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CLASSIFICATION OF SIX-DIMENSIONAL REAL DRINFELD DOUBLES

LIBOR ŠNOBL* and LADISLAV HLAVATÝ†

Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University, Břehová 7, Prague 1, 115 19, Czech Republic
**snobl@newton.fjfi.cvut.cz*
†hlavaty@br.fjfi.cvut.cz

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Starting from the classification of real Manin triples we look for those that are isomorphic as six-dimensional Drinfeld doubles i.e. Lie algebras with the ad-invariant form used for construction of the Manin triples. We use several invariants of the Lie algebras to distinguish the nonisomorphic structures and give the explicit form of maps between Manin triples that are decompositions of isomorphic Drinfeld doubles. The result is a complete list of six-dimensional real Drinfeld doubles. It consists of 22 classes of nonisomorphic Drinfeld doubles.

Keywords: Drinfeld doubles; Manin triples; Lie algebras; Lie bialgebras; T-duality.

1. Introduction

In recent years, the study of T-duality in string theory has led to the discovery of Poisson–Lie T-dual sigma models. Klimčík and Ševera have found a procedure allowing us to construct the dual models from Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$, i.e. a decompositions of a Lie algebra $\mathcal{D}$ (it must be even-dimensional) into two maximally isotropic subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ w.r.t. a bilinear form. The construction of the Poisson–Lie T-dual sigma models is described in Refs. 1 and 2.

The Lie group possessing a Lie algebra that can be written as a Manin triple is called the Drinfeld double. The classification of the two-dimensional Drinfeld doubles is trivial and the four-dimensional Drinfeld doubles can be found e.g. in the paper Ref. 3 together with the corresponding two-dimensional T-dual models. Examples of six-dimensional Drinfeld doubles and three-dimensional dual models were given e.g. in Refs. 4–6. There was an attempt to classify the six-dimensional Drinfeld doubles by the Bianchi forms of their three-dimensional isotropic subalgebras in Ref. 6 but it is not sufficient for the specification of the Drinfeld double.

As we shall see Manin triples are equivalent to Lie bialgebras and the classification of the three-dimensional Lie bialgebras (i.e. six-dimensional Manin triples) was given in Ref. 7. Without knowledge of this this work we have performed a classification of the six-dimensional Manin triples in Ref. 8. The consequent comparison proved that the results are identical even though we have started from a different description of the three-dimensional algebras and used a completely different method. It means that in Ref. 8 we have done an independent check of Ref. 7 and on the other hand, expressed the results in a different form, namely as Manin triples.

The goal of this paper is to find which of the Manin triples represent decomposition of the same (or more precisely isomorphic) Drinfeld doubles. We use the notation of Ref. 8 because the less compact sorting of the triples into parametrized classes turned out more appropriate for the classification. The result is a complete list of the real nonisomorphic six-dimensional Drinfeld doubles. Let us note that the Drinfeld double is defined not only by its Lie structure but also by a bilinear form. There are e.g. two classes of Drinfeld doubles for so(1, 3) as we shall see.

In the following sections, we first recall the definitions of Manin triple, Lie bialgebra and Drinfeld double, then briefly explain the approach we have used to distinguish the nonisomorphic structures. The main result of the paper is the classification theorem in Sec. 3. Explicit forms of maps between Manin triples that are decompositions of the isomorphic Drinfeld doubles are contained in the proof of the theorem.

2. Manin Triples, Lie Bialgebras, Drinfeld Doubles

The Drinfeld double D is defined as a connected Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ can be decomposed into a pair of subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ maximally isotropic w.r.t. $\langle\cdot,\cdot\rangle$ and $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. This ordered triple of algebras $(\mathcal{D},\mathcal{G},\tilde{\mathcal{G}})$ is called Manin triple.

One can see that the dimensions of the subalgebras are equal and that bases $\{X_i\}$, $\{\tilde{X}^i\}$, $i = 1, 2, 3$ in the subalgebras can be chosen so that

$$\langle X_i, X_j\rangle = 0\,, \quad \langle X_i, \tilde{X}^j\rangle = \langle \tilde{X}^j, X_i\rangle = \delta_i^j\,, \quad \langle \tilde{X}^i, \tilde{X}^j\rangle = 0\,. \tag{1}$$

This canonical form of the bracket is invariant with respect to the transformations

$$X'_i = X_k A_i^k\,, \qquad \tilde{X}'^j = (A^{-1})^j_k \tilde{X}^k\,. \tag{2}$$

The Manin triples that are related by the transformation (2) are considered isomorphic. Due to the ad-invariance of $\langle\cdot,\cdot\rangle$ the algebraic structure of $\mathcal{D}$ is determined by the structure of the maximally isotropic subalgebras because in the basis $\{X_i\}$, $\{\tilde{X}^i\}$ the Lie product is given by

$$\begin{aligned}
[X_i, X_j] &= f_{ij}{}^k X_k\,,\\
[\tilde{X}^i, \tilde{X}^j] &= \tilde{f}^{ij}{}_k \tilde{X}^k\,,\\
[X_i, \tilde{X}^j] &= f_{ki}{}^j \tilde{X}^k + \tilde{f}^{jk}{}_i X_k\,.
\end{aligned} \tag{3}$$

It is clear that to any Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ one can construct the dual one by interchanging $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$, i.e. interchanging the structure coefficients ${f_{ij}}^k \leftrightarrow \widetilde{f^{ij}}_k$. All properties of Lie algebras (the nontrivial being the Jacobi identities) remain to be satisfied. On the other hand for given Drinfeld double more than two Manin triples can exist and we shall see many examples of that.

One can rewrite the structure of a Manin triple also in another, equivalent, but for certain considerations more suitable, form of a Lie bialgebra defined as a Lie algebra g equipped also by a Lie cobracket[a] $\delta : g \to g \otimes g : \delta(x) = \sum x_{[1]} \otimes x_{[2]}$ such that

$$\sum x_{[1]} \otimes x_{[2]} = -\sum x_{[2]} \otimes x_{[1]}\,, \tag{4}$$

$$(id \otimes \delta) \circ \delta(x) + \text{cyclic permutations of tensor indices} = 0\,, \tag{5}$$

$$\begin{aligned}\delta([x,y]) = \sum &[x, y_{[1]}] \otimes y_{[2]} + y_{[1]} \otimes [x, y_{[2]}] \\ &- [y, x_{[1]}] \otimes x_{[2]} - x_{[1]} \otimes [y, x_{[2]}]\end{aligned} \tag{6}$$

(for detailed account on Lie bialgebras see e.g. Ref. 9 or 10, Chapter 8).

The correspondence between a Manin triple and a Lie bialgebra can now be formulated in the following way. Because both subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ of the Manin triple are of the same dimension and are connected by nondegenerate pairing, it is natural to consider $\tilde{\mathcal{G}}$ as a dual $\mathcal{G}^*$ to $\mathcal{G}$ and to use the Lie bracket in $\tilde{\mathcal{G}}$ to define the Lie cobracket in $\mathcal{G}$; $\delta(x)$ is given by $\langle \delta(x), \tilde{y} \otimes \tilde{z} \rangle = \langle x, [\tilde{y}, \tilde{z}] \rangle$, $\forall \tilde{y},\ \tilde{z} \in \mathcal{G}^*$, i.e. $\delta(X_i) = \widetilde{f_i^{jk}} X_j \otimes X_k$. The Jacobi identities in $\tilde{\mathcal{G}}$

$$\widetilde{f_m^{kl}}\widetilde{f_l^{ij}} + \widetilde{f_m^{il}}\widetilde{f_l^{jk}} + \widetilde{f_m^{jl}}\widetilde{f_l^{ki}} = 0 \tag{7}$$

are then equivalent to the property of cobracket (5) and the $\tilde{\mathcal{G}}$-component of the mixed Jacobi identities[b]

$$\widetilde{f^{jk}}_l {f_{mi}}^l + \widetilde{f^{kl}}_m {f_{li}}^j + \widetilde{f^{jl}}_i {f_{lm}}^k + \widetilde{f^{jl}}_m {f_{il}}^k + \widetilde{f^{kl}}_i {f_{lm}}^j = 0 \tag{8}$$

are equivalent to (6).

