imposed on the form of the potential, no superintegrable system with nonconstant magnetic field exists in dimension 2.

Introduction - classical Hamiltonian

Inspired by the approach used there we consider the Hamiltonian describing motion of 0–spin particle in three dimensions in a nonvanishing magnetic field, i.e. classically

$$H = \frac{1}{2}(\vec{p} + \vec{A})^2 + V(\vec{x}) \tag{1}$$

where $\vec{p}$ is the momentum, $\vec{A}$ is the vector potential and V is the electrostatic potential. The magnetic field $\vec{B} = \nabla \times \vec{A}$ is assumed to be nonvanishing so that the system is not gauge equivalent to a system with only the scalar potential. We choose the units in which the mass of the particle has the numerical value 1 and the charge of the particle is -1 (having an electron in mind as the prime example).

Introduction - gauge invariance

We recall that the equations of motion of the Hamiltonian (1) are gauge invariant, i.e. that they are the same for the potentials

$$\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \nabla\chi, \qquad V'(\vec{x}) = V(\vec{x}) \tag{2}$$

for any choice of the function $\chi(\vec{x})$ (we are considering only the static situation here). Thus, the physically relevant quantity is the magnetic field

$$\vec{B} = \nabla \times \vec{A}, \qquad \text{i.e.} \qquad B_j = \epsilon_{jkl} \frac{\partial A_l}{\partial x_k} \tag{3}$$

rather than the vector potential $\vec{A}(\vec{x})$.

Introduction - quantum Hamiltonian

We shall also consider the quantum Hamiltonian defined as the (properly symmetrized) analogue of (1) in terms of the operators of the linear momenta $\hat{P}_j = -\mathrm{i}\hbar\frac{\partial}{\partial x_j}$ and coordinates $\hat{X}_j = x_j$:

$$\begin{aligned} \hat{H} &= \frac{1}{2}\sum_j \left(\hat{P}_j + \hat{A}_j(\vec{x})\right)^2 + \hat{V}(\vec{x}) \qquad (4) \\ &= \frac{1}{2}\sum_j \left(\hat{P}_j\hat{P}_j + \hat{P}_j\hat{A}_j(\vec{x}) + \hat{A}_j(\vec{x})\hat{P}_j + \hat{A}_j(\vec{x})^2\right) + \hat{V}(\vec{x}). \end{aligned}$$

The operators $\hat{A}_j(\vec{x})$ and $\hat{V}(\vec{x})$ act on wavefunctions as multiplication by the functions $A_j(\vec{x})$ and $V(\vec{x})$, respectively.

Introduction - quantum gauge invariance

On the quantum level, the gauge transformation demonstrates itself as a unitary transformation of the Hilbert space. Namely, let us take

$$\hat{U}\psi(\vec{x}) = \exp\left(\frac{\mathrm{i}}{\hbar}\chi(\vec{x})\right) \cdot \psi(\vec{x}). \tag{5}$$

Applying (5) on the states and the observables we get an equivalent description of the same physical reality in terms of

$$\psi \to \psi' = \hat{U}\psi, \qquad \hat{O} \to \hat{O}' = \hat{U}\hat{O}\hat{U}^{\dagger}. \tag{6}$$

In particular, the following observables transform covariantly

$$(\hat{P}_j + \hat{A}_j) \to \hat{U}(\hat{P}_j + \hat{A}_j)\hat{U}^{\dagger} = P_j + \hat{A}'_j, \qquad \hat{V} \to \hat{U}\hat{V}\hat{U}^{\dagger} = \hat{V}. \tag{7}$$

Outline of the talk

We study the conditions on the structure of the integrals of motion of the first and second order in momenta, in particular how they are influenced by the gauge invariance of the problem.

Next, we concentrate on the several possibilities for integrability arising from low (i.e. first & second) order integrals.

The assumed structure of the integrals of motion

Let us consider integrals of motion which are at most second order in the momenta. Because our system is gauge invariant (2), (7) we find it convenient to express the integrals in terms of gauge covariant expressions

$$p_j^A = p_j + A_j, \qquad \hat{P}_j^A = \hat{P}_j + \hat{A}_j \tag{8}$$

rather than the momenta themselves. The operators (8) no longer commute among each other. They satisfy

$$[\hat{P}_j^A, \hat{P}_k^A] = -\mathrm{i}\hbar\epsilon_{jkl}\hat{B}_l, \qquad [\hat{P}_j^A, \hat{X}_k] = -\mathrm{i}\hbar\mathbf{1}, \tag{9}$$

where $\hat{B}_l$ is the operator of the magnetic field strength,

$$\hat{B}_j\psi(\vec{x}) = B_j(\vec{x})\psi(\vec{x}) = \epsilon_{jkl}\frac{\partial A_l}{\partial x_k}\psi(\vec{x})$$

and ϵ_{jkl} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.

