CZECH TECHNICAL UNIVERSITY
IN PRAGUE

Libor Šnobl

On Integrability and T–duality of Principal Models

DOCTORAL THESIS STATEMENT

Czech Technical University in Prague

Faculty of Nuclear Sciences and Physical Engineering

Department of Physics

Libor Šnobl

On Integrability and T–duality of Principal Models

PhD Programme : Applied Natural Sciences

Branch of study: Mathematical Engineering

Doctoral thesis statement for obtaining the academic title of "Doctor", abbreviated to "Ph.D."

Prague, July 2002

The doctoral thesis was produced in full-time manner of PhD study at the Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, CTU in Prague.

Candidate: Name: Ing. Libor Šnobl
Department of Physics, Faculty of Nuclear
Sciences and Physical Engineering of the CTU
Address: Břehová 7, 115 19 Prague 1

Supervisor: Name: Prof. RNDr. Ladislav Hlavatý, DrSc.
Department of Physics, Faculty of Nuclear
Sciences and Physical Engineering of the CTU
Address: Břehová 7, 115 19 Prague 1

Opponents: Prof. RNDr. Čestmír Burdík, DrSc., FNSPE CTU Prague
Assoc. Prof. Rikard von Unge, Ph.D., FSci MU Brno
Prof. Pavel Winternitz, Ph.D., CRM Université de Montréal

The doctoral thesis statement was distributed on ....

The defence of the doctoral thesis will be held on ....

at .... before the Board for the Defence of the Doctoral Thesis in the branch of study Mathematical Engineering in the meeting room No. ...... of the Faculty of Nuclear Sciences and Physical Engineering of the CTU in Prague.

It is possible to look into the doctoral thesis at the Dean's Office of the Faculty of Nuclear Sciences and Physical Engineering CTU in Prague, the Department of Science and Research, Břehová 7, 115 19 Prague 1.

Prof. Ing. Jiří Tolar, DrSc.
Chairperson of the Board for the Defence of the Doctoral Thesis in the branch of study Mathematical Engineering
Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University in Prague,
Břehová 7, 115 19 Prague 1

Contents

My doctoral thesis is devoted to two areas of research, the integrability of principal chiral models and the classification of algebraic structures involved in the Poisson–Lie T–duality of σ–models, namely Manin triples and Drinfeld doubles. It consists of the papers

- L. Hlavatý, and L. Šnobl: Principal chiral models on non–semisimple groups, *J. Phys A* 34 (2001) 7795–7809.
- L. Šnobl, L. Hlavatý: Principal chiral models with non–constant metric, *Czech. J. of Phys.* 51 (2001) 1441–1446.
- L. Hlavatý, L. Šnobl: Poisson–Lie T–dual models with two–dimensional targets, *Mod. Phys. Lett. A* 17 (2002) 429–434.
- L. Šnobl, L. Hlavatý: Classification of 6–dimensional real Drinfeld doubles, accepted for publication in *Int. J. of Mod. Phys. A*.

and the preprint

- L. Hlavatý, L. Šnobl: Classification of 6–dimensional Manin triples. e–preprint math.QA/0202209.

In the following sections I provide firstly a brief review of the known facts about integrability of σ–models and about Poisson–Lie T–duality, secondly a summary of my results contained in each of the papers. At the end I recollect several open questions and propose possible directions for the future research.

1 Introduction

1.1 σ–models and principal chiral models

σ–models are encountered quite often in modern theoretical physics, either as a corner–stone of a theory, e.g. the string theory or as toy models reproducing some properties of more realistic and more complex systems. They are in general field theoretical models on d–dimensional Minkowski spacetime M with values in D–dimensional target manifold T, whose action written in terms of fields

$$\phi^a : \ M \to T, \ a \in \{1, \dots, D\}$$

is

$$S = \int d^d x \mathcal{L} = \int d^d x G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b, \tag{1}$$

G_{ab} is interpreted as a given metric on the target space T and that's why it is assumed to be nondegenerate and symmetric, greek indices are raised and lowered using the Minkowski metric $\eta = diag(+1, -1, \dots, -1)$. In the following I consider only dim$M = d = 2$, i.e. $\eta = diag(+1, -1)$. The equations of motion are derived by the variation of the action

$$0 = \partial_\mu \partial^\mu \phi^b + (G^{-1})^{bc} \frac{\partial G_{cd}}{\partial \phi^e} \partial_\mu \phi^d \partial^\mu \phi^e - \frac{1}{2} (G^{-1})^{bc} \frac{\partial G_{de}}{\partial \phi^c} \partial_\mu \phi^d \partial^\mu \phi^e.$$

