Classification of solvable algebras with the given nilradical - can the knowledge of solvable extensions of its nilpotent subalgebra be useful?

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Outline

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Classification of Lie algebras - general approach

What is known in general about the classification of finite–dimensional Lie algebras over the fields of complex and real numbers?

Indecomposable vs. decomposable Lie algebras

If $\mathfrak{g}$ is decomposable into a direct sum of ideals, it should be explicitly decomposed into components that are further indecomposable

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k. \tag{1}$$

Levi decomposition

Let $\mathfrak{g}$ denote an (indecomposable) Lie algebra. A fundamental theorem due to E. E. Levi[1] tells us that any Lie algebra can be represented as the semidirect sum

$$\mathfrak{g} = \mathfrak{r} \ni\!\!\!\!+ \mathfrak{l}, \;\; [\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}, \;\; [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}, [\mathfrak{l}, \mathfrak{r}] \subseteq \mathfrak{r}, \tag{2}$$

where $\mathfrak{l}$ is semisimple subalgebra and $\mathfrak{r}$ is the radical of $\mathfrak{g}$, i.e. its maximal solvable ideal.

We note that by virtue of Jacobi identities $\mathfrak{r}$ is a representation space for $\mathfrak{l}$ and that $\mathfrak{l}$ is isomorphic to a subalgebra of the derivations of $\mathfrak{r}$. These observations put a rather stringent compatibility conditions on the possible pairs of $\mathfrak{l}, \mathfrak{r}$ and can be employed in the classification of Levi decomposable algebras.

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Semisimple Lie algebras over the field of complex numbers $\mathbb{C}$ have been completely classified by Cartan[2], over the field of real numbers $\mathbb{R}$ by Gantmacher[3].

Algorithms realizing decompositions (1),(2) exist[4].

BUT not all solvable Lie algebras are known.

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Classification of solvable Lie algebras

There are two ways of proceeding in the classification of solvable Lie algebras: by dimension, or by structure.

The dimensional approach for real Lie algebras:

- dimension 2 and 3: Bianchi L 1918 *Lezioni sulla teoria dei gruppi continui finite di trasformazioni*, (Pisa: Enrico Spoerri Editore) p 550–557
- dimension 4: Kruchkovich GI 1954, *Usp. Mat. Nauk* **9** 59
- nilpotent up to dimension 6: Morozov V V 1958 *Izv. Vys. Uchebn. Zav. Mat.* **4** (5) 161–71
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The classification of low–dimensional Lie algebras over $\mathbb{C}$ was started earlier by S. Lie himself (Lie S and Engel F 1893 *Theorie der Transformationsgruppen III*, Leipzig: B.G. Teubner).

It seems to be neither feasible, nor fruitful to proceed by dimension in the classification of Lie algebras $\mathfrak{g}$ beyond $\dim\, \mathfrak{g} = 6$. It is however possible to proceed by structure, i.e. to classify solvable Lie algebras with the nilradical of a given type.

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- Heisenberg nilradicals: Rubin J and Winternitz P 1993 *J. Phys. A* **26** 1123–38,
- Abelian nilradicals: Ndogmo JC and Winternitz P 1994 *J. Phys. A* **27** 405–23,
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Basic concepts and notation

Three series of subalgebras – characteristic series of $\mathfrak{g}$:

- derived series $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined
$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}],\ \mathfrak{g}^{(0)} = \mathfrak{g}.$$
If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is solvable.
- lower central series $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined
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If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ nilpotent. The lowest value of k s.t. $\mathfrak{g}^k = 0$ is the degree of nilpotency.
- upper central series $\mathfrak{z}_1 \subseteq \ldots \subseteq \mathfrak{z}_k \subseteq \ldots \subseteq \mathfrak{g}$ where $\mathfrak{z}_1$ is the center of $\mathfrak{g}$, $\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0,\ \forall y \in \mathfrak{g}\}$ and $\mathfrak{z}_k$ are the higher centers defined recursively through
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Any solvable Lie algebra $\mathfrak{s}$ has a uniquely defined nilradical $\mathrm{NR}(\mathfrak{s})$, i.e. maximal nilpotent ideal. Its dimension satisfies

$$\dim\ \mathrm{NR}(\mathfrak{s}) \geq \frac{1}{2}\,(\dim\ \mathfrak{s} + \dim\ C(\mathfrak{s})). \qquad (3)$$