The assumed structure of the integrals of motion

Classically, we write a general second order integral of motion as

$$X=\sum_{j=1}^{3} h_j(\vec{x}) p_j^A p_j^A + \sum_{j,k,l=1}^{3} \frac{1}{2}|\epsilon_{jkl}| n_j(\vec{x}) p_k^A p_l^A + \sum_{j=1}^{3} s_j(\vec{x}) p_j^A + m(\vec{x}). \tag{10}$$

The condition that the Poisson bracket

$$\{a(\vec{x},\vec{p}), b(\vec{x},\vec{p})\}_{P.B.} = \sum_{j=1}^{3} \left(\frac{\partial a}{\partial x_j}\frac{\partial b}{\partial p_j} - \frac{\partial b}{\partial x_j}\frac{\partial a}{\partial p_j} \right) \tag{11}$$

of the integral (10) with the Hamiltonian (1) vanishes

$$\{H, X\}_{P.B.} = 0 \tag{12}$$

leads to terms of order $3, 2, 1$ and 0 in the momenta and respectively to the following equations:

The conditions for the integrals of motion

Third order terms

$$
\begin{aligned}
&\partial_x h_1 = 0, && \partial_y h_1 = -\partial_x n_3, && \partial_z h_1 = -\partial_x n_2,\\
&\partial_x h_2 = -\partial_y n_3, && \partial_y h_2 = 0, && \partial_z h_2 = -\partial_y n_1,\\
&\partial_x h_3 = -\partial_z n_2, && \partial_y h_3 = -\partial_z n_1, && \partial_z h_3 = 0,\\
& && \nabla \cdot \vec{n} = 0. &&
\end{aligned}
\tag{13}
$$

Second order terms

$$
\begin{aligned}
\partial_x s_1 &= n_2 B_2 - n_3 B_3,\\
\partial_y s_2 &= n_3 B_3 - n_1 B_1,\\
\partial_z s_3 &= n_1 B_1 - n_2 B_2, \qquad \text{i.e.} \qquad \nabla \cdot \vec{s} = 0,\\
\partial_y s_1 + \partial_x s_2 &= n_1 B_2 - n_2 B_1 + 2(h_1 - h_2) B_3,\\
\partial_z s_1 + \partial_x s_3 &= n_3 B_1 - n_1 B_3 + 2(h_3 - h_1) B_2,\\
\partial_y s_3 + \partial_z s_2 &= n_2 B_3 - n_3 B_2 + 2(h_2 - h_3) B_1.
\end{aligned}
\tag{14}
$$

The conditions for the integrals of motion, cont'd

First order terms

$$\begin{aligned}\partial_x m &= 2h_1\partial_x V + n_3\partial_y V + n_2\partial_z V + s_3 B_2 - s_2 B_3,\\ \partial_y m &= n_3\partial_x V + 2h_2\partial_y V + n_1\partial_z V + s_1 B_3 - s_3 B_1,\\ \partial_z m &= n_2\partial_x V + n_1\partial_y V + 2h_3\partial_z V + s_2 B_1 - s_1 B_2.\end{aligned} \tag{15}$$

Zero order term

$$\vec{s}\cdot\nabla V = 0. \tag{16}$$

Equations (13) are the same as for the system with vanishing magnetic field and their explicit solution is known - they imply that the highest order terms in the integral (10) are linear combinations of products of the generators of the Euclidean group $p_1, p_2, p_3, l_1, l_2, l_3$ where $l_j = \sum_{l,k} \epsilon_{jkl} x_k p_l$, i.e. $\vec{h}$, $\vec{n}$ can be expressed in terms of 20 constants α_{ab}, $1 \leq a \leq b \leq 6$.

