

## CLASSIFICATION OF POISSON–LIE T–DUAL MODELS WITH TWO DIMENSIONAL TARGETS

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Four–dimensional Manin triples and Drinfeld doubles are classified and corresponding two–dimensional Poisson–Lie T–dual sigma models on them are constructed. The simplest example of a Drinfeld double allowing decomposition into two nontrivially different Manin triples is presented.

*Keywords:* Poisson–Lie T–duality, sigma models, Drinfeld doubles, Manin triples, string theory

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### 1. Introduction

A very important symmetry of string theories, or more specifically, two–dimensional sigma models is the T–duality. In the pioneering work,<sup>1</sup> Klimčík and Ševera introduced its nonabelian version – the Poisson–Lie T–duality and showed that the dual sigma models can be formulated on Drinfeld doubles. The explicit form of dual models on the nonabelian double  $GL(2|\mathbb{R})$  was presented in the following work.<sup>2</sup> Other dual models were given in a series of forthcoming papers, see e.g. Refs. 3–5. An attempt to classify all dual principal sigma models with three–dimensional target space<sup>6</sup> made us to revisit the models with the two–dimensional targets and classify them. In the following we classify all four–dimensional Drinfeld doubles and the Poisson–Lie T–dual models on them.

### 2. Classification of four–dimensional Drinfeld doubles

The Drinfeld double  $D$  is defined as a Lie group such that its Lie algebra  $\mathcal{D}$  equipped by a symmetric ad–invariant nondegenerate bilinear form  $\langle \cdot, \cdot \rangle$  can be decomposed into a pair of maximally isotropic subalgebras  $\mathcal{G}, \tilde{\mathcal{G}}$  such that  $\mathcal{D}$  as a vector space is the direct sum of  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$ . Any such decomposition written as an ordered set

$(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$  is called a Manin triple. It is clear that to any Drinfeld double exist at least two Manin triples  $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ ,  $(\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ . Later we show an example of Drinfeld double with more than two possible decomposition into Manin triples.

One can see that the dimensions of the subalgebras are equal and that bases  $\{T_i\}, \{\tilde{T}^i\}$  in the subalgebras can be chosen so that

$$\langle T_i, T_j \rangle = 0, \quad \langle T_i, \tilde{T}^j \rangle = \langle \tilde{T}^j, T_i \rangle = \delta_i^j, \quad \langle \tilde{T}^i, \tilde{T}^j \rangle = 0. \tag{1}$$

This canonical form of the bracket is invariant with respect to the transformations

$$T'_i = T_k A_i^k, \quad \tilde{T}'^j = (A^{-1})^j_k \tilde{T}^k. \tag{2}$$

Due to the ad-invariance of  $\langle \cdot, \cdot \rangle$  the algebraic structure of  $\mathcal{D}$  is

$$\begin{aligned} [T_i, T_j] &= f_{ij}^k T_k, & [\tilde{T}^i, \tilde{T}^j] &= \tilde{f}^{ij}_k \tilde{T}^k, \\ [T_i, \tilde{T}^j] &= f_{ki}^j \tilde{T}^k + \tilde{f}^{jk}_i T_k. \end{aligned} \tag{3}$$

From the above given facts it is clear that the subalgebras  $\mathcal{G}, \tilde{\mathcal{G}}$  of the four-dimensional Drinfeld double are two-dimensional and surprisingly the Jacobi identities do not impose any condition on coefficients  $f_{ij}^k, \tilde{f}^{ij}_k$  in this case. Each of the subalgebras is solvable and due to the invariance of (1) w.r.t. (2), the basis  $\{T_1, T_2\}$  can be chosen so that the nontrivial Lie bracket in the first subalgebra is

$$[T_1, T_2] = nT_2 \tag{4}$$

where  $n = 0$  or  $1$ . However, the Lie bracket in the second subalgebra in general cannot be written in a similar way without breaking the canonical form (1) of the bracket  $\langle \cdot, \cdot \rangle$  or the canonical form (4) of the subalgebra  $\mathcal{G}$ . Nevertheless, we can use the transformations (2) with

$$A = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}, \tag{5}$$

that preserve (4) to bring the Lie bracket of the second subalgebra to one of the following form

$$[\tilde{T}^1, \tilde{T}^2] = \beta \tilde{T}^2, \quad \beta \in \mathbb{R} \quad \text{or} \quad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^1. \tag{6}$$

In summary, *there are just four types of nonisomorphic four-dimensional Manin triples.*