Defining the Levi–Civita connection

$$\Gamma^a_{bc} = \frac{1}{2}(G^{-1})^{ad}\left(\frac{\partial G_{dc}}{\partial \phi^b} + \frac{\partial G_{bd}}{\partial \phi^c} - \frac{\partial G_{bc}}{\partial \phi^d}\right)$$

and using the symmetry of $\partial_\mu \phi^a \partial^\mu \phi^b$ one can write the equations of motion in the form

$$\partial_\mu \partial^\mu \phi^a + \Gamma^a_{bc} \partial_\mu \phi^b \partial^\mu \phi^c = 0. \tag{2}$$

(Generalized) Principal chiral models or principal σ–models form a special subclass of σ–models, namely those with the target T not only a manifold but moreover a Lie group G. In this case one may choose a basis $\{t_a\}$ of the Lie algebra $\mathcal{G}$ and express the action in a more geometrical way in terms of currents J^μ instead of derivatives of the fields

$$S = \int d^d x L_{ab}(\phi) J^{a,\mu} J^b{}_{,\mu}$$

where

$$J^\mu = \mathrm{L}_{\phi^{-1}*} \partial^\mu \phi \equiv \phi^{-1} \partial^\mu \phi, \quad J^\mu = J^{a,\mu} t_a.$$

This notation is especially useful when the Lagrangian is left–invariant on the group, then L_{ab} are just constants.

1.2 Integrability of principal chiral models

The simplest example of a principal chiral model is the model on the abelian group $G = (R^+, .)$ with $L_{ab} = const$. The equation of motion when written in terms of θ, $\phi = \exp(\theta)$ is just the wave equation and the model is explicitly solvable. One may easily find other similarly trivial solvable principal chiral models, e.g. $G = (R^n, +)$, $L_{ab} = (const.)_{ab}$. A question naturally arises whether there are principal chiral models that are not of this trivial kind but are integrable using inverse spectral transformation. Since 1978 it is known that the answer is positive, the most famous example is the principal chiral model on semisimple group with the Killing form taken as the metric [1] but also other examples are known [2],[3].

The first step in the search for spectral and inverse spectral transformation is the reformulation of equations of motion as a condition of vanishing commutator of two so–called Lax operators. The form of Lax operators used in most known cases of integrable systems is

$$X_0 = \partial_0 + M(\phi, \lambda), \; X_1 = \partial_1 + L(\phi, \lambda)$$

where M, L are matrices (elements of some Lie algebra) depending on the fields ϕ, their derivatives and the spectral parameter λ (see e.g. [4]). The Lax equation then reads

$$[\partial_0 + M(\phi, \lambda), \partial_1 + L(\phi, \lambda)] = 0. \tag{3}$$

Since a Lie group G in principal chiral models is given from the beginning, one may assume that M, L have values in its Lie algebra $\mathcal{G}$.

It is not *apriori* clear of what kind the functional dependence of M, L on the field ϕ might be. In order to be able to look for possible integrable principal chiral models, one is forced to make some assumptions (ansatz) about it. Since the equations of motion are linear in derivatives of $J^{a,\mu}$ and quadratic in $J^{a,\mu}$, the simplest reasonable ansatz proposed by N. Sochen in [5] for M, L is linear in currents $J^{a,\mu}$. Then the Lax equation might be equivalent to the equations of motion since it contains linearly derivatives of L and M and the commutator term, which is then quadratic in $J^{a,\mu}$. Using this ansatz N. Sochen reobtained the known Lax pairs for the principal chiral models studied in [1], [3], both of them have constant metric L_{ab}.

In [6], [7] using the same ansatz L. Hlavatý studied the existence of integrable principal chiral models with non–constant metric L_{ab} on the simple group $SU(2)$ and on the group of 1–dimensional affine transformations $Af(1)$. When the metric is non–constant, the coefficients of the currents $J^{a,\mu}$ in the ansatz for Lax operators might depend on ϕ, but in the $SU(2)$ case the equivalence of (3) with equations of motion enforces the constancy of the coefficients, in the $Af(1)$ case the consideration of non–constant coefficients leads to rather non–trivial partial differential equations for them (in addition to algebraic relations (16)–(21) of the paper 2.1). A method of solution of this complicated system is unknown and consequently only the constant coefficients were investigated. In the $SU(2)$ case [6] L. Hlavatý obtained necessary conditions for integrability on the metric but didn't find any example of integrable principal chiral models with non–constant metric. In the $Af(1)$ case [7] a nontrivial example of a model with non–constant metric seemingly allowing Lax formulation was presented. Later it turned out that it hadn't the spectral parameter since it could be trivially transformed away.