The derived algebra of a solvable Lie algebra $\mathfrak{s}$ is contained in the nilradical, i.e.

$$[\mathfrak{s},\mathfrak{s}] \subseteq \mathrm{NR}(\mathfrak{s}). \qquad (4)$$

The centralizer $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

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A derivation D of a given Lie algebra $\mathfrak{g}$ is a linear map

$$D: \ \mathfrak{g} \to \mathfrak{g}$$

such that for any pair x, y of elements of $\mathfrak{g}$

$$D([x,y]) = [D(x), y] + [x, D(y)]. \tag{6}$$

If an element $z \in \mathfrak{g}$ exists, such that

$$D = \mathrm{ad}_z, \ \ \text{i.e. } D(x) = [z, x], \ \forall x \in G,$$

the derivation is inner, any other one is outer.

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Classification of solvable Lie algebras with the given nilradical

We assume that the nilradical $\mathfrak{n}$, $\dim \mathfrak{n} = n$ is known. That is, in some basis $(e_1, \ldots, e_n)$ we know the Lie brackets

$$[e_a, e_b] = N_{ab}{}^c e_c. \tag{8}$$

We wish to extend the nilpotent algebra $\mathfrak{n}$ to all possible indecomposable solvable Lie algebras $\mathfrak{s}$ having $\mathfrak{n}$ as their nilradical. Thus, we add further elements $f_1, \ldots, f_p$ to the basis $(e_1, \ldots, e_n)$ which together will form a basis of $\mathfrak{s}$. It follows from (4) that

$$\begin{aligned} [f_i, e_a] &= (A_i)_a^b e_b, \ 1 \leq i \leq p, \ 1 \leq a \leq n, \\ [f_i, f_j] &= \gamma_{ij}^a e_a, \ 1 \leq i, j \leq p. \end{aligned} \tag{9}$$

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We have

- Jacobi identities between $(f_i, e_a, e_b) \Longrightarrow$ linear relations on the matrix elements of A_i
- Jacobi identities between $(f_i, f_j, e_a) \Longrightarrow$ linear relations on γ_{ij}^a in terms of the commutators of A_i and A_j.
- Jacobi identities between $(f_i, f_j, f_k) \Longrightarrow$ bilinear compatibility conditions on γ_{ij}^a and A_i.

Since $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{s}$, no nontrivial linear combination of A_i can be a nilpotent matrix, i.e. they are linearly nil–independent.

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Let us consider the adjoint representation of $\mathfrak{s}$ restricted to the nilradical $\mathfrak{n}$. Then $\mathrm{ad}|_{\mathfrak{n}}(f_k)$ is a derivation of $\mathfrak{n}$. In other words, finding all sets of matrices A_i in (9) is equivalent to finding all sets of outer nil–independent derivations of $\mathfrak{n}$

$$D^1 = \mathrm{ad}|_{\mathfrak{n}}(f_1), \ldots, D^p = \mathrm{ad}|_{\mathfrak{n}}(f_p), \tag{10}$$

such that $[D^j, D^k]$ are inner derivations. γ_{ij}^a are then determined up to elements in the center $C(\mathfrak{n})$ of $\mathfrak{n}$, i.e. the knowledge of all sets of such derivations almost amounts to the knowledge of all solvable Lie algebras with the given nilradical $\mathfrak{n}$.