From now on, we will use the formulation in terms of Manin triples, Lie bialgebra formulation of all results can be easily derived from it. We also consider only algebraic structure, the Drinfeld doubles as the Lie groups can be obtained in principle by means of exponential map and usual theorems about relation between Lie groups and Lie algebras apply, e.g. there is a one to one correspondence between (finite-dimensional) Lie algebras and connected and simply connected Lie groups. The group structure of the Drinfeld double can be deduced e.g. by taking matrix exponential of adjoint representation of its algebra.

[a]Summation index is suppressed.
[b]The Jacobi identities $[X_i, [\tilde{X}^j, \tilde{X}^k]] + \text{cyclic} = 0$ lead to both (8) (terms proportional to $\tilde{X}^l$) and (7) (terms proportional to X_l).

We shall consider two Drinfeld doubles isomorphic if they have isomorphic algebraic structure and there is an isomorphism transforming one ad-invariant bilinear form to the other. As mentioned above we can always choose a basis so that the bilinear form have canonical form (1) and the Lie product is then given by (3). The Drinfeld doubles $\mathcal{D}$ and $\mathcal{D}'$ with these special bases $Y_a = (X_1, X_2, X_3, \tilde{X}^1, \tilde{X}^2, \tilde{X}^3)$, $Y'_a = (X'_1, X'_2, X'_3, \tilde{X}'^1, \tilde{X}'^2, \tilde{X}'^3)$ are isomorphic iff there is an invertible 6×6 matrix ${C_a}^b$ such that the linear map given by

$$Y'_a = {C_a}^b Y_b \tag{9}$$

transforms the Lie multiplication of $\mathcal{D}$ into that of $\mathcal{D}'$ and preserves the canonical form of the bilinear form $\langle\cdot,\cdot\rangle$. This is equivalent to

$${C_a}^p {C_b}^q B_{pq} = B_{ab}\,, \qquad {C_a}^p {C_b}^q {F_{pq}}^r = {F'_{ab}}^c {C_c}^r\,, \tag{10}$$

where ${F_{ab}}^c$, ${F'_{ab}}^c$, a, b, $c = 1,\ldots,6$ are structure coefficients of the doubles $\mathcal{D}$ and $\mathcal{D}'$ and

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

3. Method and Result of Classification

As mentioned in the Introduction, there are 78 nonisomorphic classes of Manin triples.[8] If we take into account the duality transformation $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}}) \mapsto (\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ the number is reduced to 44. Their explicit form is given in App. B. It follows from (1) and (3) that the structure of the Manin triple can be given by the structure coefficients f_{ij}^k, $\widetilde{f^{ij}}_k$ of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ in the special basis where relations (1) hold. That is why we usually denote the Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ by $(\mathcal{G}|\tilde{\mathcal{G}})$ or $(\mathcal{G}|\tilde{\mathcal{G}}|b)$ when a scaling parameter b occurs in the definition of the Lie product. Let us note that $(\mathcal{G}|\tilde{\mathcal{G}}|b)$ and $(\mathcal{G}|\tilde{\mathcal{G}}|b')$ are isomorphic up to rescaling of $\langle\cdot,\cdot\rangle$.

It is clear that a direct check which of 44 Manin triples are decomposition of isomorphic Drinfeld doubles is a tremendous task. That is why we first evaluate as many invariants of the algebras as possible and then sort them into smaller subsets according to the values of the invariants. It is clear that only the Manin triples in these subsets can be decomposition of the same Drinfeld double. The invariants we have used are:

- signature (numbers of positive, negative and zero eigenvalues) of the Killing form,
- dimensions of the comutant $[\mathcal{D},\mathcal{D}] \equiv \mathcal{D}^1 \equiv \mathcal{D}_1$ and subalgebras created by the repeated Lie multiplication $\mathcal{D}^{i+1} = [\mathcal{D}^i, \mathcal{D}]$, (up to $i = 3$, it turns out that for $i \geq 3$ $\mathcal{D}^{i+1} = \mathcal{D}^i$). (We have for completeness determined also dimensions of $\mathcal{D}_{i+1} = [\mathcal{D}^i, \mathcal{D}^i]$, but they does not lead to refinement of our partition.)

Table 1. Invariants of Manin triples.

Signature of K	Dim. of $[\mathcal{D},\mathcal{D}]$	Dim. of $\mathcal{D}^2$, $\mathcal{D}^3$	Dim. of $\mathcal{D}_2$, $\mathcal{D}_3$	Manin triples
$(3,3,0)$	6	$6,6$	$6,6$	$(9\|5\|b)$, $(8\|5.ii\|b)$, $(7_a\|7_{1/a}\|b)$, $(7_0\|5.ii\|b)$
$(4,2,0)$	6	$6,6$	$6,6$	$(8\|5.i\|b)$, $(6_a\|6_{1/a}.i\|b)$, $(6_0\|5.iii\|b)$
$(0,3,3)$	6	$6,6$	$6,6$	$(9\|1)$
$(2,1,3)$	6	$6,6$	$6,6$	$(8\|1)$, $(8\|5.iii)$, $(7_0\|4\|b)$, $(7_0\|5.i)$, $(6_0\|4.i\|b)$, $(6_0\|5.i)$, $(5\|2.ii)$, $(4\|2.iii\|b)$,
	3	$3,3$	$3,3$	$(3\|3.i)$
$(1,0,5)$	5	$5,5$	$1,0$	$(7_a\|1)$, $(7_a\|2.i)$, $(7_a\|2.ii)$, $a>1$ $(6_a\|1)$, $(6_a\|2)$, $(6_a\|6_{1/a}.ii)$, $(6_a\|6_{1/a}.iii)$, $(6_0\|1)$, $(6_0\|2)$, $(6_0\|4.ii)$, $(6_0\|5.ii)$, $(5\|1)$, $(5\|2.i)$, $(4\|1)$, $(4\|2.i)$, $(4\|2.ii)$
	3	$3,3$	$1,0$	$(3\|1)$, $(3\|2)$, $(3\|3.ii)$, $(3\|3.iii)$
$(0,1,5)$	5	$5,5$	$1,0$	$(7_a\|1)$, $(7_a\|2.i)$, $(7_a\|2.ii)$, $a<1$ $(7_0\|1)$, $(7_0\|2.i)$, $(7_0\|2.ii)$
$(0,0,6)$	5	$5,5$	$1,0$	$(7_a\|1)$, $(7_a\|2.i)$, $(7_a\|2.ii)$, $a=1$
	3	$0,0$	0,0	$(2\|2)$
		$2,0$	$0,0$	$(2\|2.i)$, $(2\|2.ii)$
	0	$0,0$	$0,0$	$(1\|1)$

The partition of the list of Manin triples according to the values of invariants is in Table 1. The final distinction between nonisomorphic Drinfeld doubles and their decomposition into Manin triples provides the following theorem.