1.3 T–duality

1.3.1 Dualities in general

The notion of duality is nowadays often used in physics, especially in connection with the superstring theory, like S– and T–duality, AdS/CFT duality etc. The general idea behind duality is quite simple: one assumes that there are two descriptions of the same physical situation and uses duality to translate the conclusions from one description into the other, e.g. the solutions of equations of motion. A simple example (examples in this section are taken from [8]) is provided in 4–dimensional

Minkowski space by a massles scalar field $\Phi(x)$ with the action[1]

$$S_1 = \int d^4x F_\mu F^\mu = \int d^4x F \wedge *F, \quad F = d\Phi$$

and a massless antisymmetric second–rank tensor A

$$S_2 = \frac{1}{3!} \int d^4x \tilde{F}_{\mu\nu\rho} \tilde{F}^{\mu\nu\rho} = \int d^4x \tilde{F} \wedge *\tilde{F}, \quad \tilde{F} = dA$$

The equations of motion and Bianchi identities are

$$d * F = 0,\ dF = 0 \text{ resp. } d * \tilde{F} = 0,\ d\tilde{F} = 0,$$

the duality transformation is just the Hodge dual $\tilde{F} = *F$. Dualities are often of this kind, interchanging the rôles of equations of motion and Bianchi identities for respective theories. This duality can be derived from the so–called parent action for two independent fields $\tilde{F}, \phi$

$$S_P = \int d^4x \left(\tilde{F} \wedge *\tilde{F} + \phi d\tilde{F}\right)$$

The variation of S_P w.r.t. ϕ gives the Bianchi identity $d\tilde{F} = 0$, i.e. $\tilde{F}$ is locally exact, $\tilde{F} = dA$ and plugging this back into action one reobtains S_2. Similarly by the variation of S_P w.r.t. $\tilde{F}$ one finds $\tilde{F} = - * d\phi$ and after putting this into S_P one gets the action S_1 for ϕ. It is also worth mentioning that the duality connects algebraically field strengths, the connection between the original and dual fields is expressed as a differential equation

$$dA = - * d\Phi.$$

1.3.2 Abelian T–duality

The T–duality, or the target space duality, denotes a special duality between σ–models, in general connecting σ–models with different target manifolds. Its simplest example is the following, so-called abelian T–duality.

Firstly, in order not to get out of the considered class of models, one is forced to introduce except the metric also antisymmetric terms into the action

$$S = \int d^2x \left(G_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b + \epsilon^{\mu\nu}B_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b\right). \tag{4}$$

In the light–cone coordinates $z = x^0 - x^1$, $\bar{z} = x^0 + x^1$ one can unite G_{ab} and B_{ab} into $F_{ab} = G_{ab} + B_{ab}$ and write equivalently the action

$$S = \int dz d\bar{z} F_{ab}(\phi)\partial\phi^a\bar{\partial}\phi^b, \text{ where } \partial \equiv \frac{\partial}{\partial z},\ \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}. \tag{5}$$

[1] I use the conventions

$$F = \frac{1}{r!} F_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r},\ \forall F \in \Lambda^r(M),$$

$$*F = \frac{1}{(d-r)!}\frac{1}{r!}\epsilon_{\mu_1\ldots\mu_{d-r}\nu_1\ldots\nu_r}F^{\nu_1\ldots\nu_r}dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{d-r}},\ \forall F \in \Lambda^r(M)$$

and

$$\epsilon_{0123} = -1.$$

Now it is possible to construct a T–duality transformation between two such models if one assumes that an isometry of both G_{ab} and B_{ab} exists, i.e. exists a vector field $\mathbf{e} = e^a \frac{\partial}{\partial \phi^a}$ on the target manifold T such that

$$(Lie_{\mathbf{e}} G)_{ab} = G_{ac} \frac{\partial e^c}{\partial \phi^b} + G_{cb} \frac{\partial e^c}{\partial \phi^a} + \frac{\partial G_{ab}}{\partial \phi^c} e^c = 0, \tag{6}$$

and similarly for B. In suitably chosen coordinates one has

$$Lie_{\frac{\partial}{\partial \phi^0}} G_{ab} = \frac{\partial G_{ab}}{\partial \phi^0} = 0, \; Lie_{\frac{\partial}{\partial \phi^0}} B_{ab} = \frac{\partial B_{ab}}{\partial \phi^0} = 0. \tag{7}$$