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Isomorphic Lie algebras with the given nilradical

If we

1. add any inner derivation to D^k, i.e. we consider outer derivations modulo inner derivations,
2. perform a change of basis in $\mathfrak{n}$ such that the Lie brackets (8) are not changed, i.e. we consider only conjugacy classes of sets of outer derivations (modulo inner derivations)
3. change the basis in the space $\mathrm{span}\{D^1, \dots, D^p\}$,

the resulting Lie algebra is isomorphic to the original one.

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Our nilradical $\mathfrak{n}_{n,3}$

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\begin{aligned}
[e_2, e_n] &= e_1, \\
[e_3, e_{n-1}] &= e_1, \\
[e_4, e_n] &= e_2, \\
[e_k, e_n] &= e_{k-1}, \qquad 5 \leq k \leq n-2, \\
[e_{n-1}, e_n] &= -e_3.
\end{aligned} \tag{11}
$$

It has the following complete flag of invariant ideals

$$
\begin{aligned}
0 &\subset \mathfrak{n}^{n-3} \subset \mathfrak{n}^{n-4} \subset \mathfrak{z}_2 \subset \mathfrak{z}_3 \cap \mathfrak{n}^2 \subset \ldots \subset \mathfrak{z}_{n-5} \cap \mathfrak{n}^2 \subset \mathfrak{n}^2 \\
&\subset (\mathfrak{z}_2)_{\mathfrak{n}} \subset (\mathfrak{n}^{n-4})_{\mathfrak{n}} \subset \mathfrak{n}
\end{aligned} \tag{12}
$$

and a subalgebra isomorphic to $\mathfrak{n}_{n-2,1}$ expressed as

$$\tilde{\mathfrak{n}}_{n-2,1} = \mathrm{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}.$$

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and a subalgebra isomorphic to $\mathfrak{n}_{n-2,1}$ expressed as

$$\tilde{\mathfrak{n}}_{n-2,1} = \mathrm{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}.$$

Our nilradical $\mathfrak{n}_{n,3}$

$$\begin{aligned}
[e_2, e_n] &= e_1, \\
[e_3, e_{n-1}] &= e_1, \\
[e_4, e_n] &= e_2, \\
[e_k, e_n] &= e_{k-1}, \qquad 5 \leq k \leq n-2, \\
[e_{n-1}, e_n] &= -e_3.
\end{aligned} \tag{11}$$

It has the following complete flag of invariant ideals

$$\begin{aligned}
0 &\subset \mathfrak{n}^{n-3} \subset \mathfrak{n}^{n-4} \subset \mathfrak{z}_2 \subset \mathfrak{z}_3 \cap \mathfrak{n}^2 \subset \ldots \subset \mathfrak{z}_{n-5} \cap \mathfrak{n}^2 \subset \mathfrak{n}^2 \\
&\subset (\mathfrak{z}_2)_{\mathfrak{n}} \subset (\mathfrak{n}^{n-4})_{\mathfrak{n}} \subset \mathfrak{n}
\end{aligned} \tag{12}$$

and a subalgebra isomorphic to $\mathfrak{n}_{n-2,1}$ expressed as

$$\tilde{\mathfrak{n}}_{n-2,1} = \mathrm{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}.$$

What is important for us is that the solvable extensions of $\mathfrak{n}_{n-2,1}$ were fully investigated in Šnobl L and Winternitz P 2005, A class of solvable Lie algebras and their Casimir invariants, *J. Phys. A* **38** 2687. At the same time, the group of automorphisms of $\mathfrak{n}_{n-2,1}$ is almost the same as the one induced on $\tilde{\mathfrak{n}}_{n-2,1}$ by automorphisms of $\mathfrak{n}_{n,3}$. More precisely, locally they are identical, globally they differ by one reflection allowed in $\mathfrak{n}_{n-2,1}$ but not in $\tilde{\mathfrak{n}}_{n-2,1}$.

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Our current approach is as follows

1. Check which conjugacy classes of elements in $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ can be represented by derivations which preserve $\tilde{\mathfrak{n}}_{n-2,1}$.
2. For these find all solvable extensions of $\tilde{\mathfrak{n}}_{n-2,1}$ and extend them to solvable extensions of $\mathfrak{n}_{n,3}$. In this process some new parameters may arise, i.e. the extension is not necessarily unique.
3. Consider the classes of derivations whose no representative preserves $\tilde{\mathfrak{n}}_{n-2,1}$ and construct the corresponding solvable extensions.