Theorem 1. *Any six-dimensional real Drinfeld double belongs just to one of the following* 22 *classes and allows decomposition into all Manin triples listed in the class and their duals* $(\mathcal{G} \leftrightarrow \tilde{\mathcal{G}})$. *If the class contains parameter a or b, the Drinfeld doubles with different values of this parameter are nonisomorphic.*

(1) $(9|5|b) \cong (8|5.ii|b) \cong (7_0|5.ii|b)$, $b>0$,
(2) $(8|5.i|b) \cong (6_0|5.iii|b)$, $b>0$,
(3) $(7_a|7_{1/a}|b) \cong (7_{1/a}|7_a|b)$, $a \geq 1$, $b \in \mathbf{R}-\{0\}$,

(4) $(6_a|6_{1/a}.i|b) \cong (6_{1/a}.i|6_a|b)$, $a > 1$, $b \in \mathbf{R} - \{0\}$,
(5) $(9|1)$,
(6) $(8|1) \cong (8|5.iii) \cong (7_0|5.i) \cong (6_0|5.i) \cong (5|2.ii)$,
(7) $(7_0|4|b) \cong (4|2.iii|b) \cong (6_0|4.i|-b)$, $b \in \mathbf{R} - \{0\}$,
(8) $(3|3.i)$,
(9) $(7_a|1) \cong (7_a|2.i) \cong (7_a|2.ii)$, $a > 1$,
(10) $(6_a|1) \cong (6_a|2) \cong (6_a|6_{1/a}.ii) \cong (6_a|6_{1/a}.iii)$, $a > 1$,
(11) $(6_0|1) \cong (6_0|5.ii) \cong (5|1) \cong (5|2.i)$,
(12) $(6_0|2) \cong (6_0|4.ii) \cong (4|1) \cong (4|2.i) \cong (4|2.ii)$,
(13) $(3|1) \cong (3|2) \cong (3|3.ii) \cong (3|3.iii)$,
(14) $(7_a|1) \cong (7_a|2.i) \cong (7_a|2.ii)$, $0 < a < 1$,
(15) $(7_0|1)$,
(16) $(7_0|2.i)$,
(17) $(7_0|2.ii)$,
(18) $(7_1|1) \cong (7_1|2.i) \cong (7_1|2.ii)$,
(19) $(2|1)$,
(20) $(2|2.i)$,
(21) $(2|2.ii)$,
(22) $(1|1)$.

4. The Proof of Theorem 1

The essence of the proof is to find which of the 78 nonisomorphic Manin triples found in Ref. 8 and displayed in App. B yield isomorphic Drinfeld doubles. The isomorphisms are given by the explicit form of the transformation matrices C [see (9)] that were found by solution of Eq. (10). In this part we have used the computer programs Maple V and Mathematica 4. The solutions are not unique and here we present only a simple examples of them. The nonisomorphic Drinfeld doubles are distinguished by investigation of their various subalgebras and properties of $\langle \cdot, \cdot \rangle$ and the Killing form on them.

In the next subsection we analyze the subsets of nonisomorphic Manin triples characterized by invariants described in Sec. 3 and displayed in Table 1.

4.1. *Manin triples with the Killing form of signature* $(3, 3, 0)$

In this case the signature of the Killing form itself fixes the Lie algebra $\mathcal{D}$ of the Drinfeld double uniquely. It is the well-known so$(3,1)$ which is simple as a real Lie algebra and its complexification is semisimple; it decomposes into two copies of sl$(2,\mathbf{C})$. The Drinfeld doubles corresponding to $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$, $(7_a|7_{1/a}|b)$ can consequently differ only by the bilinear form $\langle \cdot, \cdot \rangle$.

We can find a necessary condition for equivalence of semisimple Drinfeld doubles from the fact that any invariant symmetric bilinear form on a complex simple Lie algebra is a multiple of the Killing form and that any invariant symmetric bilinear form on a semisimple Lie algebra is a sum of invariant symmetric bilinear forms on its simple components. (*Proof*: Let $\mathcal{G} = \oplus_i \mathcal{G}_i$ be the decomposition into simple components, $X \in \mathcal{G}_i$, $Y \in \mathcal{G}_j$, $i \neq j$. Then $\exists A_k,\ B_k \in \mathcal{G}_j$ s.t. $Y = \sum_k [A_k, B_k]$ and from the ad-invariance of the form $\langle X, Y\rangle = \sum_k \langle X, [A_k, B_k]\rangle = -\sum_k \langle [A_k, X], B_k\rangle = -\sum_k \langle 0, B_k\rangle = 0$.)

We therefore consider the complexification $\mathcal{D}_{\mathbf{C}}$ of the Drinfeld double algebra and write both the Killing form on $\mathcal{D}_{\mathbf{C}}$ and the bilinear form $\langle \cdot, \cdot\rangle$ in terms of Killing forms K_1, K_2 of still unspecified simple components $\mathrm{sl}(2, \mathbf{C})_1$, $\mathrm{sl}(2, \mathbf{C})_2$ ($\mathcal{D}_{\mathbf{C}} = \mathrm{sl}(2, \mathbf{C})_1 \oplus \mathrm{sl}(2, \mathbf{C})_2$)

$$K = K_1 + K_2\,, \quad \langle , \rangle = \alpha K_1 + \beta K_2\,.$$

We trivially extend the Killing forms K_1, K_2 to the whole Drinfeld double algebra $\mathcal{D}_{\mathbf{C}}$ and express them as

$$K_1 = \frac{\langle , \rangle - \beta K}{\alpha - \beta}\,, \qquad K_2 = \frac{\alpha K - \langle , \rangle}{\alpha - \beta}\,.$$

Because K_1, K_2 are trivially extended Killing forms, they must have three-dimensional nullspace [$\mathrm{sl}(2, \mathbf{C})_2$ in the case of K_1 and $\mathrm{sl}(2, \mathbf{C})_1$ in the case of K_2]. These two conditions on dimensions of nullspaces fix the coefficients α, β uniquely up to a permutation. Therefore, *the necessary condition for equivalence of two semisimple six-dimensional Drinfeld doubles is the equality of their sets of coefficients* $\{\alpha, \beta\}$.

We compute the coefficients α, β for the Manin triples in this class and find that in three cases $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ is

$$\{\alpha, \beta\} = \left\{\frac{i}{4b}, -\frac{i}{4b}\right\}$$

and for $(7_a|7_{1/a}|b)$ is

$$\{\alpha, \beta\} = \left\{\frac{ia}{4b(a-i)^2}, -\frac{ia}{4b(i+a)^2}\right\}.$$

We see that the Manin triple $(7_a|7_{1/a}|b)$ defines for any a, b Drinfeld doubles different from any of the Drinfeld doubles associated to the Manin triples $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ and that Drinfeld doubles corresponding to $(7_a|7_{1/a}|b)$ with different values of a and b are different except the case $a' = 1/a$, $b' = b$. The Manin triples $(7_a|7_{1/a}|b)$ and $(7_{1/a}|7_a|b)$ are mutually dual, correspond to $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$ and therefore give the same Drinfeld double. The Manin triple $(7_1|7_1|b)$ is of course self-dual.

Also one sees that the Manin triples $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ with different b cannot lead to the same Drinfeld double. For the Manin triples $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ with equal b, the transformations (9) between Drinfeld doubles exist, but may contain complex numbers since up to now we have considered only complexifications of the original Manin triples.

However, one can check that the following real transformation matrices C guarantee the equivalence of the Drinfeld doubles in this class for fixed value of b.

$$(9|5|b) \to (8|5.ii|b):\ C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 1 & 0 & -\frac{1}{b} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$(9|5|b) \to (7_0|5.ii|b):\ C = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2b} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2b} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 & 0 \\ -b & 0 & -b & 0 & 1 & 0 \\ 0 & b & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As mentioned in the beginning of this section, the transformation matrices are not unique; they contain several free parameters. Here and further we give them in a simple form setting the parameters zero or one.