The dual model is given by Buscher's formulae [9], [10]

$$\tilde{G}_{00} = \frac{1}{G_{00}}, \quad \tilde{G}_{0i} = \frac{1}{G_{00}} B_{0i}, \quad \tilde{B}_{0i} = \frac{1}{G_{00}} G_{0i},$$

$$\tilde{G}_{ij} = G_{ij} - \frac{1}{G_{00}} (G_{0i} G_{0j} + B_{i0} B_{0j}),$$

$$\tilde{B}_{ij} = B_{ij} + \frac{1}{G_{00}} (G_{0i} B_{0j} + B_{i0} G_{0j}).$$

It is easy to check that applying the Buscher's formulae twice one reobtains the original model. Also it is clear that starting from a model with $B_{ab} = 0$ one mostly gets a model with B nonvanishing, this is the reason for the above introduced modification of the action. If several independent commuting vector fields satisfying

$$(Lie_{\mathbf{e_j}} G)_{ab} = (Lie_{\mathbf{e_j}} B)_{ab} = 0$$

exist, i.e. there is an abelian algebra of isometries, then there are more mutually dual models. The dualization may proceed in several steps, using one vector field at each step for the dualization as prescribed by Buscher's formulae.

1.3.3 Poisson–Lie T–duality

A question naturally arises what can be done for models with non–abelian Lie algebra $\mathcal{G}$ of isometries and finally whether exists a generalization applicable also in the case with no isometries at all. If there are several isometries generated by non–commuting vector fields, one can of course choose just one of them and apply the procedure described in the previous section but cannot proceed further because of non–commutativity.

Nevertheless also another approach, the so–called non–Abelian duality, exists if the group acts freely on T [11]. Its vast generalization has led to the discovery of Poisson–Lie T–dual models by C. Klimčík and P. Ševera in [12]. For simplicity I will assume in the following that the group G acts on the target manifold not only freely but also transitively, i.e. $T \equiv G$.

Let's start from the light–cone action

$$S = \int dz d\bar{z} F_{ij}(\phi) \partial \phi^i \bar{\partial} \phi^j \tag{8}$$

and take as a basis of vector fields on G the left–invariant fields,

$$\mathbf{e}_a = e_a^i \frac{\partial}{\partial \phi^i},\ [\mathbf{e}_a, \mathbf{e}_b] = f_{ab}^c \mathbf{e}_c.$$

A variation of ϕ can now be expressed as

$$\delta \phi^i = k^a(z, \bar{z}) e_a^i.$$

Putting this variation into the action (8) one finds up to total divergencies

$$\delta S = \int dz d\bar{z} \left(k^a Lie_{\mathbf{e}_a} F_{ij} \partial \phi^i \bar{\partial} \phi^j + (\partial k^a) e_a^i F_{ij} \bar{\partial} \phi^j + (\bar{\partial} k^a) e_a^j F_{ij} \partial \phi^i \right).$$

Defining the current 1–forms on M

$$\mathcal{J}_a = J_a dz + \bar{J}_a d\bar{z},\ J_a = e_a^j F_{ij} \partial \phi^i,\ \bar{J}_a = e_a^i F_{ij} \bar{\partial} \phi^j, \tag{9}$$

one can write the equations of motion in the form

$$\partial \bar{J}_a + \bar{\partial} J_a = (Lie_{\mathbf{e}_a} F)_{ij} \partial \phi^i \bar{\partial} \phi^j. \tag{10}$$

It is clear that if the group G is a group of isometries of F, $Lie_{\mathbf{e}_a} F_{ij} = 0$, then the currents (9) are the corresponding Noether conserved currents $d * \mathcal{J}_a = 0$. A generalization of the isometry condition is obtained by demanding the $*\mathcal{J}_a$ to be not closed but to satisfy a Maurer–Cartan equation on some other group $\tilde{G}$ written in components

$$d * \mathcal{J}_a + \frac{1}{2} \tilde{f}_a^{bc} * \mathcal{J}_b \wedge * \mathcal{J}_c = (\partial \bar{J}_a + \bar{\partial} J_a - \tilde{f}_a^{bc} J_b \bar{J}_c) dz \wedge d\bar{z} = 0, \tag{11}$$