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Some details

The most general derivation has the form

$$
\begin{aligned}
D(e_{n-2}) &= (2c_{n-1} + (5-n)d_n)e_{n-2} + \sum_{k=4}^{n-3} b_k e_k + b_2 e_2 + b_1 e_1, \\
D(e_{n-1}) &= c_{n-1} e_{n-1} + d_{n-1} e_4 + \sum_{k=1}^{3} c_k e_k, \qquad (13) \\
D(e_n) &= \sum_{k=1}^{n} d_k e_k.
\end{aligned}
$$

The action of D on the remaining basis elements $e_1, \ldots, e_{n-3}$ is found using $D([x,y]) = [D(x), y] + [x, D(y)]$.

In the $2n$–dimensional algebra of derivations $\mathfrak{Der}(\mathfrak{n}_{n,3})$ we have $(n-1)$–dimensional ideal of inner derivations $\mathfrak{Inn}(\mathfrak{n}_{n,3})$ of the form

$$\begin{aligned} D(e_{n-2}) &= -c_3 e_{n-3}, \\ D(e_{n-1}) &= c_3 e_3 + c_1 e_1, \\ D(e_n) &= \sum_{k=1}^{n-3} d_k e_k. \end{aligned} \tag{14}$$

The elements of $\mathfrak{Der}(\mathfrak{n}_{n,3})/\mathfrak{Inn}(\mathfrak{n}_{n,3})$ can be uniquely represented by outer derivations of the form

$$\begin{aligned} D(e_{n-2}) &= (2c_{n-1} + (5-n)d_n)e_{n-2} + \sum_{k=4}^{n-4} b_k e_k + b_2 e_2 + b_1 e_1, \\ D(e_{n-1}) &= c_{n-1}e_{n-1} + d_{n-1}e_4 + c_3 e_3 + c_2 e_2, \qquad (15) \\ D(e_n) &= d_n e_n + d_{n-1}e_{n-1} + d_{n-2}e_{n-2}. \end{aligned}$$

The derivation of the form (15) leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant if and only if $d_{n-1} = 0$. We conjugate a given derivation D by the automorphism defined by

$$e_{n-2} \to e_{n-2},\ e_{n-1} \to e_{n-1} + \frac{d_{n-1}}{d_n - c_{n-1}} e_4,\ e_n \to e_n + \frac{d_{n-1}}{d_n - c_{n-1}} e_{n-1}$$

whenever possible, i.e. when $d_n \neq c_{n-1}$.

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whenever possible, i.e. when $d_n \neq c_{n-1}$.

Now we have $\hat{d}_{n-1} = 0$, i.e. $D_\Phi \equiv \hat{D}$ leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant and we can proceed by investigation of its solvable extensions.

We find that the extension of a solvable algebra with the nilradical $\tilde{\mathfrak{n}}_{n-2,1}$ to a solvable extension of the nilradical $\mathfrak{n}_{n,3}$ is unique when $d_n \neq 0$ and $c_{n-1} \neq 0$; otherwise, several non–equivalent extensions do exist.

The case when none of the conjugate derivations D_Φ leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant necessarily means $d_n = c_{n-1} \to 1$, $d_{n-1} \neq 0$ and leads to a unique solvable algebra $\mathfrak{s}_{n+1,9}$ in the list below.

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The case when none of the conjugate derivations D_Φ leaves $\tilde{\mathfrak{n}}_{n-2,1}$ invariant necessarily means $d_n = c_{n-1} \rightarrow 1$, $d_{n-1} \neq 0$ and leads to a unique solvable algebra $\mathfrak{s}_{n+1,9}$ in the list below.