4.2. *Manin triples with the Killing form of signature* $(4, 2, 0)$

In this case the signature of the Killing form again fixes the Lie algebra $\mathcal{D}$ of the Drinfeld double uniquely, it is $\mathrm{sl}(2, \mathbf{R}) \oplus \mathrm{sl}(2, \mathbf{R})$, and the Drinfeld doubles may again differ only by the bilinear form $\langle \cdot, \cdot \rangle$. We use the criterion developed in the previous subsection for semisimple Drinfeld doubles and find

- $(8|5.i|b)$, $(6_0|5.iii|b) : \{\alpha, \beta\} = \left\{\frac{1}{4b}, -\frac{1}{4b}\right\}$,
- $(6_a|6_{1/a}.i|b) : \{\alpha, \beta\} = \left\{\frac{a}{4b(a-1)^2}, -\frac{a}{4b(1+a)^2}\right\}$.

This shows that the Manin triples might specify isomorphic Drinfeld doubles only in the following two cases:

(1) $(8|5.i|b)$ and $(6_0|5.iii|b)$ for the same value of b. In this case we have found the transformation matrix C

$$(8|5.i|b) \to (6_0|5.iii|b):\ C = \begin{pmatrix} 0 & 0 & -\frac{b}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{b}{2} & \frac{b}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -\frac{1}{b} \\ 0 & 0 & 1 & -\frac{1}{b} & 0 & 0 \\ 0 & 0 & b & 0 & -1 & 0 \end{pmatrix}.$$

This transformation is real and therefore the Drinfeld doubles are isomorphic, $(8|5.i|b) \cong (6_0|5.iii|b)$.

(2) $(6_a|6_{1/a}.i|b)$ and $(6_{1/a}|6_a.i|b)$. One can easily see that these Manin triples are dual (i.e. can be obtained one from the other by the interchange $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) and the Drinfeld doubles are therefore isomorphic.

4.3. *Manin triples with the Killing form of signature* $(0, 3, 3)$

This class contains only one Manin triple $(9|1)$ and its dual; the corresponding Drinfeld double is isomorphic to $\mathrm{so}(3) \triangleright \mathbf{R}^3$ since the Killing form has the signature $(0, 3, 3)$ and $\dim[\mathcal{D}, \mathcal{D}] = 3$.

4.4. *Manin triples with the Killing form of signature* $(2, 1, 3)$

We consider only the Manin triples with $\dim[\mathcal{D}, \mathcal{D}] = 6$, the other set in this class contains only one Manin triple $(3|3.i)$, which is isomorphic as a Lie algebra to $\mathrm{sl}(2, \mathbf{R}) \oplus \mathbf{R}^3$ since the Killing form has the signature $(2, 1, 3)$ and $\dim[\mathcal{D}, \mathcal{D}] = 3$.

The Manin triples in this set $(8|1)$, $(8|5.iii)$, $(7_0|4|b)$, $(7_0|5.i)$, $(6_0|4.i|b)$, $(6_0|5.i)$, $(5|2.ii)$, $(4|2.iii|b)$, are neither semisimple ($\mathrm{rank} K \neq 6$) nor solvable ($[\mathcal{D}, \mathcal{D}] = \mathcal{D}$). Therefore they have a nontrivial Levi–Maltsev decomposition into semidirect sum of a semisimple subalgebra S and radical N

$$\mathcal{D} = S \triangleright N\,,$$

both of them are three-dimensional. Knowledge of this decomposition turns out to be helpful in the investigation of equivalence of the Drinfeld doubles.

A rather simple computation shows that the radical is in all these Manin triples Abelian and maximally isotropic, e.g. for $(8|1)$ the radical is $N = \mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$, for $(4|2.iii|b)$ the radical is $N = \mathrm{span}\{X_3, \tilde{X}^1, \tilde{X}^2\}$.

Next we find the semisimple component. It turns out that the semisimple subalgebra S is in all cases $\mathrm{sl}(2, \mathbf{R})$, e.g. for $(8|1)$ it can be evidently chosen $S = \mathrm{span}\{X_1, X_2, X_3\}$, for $(4|2.iii|b)$ the most general form of the semisimple subalgebra is $S = \mathrm{span}\big\{2X_1 - 2\alpha X_3 - \frac{2}{b}\tilde{X}^1 - 2\beta\tilde{X}^2, -\frac{2}{b}X_2 - \frac{2\gamma}{b}X_3 - \frac{2\beta}{b}\tilde{X}^1, \alpha\tilde{X}^1 + (2-\gamma)\tilde{X}^2 + \tilde{X}^3\big\}$ for any values of α, β, γ.

One can restrict the form $\langle\cdot,\cdot\rangle$ to the semisimple subalgebra S and finds that for $(8|1)$ $\langle\cdot,\cdot\rangle_S = 0$, i.e. S is maximally isotropic, whereas for $(4|2.iii|b)$ and any choice of α, β, γ is $\langle\cdot,\cdot\rangle_S = -1/bK_S$, K_S being the Killing form on S. This shows that as Drinfeld doubles $(8|1)$ and $(4|2.iii|b)$ and similarly $(4|2.iii|b)$ for different values of b are not isomorphic.

Performing the same computation for all Manin triples in this set, we find that they divide into two subsets.

(1) $(8|1)$, $(8|5.iii)$, $(7_0|5.i)$, $(6_0|5.i)$, $(5|2.ii) : \langle\cdot,\cdot\rangle_S = 0$
(2) $(7_0|4|b)$, $(6_0|4.i| - b)$, $(4|2.iii|b) : \langle\cdot,\cdot\rangle_S = -1/bK_S$, $b \in \mathbf{R} - \{0\}$

We find the transformation matrices for Manin triples in these subsets and prove the equivalence of the corresponding Drinfeld doubles:

$$(8|1) \to (8|5.iii):\ C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(8|1) \to (7_0|5.i):\ C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$(8|1) \to (6_0|5.i):\ C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$(8|1) \to (5|2.ii):\ C = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix},$$

respectively

$$(4|2.iii|b) \to (7_0|4|b):\ C = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & -\frac{1}{2b} & 0 & 1 & 0 \\ 0 & \frac{1}{2b} & 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$(4|2.iii|b) \to (6_0|4.i|-b):\ C = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{2b} & 0 & 1 & 0 \\ 0 & -\frac{1}{2b} & 0 & 0 & 0 & 1 \\ -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Concerning the Lie structure of these Drinfeld doubles, it follows from the signature of the Killing form and dimension of $[\mathcal{D},\mathcal{D}]$ that the Lie algebra of D is isomorphic in both cases to $\mathrm{sl}(2,\mathbf{R}) \triangleright \mathbf{R}^3$ where commutation relations between subalgebras are given by the unique irreducible representation of $\mathrm{sl}(2,\mathbf{R})$ on $\mathbf{R}^3$.