The conditions for the integrals of motion, cont'd

In the quantum case we have to consider a properly symmetrized analogue of (10). We choose the following convention

$$\hat{X} = \sum_{j=1}^{3}\{h_j(\vec{x}), \hat{P}_j^A\hat{P}_j^A\} + \sum_{j,k,l=1}^{3}\frac{|\epsilon_{jkl}|}{2}\{n_j(\vec{x}), \hat{P}_k^A\hat{P}_l^A\} + \\ + \sum_{j=1}^{3}\{s_j(\vec{x}), \hat{P}_j^A\} + m(\vec{x}), \qquad (17)$$

where $\{\ ,\ \}$ denotes the symmetrization.

Only (16) obtains an $\hbar^2$–proportional correction

$$\vec{s}\cdot\nabla V + \frac{\hbar^2}{4}\left(\partial_z n_1\partial_z B_1 - \partial_y n_1\partial_y B_1 + \partial_x n_2\partial_x B_2 - \partial_z n_2\partial_z B_2 + \right. \\ \left. + \partial_y n_3\partial_y B_3 - \partial_x n_3\partial_x B_3 + \partial_x n_1\partial_y B_2 - \partial_y n_2\partial_x B_1\right) = 0. \qquad (18)$$

Integrable Hamiltonians

Let us now turn our attention to the situation when the Hamiltonian (1) or (4) is integrable in the Liouville sense, with at most quadratic integrals. That means that in addition to the Hamiltonian itself there must be at least two independent integrals of motion of the form (10) or (17) which commute in the sense of Poisson bracket or Lie commutator, respectively. Independence is to be understood as functional independence in the classical situation and in the sense that no nontrivial fully symmetrized polynomial in the given operators vanishes in the quantum case.

Keeping in mind that our main goal is to arrive at examples of superintegrable systems with nonvanishing magnetic field we shall assume that the integrability arises in the simplest way possible. Firstly, let us assume that there are at least two independent first order integrals for our Hamiltonian.

Integrals related to the Euclidean group

Under the assumption that the integral is of first order in momenta the conditions (13), (14), (15) and (16) simplify tremendously. We have $\vec{h} = \vec{n} = 0$ thus the first order term in X must lie in the enveloping algebra of the Euclidean algebra, i.e. be a linear combination of linear and angular momenta

$$X_1 = \gamma_1^i l_i^A + \beta_1^i p_i^A + m_1(\vec{x}), \qquad X_2 = \gamma_2^i l_i^A + \beta_2^i p_i^A + m_2(\vec{x}). \quad (19)$$

We may use the Euclidean transformations to simplify X_1, X_2. Another allowed transformation is replacing X_1 or X_2 by an arbitrary regular linear combination of them. For convenience, we redefine the yet unknown functions $m_1(\vec{x}), m_2(\vec{x})$ as needed without renaming them.

We arrive at the following possibilities

- If we have $\vec{\gamma}_1 = \vec{\gamma}_2 = 0$ then we can set

$$X_1 = p_1^A + m_1(\vec{x}), \qquad X_2 = p_2^A + m_2(\vec{x}). \tag{20}$$

- If $\vec{\gamma}_1 \neq 0$ we can transform X_1 into $X_1 = l_3^A + \beta p_3^A + m_1(\vec{x})$.
  - Assuming that the integrability arises directly at the first order, i.e. that $\{X_1, X_2\}_{P.B.} = 0$, we arrive at a single possibility

$$X_1 = l_3^A + m_1(\vec{x}), \qquad X_2 = p_3^A + m_2(\vec{x}). \tag{21}$$

  - However, there is another option - to allow X_1 and X_2 to be not in involution and expect the second commuting integral to arise via Poisson brackets and polynomial combinations of X_1, X_2. Thus we have up to rotation and linear combination

$$X_1 = l_3^A + \beta p_3^A + m_1(\vec{x}), \quad X_2 = \sigma l_1^A + \beta_2^i p_i^A + m_2(\vec{x}), \quad \sigma = 0, 1. \tag{22}$$

In order to have nontrivial dynamics, i.e. nontrivial electric and/or magnetic field, we cannot have the full Euclidean algebra represented in terms of the integrals of motion. Thus we must require that the algebra generated by the highest order terms $l_3 + \beta p_3$ and $\sigma l_1 + \beta_2^i p_i$ in (22) via Poisson brackets closes as a proper subalgebra of the Euclidean algebra. The options are:

1. The algebra isomorphic to $\mathfrak{su}(2)$

$$\begin{aligned} X_1 = l_3^A + m_1(\vec{x}), \quad X_2 = l_1^A + m_2(\vec{x}), \\ X_3 = \{X_1, X_2\}_{P.B.} = l_2^A + m_3(\vec{x}). \end{aligned} \qquad (23)$$

2. The algebra isomorphic to l_3, p_1, p_2

$$\begin{aligned} X_1 = l_3^A + p_3^A + m_1(\vec{x}), \qquad X_2 = p_1^A + m_2(\vec{x}), \\ X_3 = \{X_1, X_2\}_{P.B.} = p_2^A + m_3(\vec{x}). \end{aligned}$$

This is, however, already included in (20) as a special subcase.