This assumption allows to express the currents as a "pure gauge" on the dual group $\tilde{G}$ with $\tilde{T}^a$ the generators of its algebra $[\tilde{T}^a, \tilde{T}^b] = \tilde{f}_c^{ab} \tilde{T}^c$

$$* \mathcal{J} = * \mathcal{J}_a \tilde{T}^a \Rightarrow * \mathcal{J} = \tilde{g}^{-1} d\tilde{g},\ \tilde{g} \in \tilde{G}.$$

The group elements $\tilde{g}$ written in some coordinates are then intepreted as the fields dual to the original ϕs, the equation of motion of the original model (10) becomes now the Bianchi identity for $*\mathcal{J}$. Also one should note that as in the previous examples of duality, the original and dual fields are connected through a differential, not algebraic equation.

It remains to investigate for which metrics F_{ij} the equivalence of (11) and (10) can be established. Comparing (10) and (11) one finds the condition

$$(Lie_{\mathbf{e}_a} F)_{ij} = \tilde{f}_a^{bc} F_{ik} e_b^k F_{lj} e_c^l. \tag{12}$$

From (12) together with the fact that Lie derivatives form a representation of the Lie algebra $\mathcal{G}$, one may express a necessary condition as

$$\tilde{f}_l^{jk} f_{mi}^l + \tilde{f}_m^{kl} f_{li}^j + \tilde{f}_i^{jl} f_{lm}^k + \tilde{f}_m^{jl} f_{il}^k + \tilde{f}_i^{kl} f_{lm}^j = 0. \tag{13}$$

The condition (13) is surprisingly just the condition on algebras $\mathcal{G}, \tilde{\mathcal{G}}$ for existence of a Manin triple on them, i.e. of a Lie algebra $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$ such that $\mathcal{G}, \tilde{\mathcal{G}}$ are its subalgebras maximally isotropic w.r.t. the form

$\langle .,. \rangle$ defined by $\langle T_a, \tilde{T}^b \rangle = \delta_a^b$. Then (13) is just the Jacobi identity for one element from $\mathcal{G}$ and two elements from $\tilde{\mathcal{G}}$ or vice versa.

The appearance of Manin triples in the description of Poisson–Lie T–dual models indicates a connection between the T–duality and Drinfeld doubles; Drinfeld double D being a connected Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad–invariant nondegenerate bilinear form $\langle .,. \rangle$ can be written as a Manin triple.

One may like to have not only some necessary conditions for existence of dual models but also a method of construction of such models, i.e. of their metrics, from some simpler structures, avoiding the need for solving the partial differential equations (12). Such a way of construction of Poisson–Lie T–dual models on Drinfeld doubles was presented by C. Klimčík and P. Ševera in [12] and [13]. The construction starts by postulating the equations of motion on the whole Drinfeld double, not depending on the choice of Manin triple:

$$\langle (\partial_\pm l) l^{-1}, \mathcal{E}^\pm \rangle = 0, \tag{14}$$

where subspaces

$$\mathcal{E}^+ = \mathrm{span}(X_i + E_{ij}(e)\tilde{X}^j),\ \mathcal{E}^- = \mathrm{span}(X_i - E_{ji}(e)\tilde{X}^j)$$

are orthogonal with respect to $\langle .,. \rangle$ and span the whole Lie algebra $\mathcal{D}$ and $\{X_i\}$, resp. $\{\tilde{X}^i\}$ form the bases of $\mathcal{G}$, resp. $\tilde{\mathcal{G}}$ such that

$$\langle X_i, X_j \rangle = 0,\ \langle \tilde{X}^i, \tilde{X}^j \rangle = 0,\ \langle X_i, \tilde{X}^j \rangle = \delta_i^j.$$

One writes $l = g.\tilde{h}$, $g \in G$, $\tilde{h} \in \tilde{G}$, resp. $l = \tilde{g}.h$, $h \in G$, $\tilde{g} \in \tilde{G}$ (such decomposition of elements of the group D exists at least at the vicinity of the unit element according to [14]) and eliminates $\tilde{h}$, resp. h by further differentiation.