4.5. *Manin triples with the Killing form of signature* $(1,0,5)$

4.5.1. *Case* $\dim[\mathcal{D},\mathcal{D}] = 5$

This set contains the greatest number of Manin triples: $(7_{a>1}|1)$, $(7_{a>1}|2.i)$, $(7_{a>1}|2.ii)$, $(6_a|1)$, $(6_a|2)$, $(6_a|6_{1/a}.ii)$, $(6_a|6_{1/a}.iii)$, $(6_0|1)$, $(6_0|5.ii)$, $(5|1)$, $(5|2.i)$, $(6_0|2)$, $(6_0|4.ii)$, $(4|1)$, $(4|2.i)$, $(4|2.ii)$. In order to shorten our considerations we firstly present the transformation matrices C showing the equivalence of following Drinfeld doubles and later we prove that the following classes of Drinfeld doubles are nonisomorphic:

(1) $(7_{a>1}|1) \cong (7_{a>1}|2.i) \cong (7_{a>1}|2.ii)$ for the same value of a

$$(7_a|1) \to (7_a|2.i):\ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2a} & 0 & 1 & 0 \\ 0 & \frac{1}{2a} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(7_a|1) \to (7_a|2.ii):\ C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2a & 0 \\ 0 & 0 & 0 & 0 & 0 & 2a \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -\frac{1}{2a} & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2a} & 0 & 1 & 0 \end{pmatrix}.$$

(2) $(6_a|1) \cong (6_a|2) \cong (6_a|6_{1/a}.ii) \cong (6_a|6_{1/a}.iii)$ for the same value of a

$(6_a|1) \to (6_a|2)$:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2a} & 0 & 1 & 0 \\ 0 & \frac{1}{2a} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$(6_a|1) \to (6_a|6_{\frac{1}{a}}.ii)$:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-a & a-1 & 0 & 0 \\ 0 & 1-a & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \frac{1}{a-1} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{a-1} & 0 & 0 & 0 & -\frac{1}{a-1} & -\frac{1}{a-1} \end{pmatrix},$$

$(6_a|1) \to (6_a|6_{\frac{1}{a}}.iii)$:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1-a & a+1 & 0 & 0 \\ 0 & -1-a & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{a+1} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{a+1} & 0 & 0 & 0 & -\frac{1}{a+1} & \frac{1}{a+1} \end{pmatrix}.$$

(3) $(5|1) \cong (5|2.i) \cong (6_0|1) \cong (6_0|5.ii)$

$$(5|1) \to (5|2.i):\; C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$(5|1) \to (6_0|1):\; C = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(5|1) \to (6_0|5.ii):\; C = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(4) $(4|1) \cong (4|2.i) \cong (4|2.ii) \cong (6_0|2) \cong (6_0|4.ii)$

$$(4|1) \to (4|2.i):\; C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(4|1) \to (4|2.ii):\; C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(4|1) \to (6_0|2):\; C = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$(4|1) \to (6_0|4.ii):\; C = \begin{pmatrix} 0 & 0 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & \frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the proof of inequivalence of the above given classes of Manin triples we exploit the fact that the Drinfeld doubles have at least one decomposition into Manin triple with the 2nd subalgebra $\tilde{\mathcal{G}}$ Abelian; we will use only these representantions $(7_a|1)$, $a > 1$, $(6_a|1)$, $(5|1)$, $(4|1)$ in our considerations.

Firstly we find all maximal isotropic Abelian subalgebras $\mathcal{A}$ of each of the given Drinfeld doubles. The dimension of any such $\mathcal{A}$ must be 3 from the maximal isotropy. The commutant is in all these cases $\mathcal{D}_1 = [\mathcal{D}, \mathcal{D}] = \mathrm{span}\{X_2, X_3, \tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$ and the centre is $Z(\mathcal{D}) = \mathrm{span}\{\tilde{X}^1\} = \mathcal{D}_2$. One can see that any element of the form $X_1 + Y$, $Y \in \mathcal{D}_1$ cannot occur in $\mathcal{A}$ because X_1 commutes only with $Z(\mathcal{D})$ and itself. Therefore, $\mathcal{A} \subset \mathcal{D}_1$. Further it follows from the maximality that $\mathcal{A}$ contains $Z(\mathcal{D})$ and we conclude that $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, Y_1, Y_2\}$ where Y_1, $Y_2 \in \mathrm{span}\{X_2, X_3, \tilde{X}^2, \tilde{X}^3\}$. Analyzing the maximal isotropy and replacing Y_1, Y_2 by their suitable linear combinations we find that $\mathcal{A}$ can be in general expressed in one of the following forms:

(1) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_2, \tilde{X}^3\}$,
(2) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_2 + \alpha\tilde{X}^3, X_3 - \alpha\tilde{X}^2\}$,
(3) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_2 + \alpha X_3, -\alpha\tilde{X}^2 + \tilde{X}^3\}$,
(4) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_3, \tilde{X}^2\}$,
(5) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$.

In the next step we check which of these subspaces really form a subalgebra of the given Manin triple.

- $(7_a|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2, X_3\}$ and $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$. One may easily construct for each of these maximal isotropic Abelian subalgebras the dual (w.r.t $\langle\cdot\rangle$) subalgebra by taking the remaining elements of the standard basis $X_1, \ldots, \tilde{X}^3$ and finds that it is isomorphic in both cases to Bianchi algebra 7_a. In other words, we have shown that this class of Drinfeld doubles is nonisomorphic to the other ones and are mutually nonisomorphic for different values of a.
- $(6_a|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2, X_3\}$, $\mathrm{span}\{\tilde{X}^1, X_2 + X_3, -\tilde{X}^2 + \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, X_2 - X_3, \tilde{X}^2 + \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$. By a slightly more complicated construction of the dual subalgebras we find that they are of the Bianchi type 6_a for the same a, i.e. this class of Drinfeld doubles is nonisomorphic to the other ones and are mutually nonisomorphic for different values of a.
- $(5|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2, \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, X_2, X_3\}$, $\mathrm{span}\{\tilde{X}^1, X_2 + \alpha X_3, -\alpha\tilde{X}^2 + \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, X_3, \tilde{X}^2\}$ and $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$.
- $(4|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2 + \alpha\tilde{X}^3, X_3 - \alpha\tilde{X}^2\}$, $\mathrm{span}\{\tilde{X}^1, X_3, \tilde{X}^2\}$ and $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$.

Already from comparison of number of possible maximal isotropic Abelian subalgebras for $(5|1)$ and $(4|1)$ one sees that the corresponding Drinfeld doubles are nonisomorphic.

It also follows that Drinfeld doubles corresponding to Manin triples $(7_a|1)$, $(6_a|1)$, $(5|1)$ and $(4|1)$ are different as Lie algebras, since any maximal isotropic Abelian subalgebra $\mathcal{A}$ of these Manin triples is in fact an Abelian ideal $\mathcal{I}$ such that $[\mathcal{D}, \mathcal{I}] = \mathcal{I}$ and any such three-dimensional ideal is maximal isotropic from ad-invariance of $\langle , \rangle$. Therefore we have in fact identified the nonisomorphic Drinfeld doubles from the knowledge of these ideals $\mathcal{I}$ (and in some cases $\mathcal{D}/\mathcal{I}$) which does not depend on the form $\langle , \rangle$ and the doubles differ already in their Lie algebra structure.

4.5.2. *Case* $\dim[\mathcal{D},\mathcal{D}] = 3$

All Manin triples of this subset are decomposition of one Drinfeld double, i.e. they can be transformed one into another by the transformation (9). Below are the corresponding matrices.

$$(3|1) \to (3|2):\ C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(3|1) \to (3|3.ii):\ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 & 1 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(3|1) \to (3|3.iii):\ C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.6. *Manin triples with the Killing form of signature* $(0,1,5)$

This set contains Manin triples $(7_{a<1}|1)$, $(7_{a<1}|2.i)$, $(7_{a<1}|2.ii)$, $(7_0|1)$, $(7_0|2.i)$, $(7_0|2.ii)$. As in the Subsec. 4.5.1 we can show that Manin triples $(7_{a<1}|1)$, $(7_{a<1}|2.i)$, $(7_{a<1}|2.ii)$ are decomposition of isomorphic Drinfeld doubles for the same a; the transformation matrices given above for $a > 1$ are meaningful also in this case. It remains to be investigated whether the Drinfeld doubles induced by $(7_0|1)$, $(7_0|2.i)$, $(7_0|2.ii)$ are isomorphic as or not.