Abelian Manin triple:

$$[T_i, T_j] = 0, \quad [\tilde{T}^i, \tilde{T}^j] = 0, \quad [T_i, \tilde{T}^j] = 0, \quad i, j = 1, 2. \tag{7}$$

Semiabelian Manin triple (only nontrivial brackets are displayed):

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [T_2, \tilde{T}^1] = T_2, \quad [T_2, \tilde{T}^2] = -T_1. \tag{8}$$

Type A nonabelian Manin triple ( $\beta \neq 0$ ):

$$\begin{aligned}
 [T_1, T_2] &= T_2, & [\tilde{T}^1, \tilde{T}^2] &= \beta \tilde{T}^2, \\
 [T_1, \tilde{T}^2] &= -\tilde{T}^2, & [T_2, \tilde{T}^1] &= \beta T_2, & [T_2, \tilde{T}^2] &= -\beta T_1 + \tilde{T}^1.
 \end{aligned}
 \tag{9}$$

Type B nonabelian Manin triple:

$$\begin{aligned}
 [T_1, T_2] &= T_2, & [\tilde{T}^1, \tilde{T}^2] &= \tilde{T}^1, \\
 [T_1, \tilde{T}^1] &= T_2, & [T_1, \tilde{T}^2] &= -T_1 - \tilde{T}^2, & [T_2, \tilde{T}^2] &= \tilde{T}^1.
 \end{aligned}
 \tag{10}$$

An interesting fact is that Drinfeld doubles corresponding to semiabelian Manin triple (8) and type B nonabelian Manin triple (10) are the same, i.e. *these Manin triples are different decomposition into maximally isotropic subalgebras of the same Lie algebra with the same invariant form.* The transformation of the dual basis between these decompositions is

$$\begin{aligned}
 X_1 &= -\tilde{T}^1 + \tilde{T}^2, & X_2 &= T_1 + T_2, \\
 \tilde{X}^1 &= T_2, & \tilde{X}^2 &= \tilde{T}^1,
 \end{aligned}
 \tag{11}$$

where  $(X_i, \tilde{X}^j)$  denote the dual basis in the type B nonabelian Manin triple and  $(T_i, \tilde{T}^j)$  is the basis in the semiabelian Manin triple. The other Manin triples specify the algebra of the Drinfeld double uniquely, i.e. there is one connected and simply connected Drinfeld double to each of these Manin triples.

### 3. Dual sigma models

Having all four–dimensional Drinfeld doubles we can construct the two–dimensional Poisson–Lie T–dual sigma models on them. The construction of the models is described in the papers.<sup>1,2</sup> The models have target spaces in the Lie groups  $G$  and  $\tilde{G}$  and are defined by the Lagrangians

$$\mathcal{L} = E_{ij}(g)(g^{-1}\partial_-g)^i(g^{-1}\partial_+g)^j,
 \tag{12}$$

$$\tilde{\mathcal{L}} = \tilde{E}_{ij}(\tilde{g})(\tilde{g}^{-1}\partial_-\tilde{g})^i(\tilde{g}^{-1}\partial_+\tilde{g})^j,
 \tag{13}$$

where

$$E(g) = (a(g) + E(e)b(g))^{-1}E(e)d(g),
 \tag{14}$$

$E(e)$  is a constant matrix and  $a(g), b(g), d(g)$  are  $2 \times 2$  submatrices of the adjoint representation of the group  $G$  on  $\mathcal{D}$  in the basis  $(T_i, \tilde{T}^j)$

$$Ad(g)^T = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}.
 \tag{15}$$

The matrix  $\tilde{E}(\tilde{g})$  is constructed analogously with

$$Ad(\tilde{g})^T = \begin{pmatrix} \tilde{d}(\tilde{g}) & \tilde{b}(\tilde{g}) \\ 0 & \tilde{a}(\tilde{g}) \end{pmatrix}, \quad \tilde{E}(\tilde{e}) = E(e)^{-1} = \begin{pmatrix} x & y \\ u & v \end{pmatrix}. \quad (16)$$