Superintegrability for the integrable system with integrals P_1, P_2

Integrals (20)

$$X_1 = p_1^A + m_1(\vec{x}), \qquad X_2 = p_2^A + m_2(\vec{x})$$

in involution imply

$$\begin{aligned} B_j(\vec{x}) &= F_j'(z), \quad B_3(\vec{x}) = 0, \quad j = 1, 2, \\ m_1(\vec{x}) &= -F_2(z), \; m_2(\vec{x}) = F_1(z), \quad V(\vec{x}) = V(z). \end{aligned} \tag{24}$$

We choose the vector potential in the form satisfying Coulomb gauge condition $\nabla \vec{A} = 0$

$$A_1(\vec{x}) = F_2(z), \qquad A_2(\vec{x}) = -F_1(z), \qquad A_3(\vec{x}) = 0. \tag{25}$$

Plugging all the information obtained about functions $\vec{A}, \vec{B}, m_j$ into the assumed form of the integrals (20) we find a very simple solution (unique up to the choice of gauge)

$$X_1 = p_1, \qquad X_2 = p_2. \tag{26}$$

Superintegrability for the integrable system with integrals P_1, P_2, cont'd

Let us now assume that our system is superintegrable, i.e. that an additional independent integral of motion exists. For simplicity, let us assume that it is of first order in momenta. Up to addition of X_1 and X_2 we have

$$X_3 = \gamma^i l_i^A + \beta p_3^A + m_3(\vec{x}). \tag{27}$$

We arrive at only two possibilities for superintegrability:

- $F_1'' = F_2'' = 0$, i.e. the magnetic field (24) is constant. Solving equations (15) and (16) we find that the electrostatic potential is constant too, i.e. we have a motion in constant magnetic field and no electric field. Such system is superintegrable and exactly solvable as follows.

Superintegrability for the integrable system with integrals P_1, P_2, cont'd

Without loss of generality we can rotate the coordinate system so that

$$\vec{B}(\vec{x}) = (B, 0, 0), \qquad \vec{A}(\vec{x}) = (0, -B\,z, 0), \qquad V(\vec{x}) = 0. \quad (28)$$

We have four independent integrals which are of first order in momenta

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = p_3 - By, \quad X_4 = l_1 + \frac{B}{2}(z^2 - y^2). \quad (29)$$

By inspection of the solution of the equations of motion one finds that this system is maximally superintegrable with the fifth independent integral not polynomial in momenta, it reads

$$X_5 = (Bz - p_2)\cos\left(\frac{Bx}{p_1}\right) - p_3 \sin\left(\frac{Bx}{p_1}\right). \quad (30)$$

Superintegrability for the integrable system with integrals P_1, P_2, cont'd

- The only superintegrable possibility for a nonconstant $\vec{B}$ is

$$\begin{aligned}\vec{A}(\vec{x}) &= A\left(\cos\left(\frac{z}{\beta}\right), \sin\left(\frac{z}{\beta}\right), 0\right), \\ \vec{B}(\vec{x}) &= -\frac{A}{\beta}\left(\cos\left(\frac{z}{\beta}\right), \sin\left(\frac{z}{\beta}\right), 0\right), \quad V(\vec{x}) = 0.\end{aligned} \tag{31}$$

The integral of motion X_3 (27) reduces to

$$X_3 = l_3 + \beta p_3 \tag{32}$$

in the gauge chosen above. In the classical mechanics the Hamiltonian is maximally superintegrable with the fifth integral expressed in terms of Jacobi elliptic functions whose arguments depend on p_1, p_2 and l_3. That is deduced from the classical solution.