We again find all maximal isotropic Abelian subalgebras of these Manin triples. We find

- $(7_0|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^3, X_1 + \alpha\tilde{X}^2, X_2 - \alpha\tilde{X}^1\}$, $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$,
- $(7_0|2.i)$ the only maximal isotropic Abelian subalgebra is $\mathrm{span}\{\tilde{X}^3, X_1, X_2\}$, the dual subalgebra to it w.r.t $\langle\cdot,\cdot\rangle$ does not exist.
- $(7_0|2.ii)$ the only maximal isotropic Abelian subalgebra is $\mathrm{span}\{\tilde{X}^3, X_1, X_2\}$, the dual subalgebra to it w.r.t $\langle\cdot,\cdot\rangle$ does not exist.

This means that Drinfeld double induced by $(7_0|1)$ has only decompositions into Manin triple $(7_0|1)$ and that Drinfeld doubles corresponding to $(7_0|2.i)$, $(7_0|2.ii)$ are not isomorphic to the Drinfeld double corresponding to $(7_{a<1}|1)$ for any value of a. To prove that also $(7_0|2.i)$, $(7_0|2.ii)$ induce nonisomorphic Drinfeld doubles, we find all isotropic subalgebras of Bianchi type 7_0 in the Manin triple $(7_0|2.ii)$. They are

$$\mathrm{span}\{Y_1, Y_2, Y_3\},$$

where

$$Y_1 = X_1 - \alpha\tilde{X}^3, \quad Y_2 = X_2 - \beta\tilde{X}^3, \quad Y_3 = X_3 + \alpha\tilde{X}^1 + \beta\tilde{X}^2, \quad \alpha,\ \beta \in \mathbf{R},$$

and the dual subalgebra w.r.t. $\langle\cdot,\cdot\rangle$ is in general

$$\mathrm{span}\{\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3\},$$

where

$$\tilde{Y}_1 = \gamma X_2 + \tilde{X}^1 - \gamma\beta\tilde{X}^3, \quad \tilde{Y}_2 = -\gamma X_1 + \tilde{X}^2 + \gamma\alpha\tilde{X}^3, \quad \tilde{Y}_3 = \tilde{X}^2, \quad \gamma \in \mathbf{R}.$$

Structure coefficients in this new basis $Y_1, \ldots, \tilde{Y}_3$ are identical with the original structure coefficients for any α, β, γ, therefore the Drinfeld double corresponding to $(7_0|2.ii)$ allows no decomposition into other Manin triples and similarly for $(7_0|2.i)$.

Concerning the Lie algebra structure, the Drinfeld doubles corresponding to $(7_0|2.i)$ and $(7_0|2.ii)$ are isomorphic as Lie algebras because they differ just by the sign of the bilinear form $\langle,\rangle$, and consequently the commutation relations implied by ad-invariance of $\langle,\rangle$ are the same. The other Drinfeld doubles specify different Lie algebras for the same reason as in Subsec. 4.5.1.

4.7. *Manin triples with the Killing form of signature* $(0, 0, 6)$

4.7.1. *Case* $\dim[\mathcal{D}, \mathcal{D}] = 5$

This set contains Manin triples $(7_1|1)$, $(7_1|2.i)$ and $(7_1|2.ii)$. They specify isomorphic Drinfeld doubles. For transformation matrices see Subsec. 4.5.1 and substitute $a = 1$.

4.7.2. *Case* $\dim[\mathcal{D},\mathcal{D}] = 3$

In this set, the only Manin triples that can lead to the same Drinfeld double are $(2|2.i)$ and $(2|2.ii)$. To see that the Drinfeld doubles are different, it is sufficent to find the centres $Z(\mathcal{D})$ of these Manin triples and restrict the form $\langle\cdot,\cdot\rangle$ to them. These restricted forms $\langle\cdot,\cdot\rangle_{Z(\mathcal{D})}$ have different signatures, therefore the Drinfeld doubles are nonisomorphic:

(1) $(2|2.i) : Z(\mathcal{D}) = \mathrm{span}\{X_1, X_2 - \tilde{X}^2, \tilde{X}^3\}$, signature of $\langle\cdot,\cdot\rangle_{Z(\mathcal{D})} = (0, 1, 2)$.
(2) $(2|2.ii) : Z(\mathcal{D}) = \mathrm{span}\{X_1, X_2 + \tilde{X}^2, \tilde{X}^3\}$, signature of $\langle\cdot,\cdot\rangle_{Z(\mathcal{D})} = (1, 0, 2)$.

These Drinfeld doubles are isomorphic as Lie algebras because they differ just by the sign of the bilinear form $\langle,\rangle$ and the commutation relations are due to the ad-invariance the same.

5. Conclusions

In this work we have constructed the complete list of six-dimensional real Drinfeld doubles up to their isomorphisms i.e. maps preserving both the Lie structure and an ad-invariant symmetric bilinear form $\langle,\rangle$ that define the double. The result is summarized in the theorem at the end of Sec. 3 and claims that there just 22 classes of the nonisomorphic Drinfeld doubles. Some of them contain one or two real parameters denoted a and b. The number 22 is in a way conditional because e.g. the classes 9,14,18 could be united into one. The reason why they are given as separate classes is that they have different values of their invariants, in this case the signature of the Killing form.

An important point that follows from the classification is that there are several different Drinfeld doubles corresponding to Lie algebras $\mathrm{so}(1,3)$, $\mathrm{sl}(2,\mathbf{R}) \oplus \mathrm{sl}(2,\mathbf{R})$, $\mathrm{sl}(2,\mathbf{R}) \triangleright \mathbf{R}^3$ whereas on solvable Lie algebras the Drinfeld double is unique (in some cases up to the sign of the bilinear form). On the other hand there are Manin triples with one isotropic subalgebra Abelian that are equivalent as Drinfeld doubles even though the other subalgebras are different [see $(6_0|1)$ and $(5|1)$]. That is why it is necessary to investigate the (non)equivalence of the Manin triples of this form. Moreover the above given examples indicate the diversity of Drinfeld double structures one may encounter in higher dimensions.

Beside that from the present classification procedure one can find whether a given six-dimensional Lie algebra can be equipped by a suitable ad-invariant bilinear form and turned into a Drinfeld double (and how many such forms exist). The decisive aspects are the signature of the Killing form and the dimensions of the ideals $\mathcal{D}_j, \mathcal{D}^j$. The necessary condition is that they have the values occuring in Table 1. The investigation then can be reduced to a direct check of equivalence with a particular six-dimensional Lie algebra (possibly after determination of Abelian ideals and the factor algebras as in the Subsec. 4.5.1).

One can see that for many Drinfeld doubles there are several decompositions into Manin triples. For each Manin triple there is a pair of dual sigma models. Their equation of motion [2]

$$\langle \partial_\pm l l^{-1}, \mathcal{E}^\pm \rangle = 0 \tag{12}$$

are given by the Drinfeld double and a three-dimensional subspace $\mathcal{E}^+ \subset \mathcal{D}$ so that all these models (for fixed $\mathcal{E}^+$) are equivalent. Moreover the scaling of $\langle , \rangle$ does not change the equations of motion (12) and consequently all the models corresponding to (nonisomorphic) Drinfeld doubles with different b are equal. We can construct the explicit forms of the equations of motion for every Drinfeld double but without a physical motivation this does not make much sense.

Let us note that the complete sets of the equivalent sigma models for a fixed Drinfeld double are given by the so called modular space of the double. The construction of all nonisomorphic Manin triples for the double is the first step in the construction of the modular spaces.