Both equations of motion of the above given lagrangian systems can be reduced from equation of motion on the whole Drinfeld double, not depending on the choice of Manin triple:

$$\langle (\partial_{\pm} l) l^{-1}, \mathcal{E}^{\pm} \rangle = 0, \quad (17)$$

where subspaces  $\mathcal{E}^+ = \text{span}(T^i + E^{ij}(e)\tilde{T}_j)$ ,  $\mathcal{E}^- = \text{span}(T^i - E^{ij}(e)\tilde{T}_j)$  are orthogonal w.r.t.  $\langle, \rangle$  and span the whole Lie algebra  $D$ . One writes  $l = g.\tilde{h}$ ,  $g \in G$ ,  $\tilde{h} \in \tilde{G}$  (such decomposition of group elements exists at least at the vicinity of the unit element) and eliminates  $\tilde{h}$  from (17), respectively  $l = \tilde{g}.h$ ,  $h \in G$ ,  $\tilde{g} \in \tilde{G}$  and eliminates  $h$  from (17). The resulting equations of motion for  $g$ , resp.  $\tilde{g}$  are the equations of motion of the corresponding lagrangian system (see Ref. 1).

The corresponding models for the Manin triples (7)–(10) are the following.

**Abelian double:** The adjoint representations of the groups  $G, \tilde{G}$  are trivial so that

$$\tilde{E}(\tilde{g}) = \tilde{E}(e) = E(g)^{-1} = E(e)^{-1}, \quad (18)$$

and the Lagrangians of the dual models are

$$\mathcal{L} = (xv - uy)^{-1} (v \partial_- \chi \partial_+ \chi - y \partial_- \chi \partial_+ \theta - u \partial_- \theta \partial_+ \chi + x \partial_- \theta \partial_+ \theta), \quad (19)$$

$$\tilde{\mathcal{L}} = x \partial_- \sigma \partial_+ \sigma + y \partial_- \sigma \partial_+ \rho + u \partial_- \rho \partial_+ \sigma + v \partial_- \rho \partial_+ \rho. \quad (20)$$

**Semiabelian double:** The adjoint representations of the groups  $G, \tilde{G}$  are

$$Ad(g)^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \end{pmatrix}, \quad Ad(\tilde{g})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\rho & e^\sigma & 0 & 0 \\ 0 & 0 & 1 & \rho e^{-\sigma} \\ 0 & 0 & 0 & e^{-\sigma} \end{pmatrix}$$

where  $(\chi, \theta)$  and  $(\sigma, \rho)$  are group coordinates of  $G$  and  $\tilde{G}$ . The Lagrangians of the dual models are

$$\begin{aligned} \mathcal{L} = & (vx - uy - u\theta + y\theta + \theta^2)^{-1} [v \partial_- \chi \partial_+ \chi - (\theta + y) \partial_- \chi \partial_+ \theta \\ & + (\theta - u) \partial_- \theta \partial_+ \chi + x \partial_- \theta \partial_+ \theta], \end{aligned} \quad (21)$$

$$\begin{aligned} \tilde{\mathcal{L}} = & (x - u\rho - y\rho + v\rho^2) \partial_- \sigma \partial_+ \sigma + (y - v\rho) \partial_- \sigma \partial_+ \rho \\ & + (u - v\rho) \partial_- \rho \partial_+ \sigma + v \partial_- \rho \partial_+ \rho. \end{aligned} \quad (22)$$

Similarly one may use the other possible decomposition of the double into maximally isotropic subalgebras, i.e. **type B nonabelian Manin triple**. In this case the adjoint representations of the groups  $G, \tilde{G}$  are

$$Ad(g)^T = \begin{pmatrix} 1 & \theta e^{-x} & 0 & 0 \\ 0 & e^{-x} & 0 & 0 \\ 0 & -1 + e^{-x} & 1 & 0 \\ -1 + e^x & \theta - \theta e^{-x} & -\theta & e^x \end{pmatrix},$$

$$Ad(\tilde{g})^T = \begin{pmatrix} e^{-\rho} & -\sigma & \sigma - e^\rho \sigma & -1 + e^{-\rho} \\ 0 & 1 & -1 + e^\rho & 0 \\ 0 & 0 & e^\rho & 0 \\ 0 & 0 & e^\rho \sigma & 1 \end{pmatrix}$$

and the Lagrangians of the dual models are

$$\begin{aligned} \mathcal{L} = & [vx + (e^x - 1 - y)(e^x - 1 + u)]^{-1} [(v + u\theta + y\theta + x\theta^2)\partial_- \chi \partial_+ \chi \\ & + (-1 + e^x - y - x\theta)\partial_- \chi \partial_+ \theta - (-1 + e^x + u + x\theta)\partial_- \theta \partial_+ \chi + x\partial_- \theta \partial_+ \theta], \end{aligned} \quad (23)$$