The resulting 2nd order equations can be also obtained by variation of actions with Lagrangians

$$\mathcal{L} = E_{ij}(g)(g^{-1}\partial_- g)^i (g^{-1}\partial_+ g)^j, \tag{15}$$

$$\tilde{\mathcal{L}} = \tilde{E}^{ij}(\tilde{g})(\tilde{g}^{-1}\partial_- \tilde{g})_i (\tilde{g}^{-1}\partial_+ \tilde{g})_j, \tag{16}$$

where

$$E(g) = (a(g) + E(e)b(g))^{-1} E(e) d(g), \tag{17}$$

$E(e)$ is a constant matrix and $a(g), b(g), d(g)$ are submatrices of the adjoint representation of the group G on $\mathcal{D}$ in the basis $(X_i, \tilde{X}^j)$

$$Ad(g)^T = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}, \tag{18}$$

the matrix $\tilde{E}(\tilde{g})$ is constructed analogously.

This construction not only shows that examples of Poisson–Lie T–duality without isometries of the target, i.e. with both G and $\tilde{G}$ non-abelian, exist but also explains that one may consider dual all models arising from the same Drinfeld double (with a given constant matrix $E^{ij}(e)$), not only the pairs of models on one Manin triple; all such models share the same original equations on the double (14).

2 My results on integrability of principal chiral models

I concentrated on the search for principal chiral models with non–constant metric that allow Lax formulation of their equations of motion on low–dimensional Lie groups. If such nontrivial model had been found (unfortunately hadn't), a natural second step would be to use the inverse spectral transformation in order to find soliton solutions of its equations of motion.

In a continuation of the works [6], [7], I and L. Hlavatý considered the $Af(1)$ case for non–diagonal metric and also investigated models on all non–semisimple 3–dimensional Lie groups. We generalized the ansatz from the linear to affine one to get a nontrivial spectral parameter. At the end, all examples of principal chiral models on these groups that allow Lax formulation of the chosen form turned out to be equivalent to a sequence of linear partial differential equations, therefore unsuitable for solution using inverse spectral transformation. It is possible to use even more general ansatz for Lax operators, e.g. allow quadratic or higher order terms in currents, but currently it appears that such generalization complicates the necessary conditions for integrability so that they cannot be solved even using the currently available computer algebra systems. The results were published in the following two papers:

2.1 L. Hlavatý, and L. Šnobl. Principal chiral models on non–semisimple groups

Journal of Physics A 34 (2001) 7795–7809.

In this paper we investigated principal chiral models on solvable 2– and 3–dimensional Lie groups. We were trying to find models whose equations of motion can be formally rewritten as a Lax pair.

It turned out that equations of such models are in most cases equivalent to a sequence of linear partial differential equations, e.g. a wave equation for one field and two linear partial differential equations for the remaining fields depending nonlinearly on the solution of the wave equation. Only one of the models appeared to be truly nonlinear, but further considerations (see the paper 2.2) showed that it can be also brought to a similar form of sequence of linear equations.

2.2 L. Šnobl, L. Hlavatý. Principal chiral models with non–constant metric

Czechoslovak Journal of Physics 51 (2001) 1441–1446.

This article is a summary of the talk at the 10th International Colloquium "Quantum Groups and Integrable Systems". It contains a brief review of results of previous paper and two new results.

Firstly, the Lax pair of the 2–dimensional nonlinear system (two second order partial differential equations) obtained before (see the paper 2.1) hinted a possible reformulation after a change of variables in terms of linear differential equations, namely as a wave equation and two first order equations depending nonlinearly on the solution of the wave equation.

Secondly, we reconsidered an older work by L. Hlavatý [6] concerning the principal models on $SU(2)$ and we proved that the technical assumption of the diagonal form of the metric, simplifying the computation to a manageable state, leads immediately to constancy of the metric. The consideration of non–diagonal metrics seemed far too complicated and was not pursued further.

3 My results on T–duality and Drinfeld doubles

I and L. Hlavatý have concentrated on investigation and classification of Drinfeld doubles in the lowest nontrivial dimensions 4 and 6. The classification of Drinfeld doubles in dimension 4 and the corresponding pairs of T–dual models are presented in the paper 3.1. The main conclusion of this paper is that even in this case a Drinfeld double with several non–isomorphic Manin triples exists and provides a motivation for investigation of higher–dimensional cases. In the dimension 6 we found all non–isomorphic Manin triples and wrote them in 78 classes (the preprint 3.2) and then investigated which of these Manin triples define isomorphic Drinfeld doubles (the paper 3.3). It turned out that in the chosen parametrization there are 22 classes of non–isomorphic 6–dimensional real Drinfeld doubles.

3.1 L. Hlavatý, L. Šnobl. Classification of Poisson–Lie T–dual models with two–dimensional targets

Modern Physics Letters A17 (2002) 429–434.