The resulting classification

Any solvable Lie algebra $\mathfrak{s}$ with the nilradical $\mathfrak{n}_{n,3}$ has dimension $\dim \mathfrak{s} = n + 1$, or $\dim \mathfrak{s} = n + 2$.

Five types of solvable Lie algebras of dimension $\dim \mathfrak{s} = n + 1$ with the nilradical $\mathfrak{n}_{n,3}$ exist for any $n \geq 7$. They are represented by the following:

$$
\begin{aligned}
[f, e_1] &= (\alpha + 2\beta)e_1, \\
[f, e_2] &= 2\beta e_2, \\
[f, e_3] &= (\alpha + \beta)e_3, \\
[f, e_k] &= ((3 - k)\alpha + 2\beta)e_k, \ 4 \leq k \leq n - 2, \\
[f, e_{n-1}] &= \beta e_{n-1}, \\
[f, e_n] &= \alpha e_n.
\end{aligned}
$$

The classes of mutually nonisomorphic algebras of this type are

$$\begin{aligned}
\mathfrak{s}_{n+1,1}(\beta): &\qquad \alpha = 1, \beta \in \mathrm{F}\backslash\{0, -\frac{1}{2}, \frac{n-5}{2}\},\\
\mathfrak{s}_{n+1,2}: &\qquad \alpha = 1, \beta = \frac{n-5}{2},\\
\mathfrak{s}_{n+1,3}: &\qquad \alpha = 1, \beta = 0,\\
\mathfrak{s}_{n+1,4}: &\qquad \alpha = 1, \beta = -\frac{1}{2},\\
\mathfrak{s}_{n+1,5}: &\qquad \alpha = 0, \beta = 1,
\end{aligned}$$

where the splitting into subcases reflects different dimensions of the characteristic series.

$$
\mathfrak{s}_{n+1,6}(\epsilon): \qquad
\begin{array}{rcl}
[f, e_1] &=& (n-3)e_1, \\
[f, e_2] &=& (n-4)e_2, \\
[f, e_3] &=& (1 + \frac{n-4}{2})e_3, \\
[f, e_k] &=& (n-1-k)e_k, \ 4 \leq k \leq n-2, \\
[f, e_{n-1}] &=& \frac{n-4}{2} e_{n-1}, \\
[f, e_n] &=& e_n + \epsilon e_{n-2}
\end{array}
$$

where $\epsilon = 1$ over $\mathbb{C}$, whereas over $\mathbb{R}$ $\epsilon = 1$ for n odd, $\epsilon = \pm 1$ for n even.

$$
\begin{aligned}
\mathfrak{s}_{n+1,7}: \qquad [f,e_1] &= e_1,\\
[f,e_2] &= 0,\\
[f,e_3] &= e_3 - e_1,\\
[f,e_k] &= (3-k)e_k,\ 4 \le k \le n-2,\\
[f,e_{n-1}] &= e_2,\\
[f,e_n] &= e_n.
\end{aligned}
$$

$$
\begin{array}{rcl}
& & \mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3}) : \\
[f, e_1] & = & e_1, \qquad [f, e_2] = e_2, \\
[f, e_3] & = & \frac{1}{2} e_3, \\
[f, e_k] & = & e_k + \sum_{l=4}^{k-2} a_{k-l+1} e_l + a_{k-2} e_2 + a_{k-1} e_1, \; 4 \leq k \leq n-2, \\
[f, e_{n-1}] & = & \frac{1}{2} e_{n-1} + a_2 e_3, \\
[f, e_n] & = & 0,
\end{array}
$$

$a_j \in \mathbb{F}$, at least one a_j satisfies $a_j \neq 0$. Over $\mathbb{C}$: the first $a_j \neq 0$ satisfies $a_j = 1$. Over $\mathbb{R}$: the first $a_j \neq 0$ for even j satisfies $a_j = 1$. If all $a_j = 0$ for j even, then the first $a_j \neq 0$ (j odd) satisfies $a_j = \pm 1$.