Superintegrability for the integrable system with integrals L_3, P_3

Performing a similar analysis for the case (21)

$$X_1 = l_3^A + m_1(\vec{x}), \qquad X_2 = p_3^A + m_2(\vec{x})$$

we find

$$\begin{aligned} & m_1(\vec{x}) = -F_2(R), \quad m_2(\vec{x}) = F_1(R), \quad R = \sqrt{x^2+y^2}, \\ & \vec{B}(\vec{x}) = \left(-F_1'\frac{y}{R}, F_1'\frac{x}{R}, \frac{1}{R}F_2'\right), \quad V(\vec{x}) = V(R), \\ & \vec{A}(\vec{x}) = \left(-\frac{y}{R^2}F_2(R), \frac{x}{R^2}F_2(R), -F_1(R)\right). \end{aligned} \tag{33}$$

Substituting (33) into our form of the integrals (21) we find that in our choice of gauge we have in fact

$$X_1 = l_3, \qquad X_2 = p_3, \tag{34}$$

i.e. the first order integrals are again of direct geometric origin.

Superintegrability for the integrable system with integrals L_3, P_3, cont'd

An explicit computation shows that the system with the potentials and the field strength (33) is not first order minimally superintegrable for any choice of the functions F_1 or F_2 other than F_1, F_2 constants, i.e. $\vec{B} = 0$. The same result applies also to the quantum case where only the difference between equations (16) and (18) needs to be considered.

Superintegrability for the integrable system with integrals L_1, L_2, L_3

Let us now turn our attention to the case when we have three first order integrals of motion (23). We cannot choose among them two in involution but we easily obtain a second order integral

$$(\vec{X})^2 = (X_1)^2 + (X_2)^2 + (X_3)^2 \tag{35}$$

which is in involution with all of them. Thus assuming that we have the integrals

$$X_1 = l_3^A + m_1(\vec{x}), \qquad X_2 = l_1^A + m_2(\vec{x})$$

we have immediately a minimally superintegrable system.
The compatibility of equations (15) for the three integrals X_1, X_2 and $X_3 = \{X_1, X_2\}_{P.B.} = l_2^A + m_3(\vec{x})$ leads to

$$\vec{B}(\vec{x}) = g \frac{\vec{x}}{|\vec{x}|^3}, \tag{36}$$

i.e. a magnetic monopole of an arbitrary strength g.

Superintegrability for the integrable system with integrals L_1, L_2, L_3, cont'd

From the condition (16) we find that the electrostatic potential $V(\vec{x})$ must be spherically symmetric,

$$V(\vec{x}) = V(|\vec{x}|). \tag{37}$$

Thus the classical Hamiltonian system (1) with the potentials and field strengths defined in (36), (37) is the only system which possesses the three first order integrals (23) and is minimally superintegrable because the Hamiltonian H is functionally independent of X_1, X_2, X_3.

Imposing that an additional independent integral X_4 of the form (10), i.e. at most second order in momenta, exists, we find only one system, namely the Coulomb potential modified by the $|\vec{x}|^{-2}$ term proportional to the strength of the magnetic monopole

$$V(\vec{x}) = \frac{g^2}{2}\frac{1}{|\vec{x}|^2} - \frac{Q}{|\vec{x}|}. \tag{38}$$

Superintegrability for the integrable system with integrals L_1, L_2, L_3, cont'd

We have three additional integrals of the given form which are the components of the Laplace-Runge-Lenz vector modified by the presence of the magnetic monopole. Of course, only one of them is functionally independent of the Hamiltonian and the integrals X_1, X_2, X_3.

The fact that the system defined by (36) and (38) is maximally superintegrable has been known for long time (see e.g. A. Peres. Phys. Rev, 167(5):1449, 1968 or S. Labelle, M. Mayrand, and L. Vinet. J. Math. Phys., 32(6):1516-1521, 1991). Here we have shown that under the restrictions imposed on the structure and order of the integrals there is no other maximally superintegrable case in this class.

Superintegrability for the integrable system with integrals L_1, L_2, L_3, cont'd

While it may be surprising that no modification of the isotropic harmonic oscillator arose in our calculation, we refer the reader to S. Labelle, M. Mayrand, and L. Vinet where it was demonstrated that it is superintegrable but of the fourth order in momenta, not at most second, as considered here.