Appendix A. Bianchi Algebras

It is known that any three-dimensional real Lie algebra can be brought to one of 11 forms by a change of basis. These forms represent nonisomorphic Lie algebras and are conventionally known as Bianchi algebras. They are denoted by $\mathbf{1}, \ldots, \mathbf{5}$, $\mathbf{6_a}$, $\mathbf{6_0}$, $\mathbf{7_a}$, $\mathbf{7_0}$, $\mathbf{8}$, $\mathbf{9}$ (see e.g. Ref. 11, in literature often uppercase roman numbers are used instead of arabic ones). The list of Bianchi algebras is given in decreasing order starting from simple algebras.

$$\mathbf{9}: \ [X_1, X_2] = X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2,\ (\text{i.e. so}(3))\,,$$

$$\mathbf{8}: \ [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2,\ (\text{i.e. sl}(2, \mathbf{R}))\,,$$

$$\mathbf{7_a}: \ [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_2 + aX_3,\ a > 0\,,$$

$$\mathbf{7_0}: \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,$$

$$\mathbf{6_a}: \ [X_1, X_2] = -aX_2 - X_3,\ [X_2, X_3] = 0\,,$$

$$[X_3, X_1] = X_2 + aX_3,\ a > 0,\ a \neq 1\,,$$

$$\mathbf{6_0}: \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$

$$\mathbf{5}: \ [X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,$$

$$\mathbf{4}: \ [X_1, X_2] = -X_2 + X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,$$

$$\mathbf{3}: \ [X_1, X_2] = -X_2 - X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_2 + X_3\,,$$

$$\mathbf{2}: \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = 0\,,$$

$$\mathbf{1}: \ [X_1, X_2] = 0,\ [X_2, X_3] = 0,\ [X_3, X_1] = 0\,.$$

One might use also another classification (used e.g. in Ref. 7). In this notation the basis of the Lie algebra is usually written as (e_0, e_1, e_2) and the classification is:

$$\mathbf{R}^3 = \mathbf{1}: \; [e_1, e_2] = 0, \; [e_0, e_i] = 0\,,$$

$$n_3 = \mathbf{2}: \; [e_1, e_2] = e_0, \; [e_0, e_i] = 0\,,$$

$$r_3(\rho): \; [e_1, e_2] = 0, \; [e_0, e_1] = e_1\,,$$

$$[e_0, e_2] = \rho e_2, \; -1 \le \rho \le 1\,.$$

This algebra is isomorphic to $\mathbf{6_0}$ for $\rho = -1$, $\mathbf{6}_{\frac{\rho+1}{\rho-1}}$ for $0 < |\rho| < 1$, $\mathbf{3}$ for $\rho = 0$ and $\mathbf{5}$ for $\rho = 1$.

$$r_3'(1) = \mathbf{4}: \; [e_1, e_2] = 0, \; [e_0, e_1] = e_1, \; [e_0, e_2] = e_1 + e_2\,,$$

$$s_3(\mu): \; [e_1, e_2] = 0, \; [e_0, e_1] = \mu e_1 - e_2, \; [e_0, e_2] = e_1 + \mu e_2, \; \mu \ge 0\,.$$

This algebra is isomorphic to $\mathbf{7_0}$ for $\mu = 0$ and $\mathbf{7}_\mu$ for $\mu > 0$.

$$\mathrm{sl}(2, \mathbf{R}) = \mathbf{8}\,, \qquad \mathrm{so}(3) = \mathbf{9}\,.$$

It is clear that this classification is more compact, on the other hand the classes in this classification contain algebras with different properties such as dimensions of commutant etc. and surprisingly the special cases of parameters we need to distinguish correspond in most cases to different Bianchi algebras. Therefore we use the Bianchi classification.

Appendix B. List of Manin Triples

We present a list of Manin triples based on Ref. 8. The label of each Manin triple, e.g. $(\mathbf{8}|\mathbf{5.ii}|\mathbf{b})$, indicates the structure of the first subalgebra $\mathcal{G}$, e.g. Bianchi algebra $\mathbf{8}$, the structure of the second subalgebra $\tilde{\mathcal{G}}$, e.g. Bianchi algebra $\mathbf{5}$; roman numbers i, ii etc. (if present) distinguish between several possible pairings $\langle\cdot,\cdot\rangle$ of the subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$ and the parameter b indicates the Manin triples differing by the rescaling of $\langle\cdot,\cdot\rangle$ (if such Manin triples are not isomorphic).

The Lie structures of the subalgebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are written out in mutually dual bases (X_1, X_2, X_3) and $(\tilde{X}^1, \tilde{X}^2, \tilde{X}^3)$ where the transformation (2) was used to bring $\mathcal{G}$ to the standard Bianchi form. Because of (3) this information specifies the Manin triple completely.

The dual Manin triples $(\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ are not written explicitly but can be easily obtained by $X_j \leftrightarrow \tilde{X}^j$.

(1) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{9}$:

$$(\mathbf{9}|\mathbf{1}): \; [X_1, X_2] = X_3, \; [X_2, X_3] = X_1, \; [X_3, X_1] = X_2\,,$$

$$[\tilde{X}^1, \tilde{X}^2] = 0, \; [\tilde{X}^2, \tilde{X}^3] = 0, \; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$

$$(\mathbf{9}|\mathbf{5}|\mathbf{b}): \; [X_1, X_2] = X_3, \; [X_2, X_3] = X_1, \; [X_3, X_1] = X_2\,,$$

$$[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2, \; [\tilde{X}^2, \tilde{X}^3] = 0, \; [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3, \; b > 0\,.$$

(2) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{8}$:

$$\begin{aligned}(\mathbf{8}|\mathbf{1}):\ & [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,\\ & \mathrm{v}[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\ (\mathbf{8}|\mathbf{5.i}|\mathbf{b}):\ & [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,\\ & [\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0\,,\\ (\mathbf{8}|\mathbf{5.ii}|\mathbf{b}):\ & [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,\\ & [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1,\ b > 0\,,\\ (\mathbf{8}|\mathbf{5.iii}):\ & [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,\\ & [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -(\tilde{X}^1 + \tilde{X}^3)\,.\end{aligned}$$

(3) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{7_a}$:

$$\begin{aligned}(\mathbf{7_a}|\mathbf{1}):\ & [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,\\ & [X_3, X_1] = X_2 + aX_3,\ a > 0\,,\\ & [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\ (\mathbf{7_a}|\mathbf{2.i}):\ & [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,\\ & [X_3, X_1] = X_2 + aX_3,\ a > 0\,,\\ & [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\ (\mathbf{7_a}|\mathbf{2.ii}):\ & [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,\\ & [X_3, X_1] = X_2 + aX_3,\ a > 0\,,\\ & [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\ (\mathbf{7_a}|\mathbf{7_{1/a}}|\mathbf{b}):\ & [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,\\ & [X_3, X_1] = X_2 + aX_3,\ a > 0\,,\\ & [\tilde{X}^1, \tilde{X}^2] = b\big(-\tfrac{1}{a}\tilde{X}^2 + \tilde{X}^3\big),\ [\tilde{X}^2, \tilde{X}^3] = 0\,,\\ & [\tilde{X}^3, \tilde{X}^1] = b\big(\tilde{X}^2 + \tfrac{1}{a}\tilde{X}^3\big),\ b \in \mathbf{R} - \{0\}\,.\end{aligned}$$