$$
\mathfrak{s}_{n+1,9}: \quad \begin{aligned}
[f, e_1] &= 3e_1, \\
[f, e_2] &= 2e_2, \\
[f, e_3] &= 2e_3 - e_2, \\
[f, e_k] &= (5-k)e_k, \ 4 \le k \le n-2, \\
[f, e_{n-1}] &= e_{n-1} + e_4, \\
[f, e_n] &= e_n + e_{n-1}.
\end{aligned}
$$

Precisely one solvable Lie algebra $\mathfrak{s}_{n+2}$ of $\dim \mathfrak{s} = n + 2$ with the nilradical $\mathfrak{n}_{n,3}$ exists. It is presented in a basis $(e_1, \ldots, e_n, f_1, f_2)$ where the Lie brackets involving f_1 and f_2 are

$$\begin{aligned}
[f_1, e_1] &= e_1,\ [f_2, e_1] = 2e_1, \\
[f_1, e_2] &= 0,\ [f_2, e_2] = 2e_2, \\
[f_1, e_3] &= e_3,\ [f_2, e_3] = e_3, \\
[f_1, e_k] &= (3-k)e_k,\ [f_2, e_k] = 2e_k,\ 4 \leq k \leq n-2, \\
[f_1, e_{n-1}] &= 0,\ [f_2, e_{n-1}] = e_{n-1}, \\
[f_1, e_n] &= e_n, [f_2, e_n] = 0,\ [f_1, f_2] = 0.
\end{aligned}$$

For $n = 5, 6$ the results are slightly different.

Generalized Casimir invariants

The term Casimir operator is usually reserved for elements of the center of the enveloping algebra of a Lie algebra $\mathfrak{g}$. These operators are in one–to–one correspondence with polynomial invariants characterizing orbits of the coadjoint representation of $\mathfrak{g}$. The search for invariants of the coadjoint representation is algorithmic and amounts to solving a system of linear first order PDEs (see[5]). In general, solutions are not necessarily polynomials and we shall call the general solutions generalized Casimir invariants or invariants of the coadjoint representation.

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Let us consider some basis $(g_1, \dots, g_n)$ of $\mathfrak{g}$, in which the structure constants are ${c_{ij}}^k$. A basis for the coadjoint representation is given by the first order differential operators

$$\hat{G}_k = g_b {c_{ka}}^b \frac{\partial}{\partial g_a}, \tag{16}$$

where the quantities g_a are commuting independent variables which can be identified with coordinates in the dual basis of the space $\mathfrak{g}^*$ (i.e. $g_a \equiv g_a^{**}$).

The generalized Casimir invariants are functions on $\mathfrak{g}^*$, solutions of the following system of partial differential equations

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$$\hat{G}_k I = 0, \; k = 1, \ldots, n. \qquad (17)$$

Considering first the nilpotent algebra $\mathfrak{n}_{n,3}$ we have the operators (16) in the form

$$\begin{aligned}\hat{E}_1 &= 0, \ \ \hat{E}_2 = e_1 \frac{\partial}{\partial e_n}, \ \ E_3 = e_1 \frac{\partial}{\partial e_{n-1}}, \ \ \hat{E}_4 = e_2 \frac{\partial}{\partial e_n}, \\ \hat{E}_k &= e_{k-1} \frac{\partial}{\partial e_n}, \ 5 \leq k \leq n-2, \ \ \hat{E}_{n-1} = -e_1 \frac{\partial}{\partial e_3} - e_3 \frac{\partial}{\partial e_n}, \\ \hat{E}_n &= -e_1 \frac{\partial}{\partial e_2} - e_2 \frac{\partial}{\partial e_4} - \sum_{k=5}^{n-2} e_{k-1} \frac{\partial}{\partial e_k} + e_3 \frac{\partial}{\partial e_{n-1}}.\end{aligned}$$

It is evident that any solution I of Eq. (17) cannot depend on e_3, e_{n-1} because of $\hat{E}_{n-1} I = \hat{E}_3 I = \hat{E}_2 I = 0$. Consequently, all considered operators $\hat{E}_j$ can be truncated to act on functions of $\tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \tilde{e}_3 = e_4, \ldots, \tilde{e}_{n-3} = e_{n-2}, \tilde{e}_{n-2} = e_n$ only.