1st & 2nd order integrals - work in progress

Next we consider another case where there is one first order integral, assumed in the form

$$X_1 = p_1^A + m_1(\vec{x}).$$

The conditions (13-16) imply that

$$B_2(\vec{x}) = -\partial_z m_1, B_3(\vec{x}) = \partial_y m_1, V(\vec{x}) = V(y,z), m_1(\vec{x}) = m_1(y,z).$$

The second integral we assume quadratic in linear momenta

$$\begin{aligned} X_2 &= \rho_{11}(p_1^A)^2 + \rho_{22}(p_2^A)^2 + \rho_{33}(p_3^A)^2 + \rho_{23}p_2^A p_3^A + \rho_{13}p_1^A p_3^A \\ &\quad + rho_{12}p_1^A p_2^A + s_{21}(\vec{x})p_1^A + s_{22}(\vec{x})p_2^A + s_{23}(\vec{x})p_3^A + m_2(\vec{x}). \end{aligned}$$

1st & 2nd order integrals - work in progress

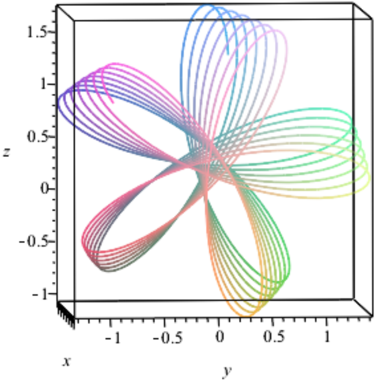
Using the residual Euclidean transformations and subtraction of X_1 and H we simplify the second integral X_2 and proceed to study various subcases. Among others, we find the following systems

$$\begin{aligned} H &= \frac{1}{2}(p_1 + K_1 z)^2 + \frac{1}{2}(p_2 - K_4 z)^2 + \frac{1}{2}p_3^2 + \frac{1}{2}K_1^2 y^2 + K_2 K_1 z, \\ X_1 &= p_1, \\ X_2 &= p_1 p_2 + K_1 p_2 z + K_2 p_2 - K_1 y p_3 - \frac{1}{2}K_1 K_4 z^2 + \frac{1}{2}K_1 K_4 y^2, \\ \vec{B}(\vec{x}) &= (K_4, K_1, 0). \end{aligned}$$

The system doesn't become quadratically superintegrable with nonvanishing $\vec{B}(\vec{x})$ for any choice of the constants.

1st & 2nd order integrals - work in progress

Its classical trajectories are not bounded and look like



1st & 2nd order integrals - work in progress

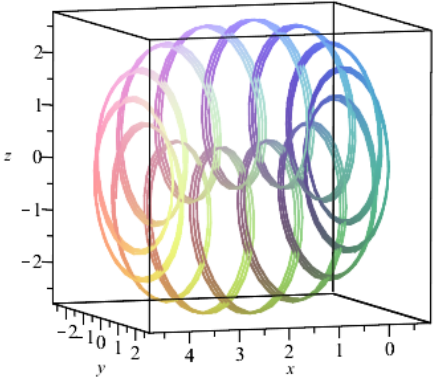
Another, related integrable system looks like

$$
\begin{aligned}
H &= \frac{1}{2}\left((p_1 - K_7 z)^2 + (p_2 - K_4 z)^2 + p_3^2\right. \\
&\quad \left. - K_1 K_7 y^2 - K_7 K_1 z^2 - K_7^2 z^2\right), \\
X_1 &= p_1, \\
X_2 &= p_1 p_2 - \frac{1}{2} K_1 K_4 z^2 + K_1 p_2 z - K_1 y p_3 + \frac{1}{2} K_1 K_4 y^2, \\
\vec{B}(\vec{x}) &= (K_4, -K_7, 0).
\end{aligned}
$$

Its classical trajectories are not bounded but by a suitable choice of initial data ($p_1 = 0$) can be constrained to a bounded region.

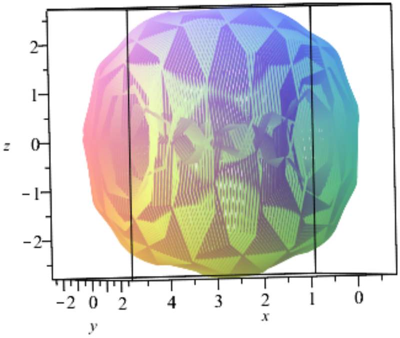
1st & 2nd order integrals - work in progress

They look like



1st & 2nd order integrals - work in progress

Waiting long enough they densely cover a self-intersecting surface in space.



1st & 2nd order integrals - work in progress

When $K_4 = 0$ there is an additional integral of motion of the form

$$X_3 = p_2^2 - K_1 K_7 y^2 \tag{39}$$

and the system becomes quadratically minimally superintegrable, with trajectories helixes or circles. However, it is not maximally quadratically superintegrable.