(4) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{7_0}$:

$$\begin{aligned}(\mathbf{7_0}|\mathbf{1}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,\\ & [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\ (\mathbf{7_0}|\mathbf{2.i}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,\\ & [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\end{aligned}$$

$$(\mathbf{7_0}|\mathbf{2.ii}):\ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0\,,$$
$$(\mathbf{7_0}|\mathbf{4}|\mathbf{b}):\ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=b(-\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3]=0\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=b\tilde{X}^3,\ b\in\mathbf{R}-\{0\}\,,$$
$$(\mathbf{7_0}|\mathbf{5.i}):\ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^2,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=\tilde{X}^3\,,$$
$$(\mathbf{7_0}|\mathbf{5.ii}|\mathbf{b}):\ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=b\tilde{X}^2,\ [\tilde{X}^3,\tilde{X}^1]=-b\tilde{X}^1\,,\ b>0\,.$$

(5) Manin triples with the first subalgebra $\mathcal{G}=\mathbf{6_a}$:

$$(\mathbf{6_a}|\mathbf{1}):\ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0\,,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0\,,$$
$$(\mathbf{6_a}|\mathbf{2}):\ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=\tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^1]=0\,,$$
$$(\mathbf{6_a}|\mathbf{6_{1/a}.i}|\mathbf{b}):\ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=-b\big(\tfrac{1}{a}\tilde{X}^2+\tilde{X}^3\big),\ [\tilde{X}^2,\tilde{X}^3]=0\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=b\big(\tilde{X}^2+\tfrac{1}{a}\tilde{X}^3\big),\ b\in\mathbf{R}-\{0\}\,,$$
$$(\mathbf{6_a}|\mathbf{6_{1/a}.ii}):\ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^3]=\tfrac{a+1}{a-1}(\tilde{X}^2+\tilde{X}^3)\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=\tilde{X}^1\,,$$
$$(\mathbf{6_a}|\mathbf{6_{1/a}.iii}):\ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^3]=\tfrac{a-1}{a+1}(-\tilde{X}^2+\tilde{X}^3)\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=-\tilde{X}^1\,.$$

(6) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{6_0}$:

$$
\begin{aligned}
(\mathbf{6_0}|\mathbf{1}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,\\
& [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\
(\mathbf{6_0}|\mathbf{2}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,\\
& [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\
(\mathbf{6_0}|\mathbf{4.i}|\mathbf{b}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,\\
& [\tilde{X}^1, \tilde{X}^2] = b(-\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2, \tilde{X}^3] = 0\,,\\
& [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b \in \mathbf{R} - \{0\}\,,\\
(\mathbf{6_0}|\mathbf{4.ii}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,\\
& [\tilde{X}^1, \tilde{X}^2] = (-\tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3)\,,\\
& [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3\,,\\
(\mathbf{6_0}|\mathbf{5.i}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,\\
& [\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^3\,,\\
(\mathbf{6_0}|\mathbf{5.ii}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,\\
& [\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^1 + \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3\,,\\
(\mathbf{6_0}|\mathbf{5.iii}|\mathbf{b}):\ & [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,\\
& [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^1,\ b > 0\,.
\end{aligned}
$$

(7) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{5}$:

$$
\begin{aligned}
(\mathbf{5}|\mathbf{1}):\ & [X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,\\
& [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\
(\mathbf{5}|\mathbf{2.i}):\ & [X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,\\
& [\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,\\
(\mathbf{5}|\mathbf{2.ii}):\ & [X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,\\
& [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0
\end{aligned}
$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{6_0}$, $\mathbf{7_0}$, $\mathbf{8}$, $\mathbf{9}$.

(8) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{4}$:

$$\begin{aligned}(\mathbf{4}|\mathbf{1}):\ & [X_1,X_2] = -X_2 + X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = 0,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{4}|\mathbf{2.i}):\ & [X_1,X_2] = -X_2 + X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{4}|\mathbf{2.ii}):\ & [X_1,X_2] = -X_2 + X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = -\tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{4}|\mathbf{2.iii}|\mathbf{b}):\ & [X_1,X_2] = -X_2 + X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = 0,\ [\tilde{X}^3,\tilde{X}^1] = b\tilde{X}^2,\ b \in \mathbf{R} - \{0\}\end{aligned}$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{6_0}$, $\mathbf{7_0}$.

(9) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{3}$:

$$\begin{aligned}(\mathbf{3}|\mathbf{1}):\ & [X_1,X_2] = -X_2 - X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_2 + X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = 0,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{3}|\mathbf{2}):\ & [X_1,X_2] = -X_2 - X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_2 + X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{3}|\mathbf{3.i}):\ & [X_1,X_2] = -X_2 - X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_2 + X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = -b(\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3] = 0\,,\\ & [\tilde{X}^3,\tilde{X}^1] = b(\tilde{X}^2 + \tilde{X}^3),\ b \in \mathbf{R} - \{0\}\,,\\ (\mathbf{3}|\mathbf{3.ii}):\ & [X_1,X_2] = -X_2 - X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_2 + X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = \tilde{X}^2 + \tilde{X}^3,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{3}|\mathbf{3.iii}):\ & [X_1,X_2] = -X_2 - X_3,\ [X_2,X_3] = 0,\ [X_3,X_1] = X_2 + X_3\,,\\ & [\tilde{X}^1,\tilde{X}^2] = \tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^3] = 0,\ [\tilde{X}^3,\tilde{X}^1] = -\tilde{X}^1\,.\end{aligned}$$

(10) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{2}$:

$$\begin{aligned}(\mathbf{2}|\mathbf{1}):\ & [X_1,X_2] = 0,\ [X_2,X_3] = X_1,\ [X_3,X_1] = 0\,,\\ & [\tilde{X}^1,\tilde{X}^2] = 0,\ [\tilde{X}^2,\tilde{X}^3] = 0,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{2}|\mathbf{2.i}):\ & [X_1,X_2] = 0,\ [X_2,X_3] = X_1,\ [X_3,X_1] = 0\,,\\ & [\tilde{X}^1,\tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3] = 0,\ [\tilde{X}^3,\tilde{X}^1] = 0\,,\\ (\mathbf{2}|\mathbf{2.ii}):\ & [X_1,X_2] = 0,\ [X_2,X_3] = X_1,\ [X_3,X_1] = 0\,,\\ & [\tilde{X}^1,\tilde{X}^2] = -\tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3] = 0,\ [\tilde{X}^3,\tilde{X}^1] = 0\end{aligned}$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{3}$, $\mathbf{4}$, $\mathbf{6_0}$, $\mathbf{6_a}$, $\mathbf{7_0}$, $\mathbf{7_a}$.

(11) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{1}$:

$$(\mathbf{1}|\mathbf{1}): \ [X_1, X_2] = 0, \ [X_2, X_3] = 0, \ [X_3, X_1] = 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0, \ [\tilde{X}^2, \tilde{X}^3] = 0, \ [\tilde{X}^3, \tilde{X}^1] = 0$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{2}$–$\mathbf{9}$.

References

1. C. Klimčík and P. Ševera, *Phys. Lett.* **B351**, 455 (1995).
2. C. Klimčík, *Nucl. Phys. B.* (*Proc. Suppl.*) **46**, 116 (1996).
3. L. Hlavatý and L. Šnobl, *Mod. Phys. Lett.* **A17**, 429 (2002).
4. M. A. Lledó and V. S. Varadarajan, *Lett. Math. Phys.* **45**, 247 (1998).
5. K. Sfetsos, *Phys. Lett.* **B432**, 365 (1998).
6. M. A. Jafarizadeh and A. Rezaei-Aghdam, *Phys. Lett.* **B458**, 470 (1999).
7. X. Gomez, *J. Math. Phys.* **41**, 4939 (2000).
8. L. Hlavatý and L. Šnobl, *e-print* math. QA/0202209.
9. V. G. Drinfeld, Quantum Groups, *Proc. Int. Congr. Math.* (Berkeley, 1986), pp. 798–820.
10. S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, Cambridge, 2000).
11. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, 1987).