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Then $\hat{E}_{3T}, \hat{E}_{n-1T}$ vanish and the remaining operators are exactly those present in the investigation of invariants of $\mathfrak{n}_{n-2,1}$.

Therefore, the generalized Casimir invariants of $\mathfrak{n}_{n,3}$ are the same as the ones for $\mathfrak{n}_{n-2,1}$ once written in an appropriate basis.

The nilpotent Lie algebra $\mathfrak{n}_{n,3}$ has $n-4$ functionally independent invariants. They can be chosen to be the following polynomials

$$\begin{aligned} \xi_0 &= e_1, \\ \xi_k &= \frac{(-1)^k k}{(k+1)!} e_2^{k+1} + \sum_{j=0}^{k-1} (-1)^j \frac{e_2^j \ e_{k+3-j} \ e_1^{k-j}}{j!}, \ 1 \le k \le n-5. \end{aligned}$$

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A similar argument works also for its solvable extensions.

The algebras $\mathfrak{s}_{n+1,1}(\beta), \dots, \mathfrak{s}_{n+1,9}$ have $n-5$ invariants each. Their form is

- $\mathfrak{s}_{n+1,1}(\beta)$, $\mathfrak{s}_{n+1,2}$, $\mathfrak{s}_{n+1,3}$, $\mathfrak{s}_{n+1,6}$, $\mathfrak{s}_{n+1,7}$ and $\mathfrak{s}_{n+1,9}$

$$\chi_k = \frac{\xi_k}{\xi_0^{(k+1)\frac{2\beta}{1+2\beta}}}, \; 1 \leq k \leq n-5. \tag{18}$$

  For $\mathfrak{s}_{n+1,2}$ is $\beta = \frac{n-5}{2}$, for $\mathfrak{s}_{n+1,3}$ and $\mathfrak{s}_{n+1,7}$ we have $\beta = 0$, for $\mathfrak{s}_{n+1,6}(\epsilon)$ we have $\beta = \frac{n-4}{2}$ and for $\mathfrak{s}_{n+1,9}$ is $\beta = 1$, respectively in Equation (18).

- $\mathfrak{s}_{n+1,4}$

$$\chi_1 = \xi_0, \; \chi_k = \frac{\xi_k^2}{\xi_1^{k+1}}, \; 2 \leq k \leq n-5. \tag{19}$$

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- $\mathfrak{s}_{n+1,5}$

$$\chi_k = \frac{\xi_k}{\xi_0^{k+1}}, \ 1 \leq k \leq n-5. \qquad (20)$$

- $\mathfrak{s}_{n+1,8}(a_2, a_3, \ldots, a_{n-3})$

$$\begin{aligned}\chi_k &= \sum_{q=0}^{[\frac{k+1}{2}]} (-1)^q \frac{(\ln \xi_0)^q}{q!} \left(\sum_{i_1+\ldots+i_q=k-2q+1} a_{i_1+3} a_{i_2+3} \ldots a_{i_q+3} \right. \\ &+ \left. \sum_{j+i_1+\ldots+i_q=k-2q-1} \frac{\xi_{j+1}}{\xi_0^{j+2}} \ a_{i_1+3} a_{i_2+3} \ldots a_{i_q+3} \right), \\ & 1 \leq k \leq n-5. \qquad (21)\end{aligned}$$

The summation indices take the values
$0 \leq j, i_1, \ldots, i_q \leq k+1$.

The Lie algebra $\mathfrak{s}_{m+2}$ has $n-6$ functionally independent invariants that can be chosen to be

$$\chi_k = \frac{\xi_{k+1}}{\xi_1^{\frac{k+2}{2}}},\ 1 \leq k \leq n-6. \tag{22}$$

Thank you for your attention