$$\begin{aligned} \tilde{\mathcal{L}} = & [vx - uy + e^\rho(u - 2vx - y + 2uy) + e^{2\rho}(1 + vx + y - u(1 + y))]^{-1} \\ & [x\partial_- \sigma \partial_+ \sigma + (vx - e^{-\rho}vx + y + e^{-\rho}uy - uy - x\sigma)\partial_- \sigma \partial_+ \rho \\ & - (vx - e^{-\rho}vx - u + e^{-\rho}uy - uy + x\sigma)\partial_- \rho \partial_+ \sigma \\ & - (u\sigma + y\sigma - v - x\sigma^2)\partial_- \rho \partial_+ \rho]. \end{aligned} \quad (24)$$

This model has the same equations of motion in the double (17) as the previous one (up to transformation of matrix  $E(e)$  induced by the change of basis of algebra) and in this sense is equivalent to it.

**Type A nonabelian doubles:** The adjoint representations of the groups  $G, \tilde{G}$  are

$$Ad(g)^T = \begin{pmatrix} 1 & \theta e^{-x} & 0 & 0 \\ 0 & e^{-x} & 0 & 0 \\ 0 & -\beta \theta e^{-x} & 1 & 0 \\ \beta \theta & \beta \theta^2 e^{-x} & -\theta & e^x \end{pmatrix},$$

$$Ad(\tilde{g})^T = \begin{pmatrix} 1 & 0 & 0 & -\beta^{-1} \rho e^{-\sigma} \\ -\rho & e^\sigma & \beta^{-1} \rho & \beta^{-1} \rho^2 e^{-\sigma} \\ 0 & 0 & 1 & \rho e^{-\sigma} \\ 0 & 0 & 0 & e^{-\sigma} \end{pmatrix}$$

where  $\beta$  parametrizes different Drinfeld doubles. The Lagrangians of the dual models are

$$\begin{aligned} \mathcal{L} = & (vx - uy + u\beta\theta - y\beta\theta + \beta^2\theta^2)^{-1} [(v + u\theta + y\theta + x\theta^2)\partial_- \chi \partial_+ \chi \\ & - (y + x\theta - \beta\theta)\partial_- \chi \partial_+ \theta - (u + x\theta + \beta\theta)\partial_- \theta \partial_+ \chi + x\partial_- \theta \partial_+ \theta], \end{aligned} \quad (25)$$

$$\begin{aligned} \tilde{\mathcal{L}} = & (\beta^2 - u\beta\rho + y\beta\rho + vx\rho^2 - uy\rho^2)^{-1} \\ & [\beta^2(x - u\rho - y\rho + v\rho^2)\partial_- \sigma \partial_+ \sigma \\ & + \beta(y\beta + vx\rho - uy\rho - v\beta\rho)\partial_- \sigma \partial_+ \rho \\ & - \beta(-u\beta + vx\rho - uy\rho + v\beta\rho)\partial_- \rho \partial_+ \sigma + v\beta^2\partial_- \rho \partial_+ \rho]. \end{aligned} \quad (26)$$

By rescaling  $E(e) \mapsto E(e)/\beta$ ,  $\mathcal{L} \mapsto \mathcal{L}\beta$ ,  $\tilde{\mathcal{L}} \mapsto \tilde{\mathcal{L}}/\beta$  we obtain the  $GL(2|\mathbb{R})$  model found in the work.<sup>2</sup> It means that even though we have a one-parametric class of nonisomorphic Drinfeld doubles of type A the corresponding dual models are equivalent.

#### 4. Conclusions

We have classified the four-dimensional Drinfeld doubles and constructed the Poisson–Lie T–dual models on them. The investigation of the Drinfeld doubles showed explicitly that neither the subalgebras  $\mathcal{G}$ ,  $\tilde{\mathcal{G}}$  *per se* specify the Drinfeld double completely (viz. (9) vs. (10)) nor the Drinfeld double fixes the subalgebras  $\mathcal{G}$ ,  $\tilde{\mathcal{G}}$  uniquely (viz. (8) and (10)). It turned out that besides the pair of dual models on  $GL(2|\mathbb{R})$  presented in Ref. 2 and the trivial abelian models, there exist two pairs of dual models (21), (22) and (23), (24) on the semiabelian double (8). This is the simplest (and the only one known to the authors) example of nontrivial modular space of  $\sigma$ -models mutually connected by Poisson–Lie T–duality transformation.

It would be very interesting to find whether any of the semiabelian or nonabelian models is integrable.

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