Conclusions

- We expressed the conditions for the existence of an integral of motion which is at most second order in momenta in a gauge invariant way.
- We looked in detail at Hamiltonians which possess two first order integrals of motion corresponding to the subgroups of the Euclidean group and some Hamiltonians possessing one first order and one second order integral. We described the implied structure of the Hamiltonian and studied the choices of the vector and scalar potential under which these integrable systems become superintegrable of first or second order in momenta.

Conclusions

- We have seen that maximal superintegrability in three spatial dimensions does not imply constant magnetic field, i.e. in 2D it is a consequence of low dimension.
- It appears that maximally superintegrable systems with integrals polynomial in momenta and nonvanishing magnetic field are more difficult to find compared to the scalar potential case. Even the explicitly solvable system with a constant magnetic field and vanishing electric field requires integrals which are not polynomials.

Thank you for your attention

Superintegrability for the integrable system with integrals P_1, P_2, cont'd

The classical equation of motion of (31) for $z(t)$ is

$$\ddot{z}(t) = -\frac{A\,p}{\beta}\sin\left(\frac{z(t)-\phi_p}{\beta}\right). \tag{40}$$

The order of this equation can be lowered, obtaining

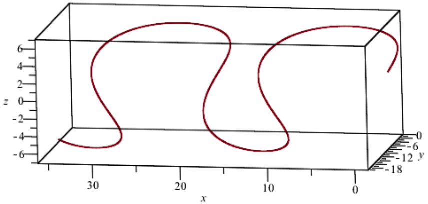
$$\frac{1}{2}\left(\dot{z}(t)\right)^2 = A\,p\left(\cos\left(\frac{z(t)-\phi_p}{\beta}\right)+\kappa\right), \qquad \kappa \geq -1 \tag{41}$$

($\kappa < -1$ is unphysical since then (41) doesn't have real solutions). The solution of (41) is expressible in terms of Jacobi elliptic function sn after we change the variables $z(t) = \phi_p + \beta\arccos(\zeta(t))$, $t = \frac{\beta}{\sqrt{2Ap}}\tau$ to get

$$(\dot{\zeta}(\tau))^2 = -(\zeta(\tau)-1)(\zeta(\tau)+1)(\zeta(\tau)+\kappa). \tag{42}$$

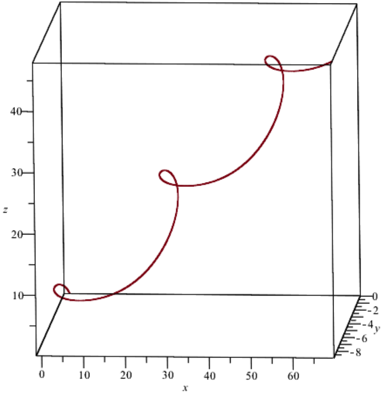
Superintegrability for the integrable system with integrals P_1, P_2, cont'd

The equations for $x(t), y(t)$ now reduce to quadratures in terms of it. Solving them numerically we obtain the trajectories for our system. For $-1 < \kappa < 1$ they are bounded in the plane perpendicular to $(p_1, p_2, 0)$ and appear like a deformed helix whose axis is parallel to the vector $(p_1, p_2, 0)$:



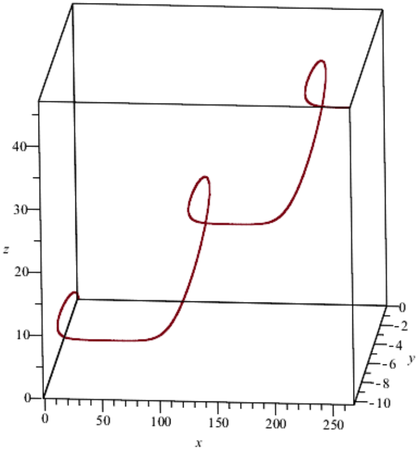
Superintegrability for the integrable system with integrals P_1, P_2, cont'd

For $1 \leq \kappa$ they are no longer bounded in the z-direction and appear like a deformed helix whose axis is no longer parallel to the xy-plane.



Superintegrability for the integrable system with integrals P_1, P_2, cont'd

The value $\kappa = 1$ appears to be a limiting case of the $\kappa > 1$ situation.



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