In this paper we constructed all 4–dimensional Manin triples and the corresponding Drinfeld doubles. Quite surprisingly, even in this low dimensional case one Drinfeld double possessing decompositions into non–isomorphic Manin triples, i.e. into non–isomorphic pairs of maximal isotropic subalgebras exists. This is the simplest example of such Drinfeld double, several 6–dimensional examples were found in [15] and a complete investigation of 6–dimensional case is contained in the next paper 3.3. Corresponding pairs of Poisson–Lie T–dual models on Manin triples were explicitly constructed.

The knowledge of explicit examples of Drinfeld doubles decomposable into non–isomorphic Manin triples might be of interest in the research on Poisson–Lie T–duality, especially it might be possible to check whether the duality between models corresponding to different Manin triples survives also in quantum theory; our considerations were only on the classical level.

3.2 L. Hlavatý, L. Šnobl. Classification of 6–dimensional real Manin triples

preprint math.QA/0202209.

In this preprint we presented the list of all 6–dimensional Manin triples. It is a significant generalization of the paper [15] of M.A. Jafarizadeh and A. Rezaei-Aghdam who have assumed both isotropic subalgebras in the Bianchi form and consequently missed quite a lot of other cases. After its submission to electronic archive, we found that an equivalent classification of 3–dimensional Lie bialgebras was already published by X. Gomez in [16], albeit using different classification of 3-dimensional Lie algebras as a starting point.

This work therefore provides an independent check of the Gomez's work. We compared the results and found that after translating the notations they are equivalent. Consequently, our work was not published.

Nevertheless there are two reasons for including it into the thesis. Firstly, we have used it as a starting point for investigation of Drinfeld doubles (the paper 3.3) and secondly the classification was performed using different, more straightforward, if more computationally demanding, method.

Our method basically starts from the first subalgebra of the Manin triple written in a basis with fixed commutation relations (the so–called Bianchi form), then gradually solving the Jacobi identities between the subalgebras and in the second subalgebra itself and finally using the remaining freedom in the choice of Bianchi basis of the first subalgebra to write the second subalgebra in the simplest possible form. (Gomez had used more theoretical approach, using the notion of twisting etc.) The intermediate results therefore contain also general commutation relations of any basis of the second subalgebra dual to the original Bianchi-type one. These intermediate results might be of interest in some applications.

3.3 L. Šnobl, L. Hlavatý. Classification of 6–dimensional real Drinfeld doubles

accepted for publication in International Journal of Modern Physics A.

In this paper we classified all 6–dimensional real Drinfeld doubles, i.e. we identified Manin triples giving rise to the same Drinfeld double.

Our investigation is based on the invariants of the underlying Lie algebra $\mathcal{D}$, i.e. rank of its Killing form and dimensions of derived subalgebras $\mathcal{D}^i$, $\mathcal{D}_i$. These invariants give a coarse sorting into classes of Manin triples that might possibly lead to the same Drinfeld doubles. Manin triples in each of these classes are then studied and the (non)equivalence of the corresponding Drinfeld doubles is rigorously proven.

Interesting conclusion is that not only rather different Manin triples might lead to the same Drinfeld double, but also the same underlying

Lie algebra $\mathcal{D}$ may be equipped, e.g. for $\mathcal{D}$ semisimple, with different bilinear forms and define different Drinfeld doubles.

4 Conclusions and future prospects

Concerning the integrability of principal σ–models I have found no interesting, truly nonlinear integrable model on any 2– and 3– dimensional non–semisimple Lie group, although given the ansatz for Lax operators the investigation seems to be complete (up to completness of results obtained using computer algebra systems, as mentioned in the paper 2.1). I don't know whether it is due to our linear ansatz for Lax operators or whether it is a general property of principal models on non–semisimple groups. One may imagine a generalization of the ansatz but it appears that the arising conditions for the Lax operators would be too complicated to solve.

The classification of principal models on the group $SU(2)$ with diagonal metric and Lax pair linear in currents can be now considered complete: only models with constant metric allow Lax pair and all models with constant metric have a Lax pair. Unfortunately a classification of integrable principal chiral models with nonconstant non-diagonal metric on $SU(2)$ seems to be technically unfeasible in the present time.

Concerning the Drinfeld doubles and Poisson–Lie T–duality we have constructed a complete list of six-dimensional real Drinfeld doubles up to their isomorphisms i.e. maps preserving both the Lie structure and the ad-invariant symmetric bilinear form $\langle .,. \rangle$ that define the double. In our parametrization there are just 22 classes of the non-isomorphic Drinfeld doubles. One can see that for many Drinfeld doubles there are several decompositions into Manin triples. We can in principle construct the explicit Lagrangians of the pairs of Poisson–Lie T–dual models for every Drinfeld double but given the large number of Manin triples this does not make much sense unless a concrete physical motivation picks up some of them. The investigation of properties of quantum analogs of different models on the same Drinfeld double and whether the connection between them survives the quantization, might be of interest in the superstring theory. As far as I know no explicit examples of such models were studied before, only recently a paper by R. von Unge on this subject appeared [17].

An important point that follows from the classification of Drinfeld doubles is that there are several different Drinfeld doubles corresponding to Lie algebras $so(1,3)$, $sl(2,\mathbf{R})\oplus sl(2,\mathbf{R})$, $sl(2,\mathbf{R})\triangleright\mathbf{R}^3$ whereas on solvable Lie algebras the Drinfeld double is unique (in some cases up to the sign of the bilinear form). It might be interesting to know whether such behaviour holds in any dimension or is just a low–dimensional artifact.

On the other hand there are Manin triples with one isotropic subalgebra abelian that are equivalent as Drinfeld doubles even though the other subalgebras are different. That's why it was necessary to investigate the (non)equivalence of the Manin triples of this form. Moreover the above given examples indicate the diversity of Drinfeld double structures one may encounter in higher dimensions. Beside that from the present classification procedure one can find whether a given six-dimensional Lie algebra can be equipped by a suitable ad-invariant bilinear form and turned into a Drinfeld double (and how many such forms exist). The investigation can be reduced to a direct check of equivalence with a particular six-dimensional Lie algebra. For example, one can see that there is no Drinfeld double on $SO(4)$.

Finally, let me note that the complete sets of equivalent σ–models for a fixed Drinfeld double are given by the so–called modular space of the double. The construction of all non-isomorphic Manin triples for the double is the first step in the construction of the modular spaces.

List of publications and preprints relating to the dissertation

1. L. Hlavatý, and L.Šnobl. Principal chiral models on non–semisimple groups. *J. Phys A* 34 (2001) 7795–7809.

2. L. Šnobl, L. Hlavatý. Principal chiral models with non–constant metric. *Czech. J. of Phys.* 51 (2001) 1441–1446.

3. L. Hlavatý, L. Šnobl. Poisson–Lie T–dual models with two–dimensional targets. *Mod. Phys. Lett. A* 17 (2002) 429, e–preprint hep–th/0110139.

4. L. Hlavatý, L. Šnobl. Classification of 6–dimensional Manin triples. e–preprint math.QA/0202209.

5. L. Šnobl, L. Hlavatý. Classification of 6–dimensional real Drinfeld doubles. accepted for publication in *International Journal of Modern Physics A*, e–preprint math.QA/0202210.

6. L. Šnobl. On modular spaces of semisimple Drinfeld double, e–preprint hep-th/0204244.

List of conference talks relating to the dissertation

1. Winter School "Geometry and Physics", Srní, January 2001; a contributed talk "Principal chiral models on non–semisimple groups".

2. 10th Colloquium "Quantum Groups and Integrable Systems", Prague, June 2001; a contributed talk "Principal chiral models with non–constant metric".

3. XX Workshop on Geometric Methods in Physics, Bialowieža, July 2001; a contributed talk "Principal chiral models with non–constant metric".

4. Winter School "Geometry and Physics", Srní, January 2002; a contributed talk "Poisson–Lie T–dual models with two dimensional targets".

List of citations

- L. Šnobl, L. Hlavatý. Classification of 6–dimensional real Drinfeld doubles. accepted for publication in *International Journal of Modern Physics A*, e–preprint math.QA/0202210 cited by
  - Rikard von Unge. Poisson–Lie T–plurality. J. High Energy Phys. JHEP 07 (2002) 014.
- L. Šnobl. On modular spaces of semisimple Drinfeld double, e–preprint hep-th/0204244, cited by
  - Rikard von Unge. Poisson–Lie T–plurality. J. High Energy Phys. JHEP 07 (2002) 014.

References

[1] V. E. Zakharov and A. V. Mikhailov. *J. Exp. Theor. Phys.* 74(6):1953–1973, 1978, in Russian.

[2] I V. Cherednik. *Jadernaja fizika* 33(1):278–282, 1981, in Russian.

[3] P I. Holod. *Theor. Math. Phys.* 70(1):18–29, 1987, in Russian.

[4] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur. *Stud. Appl. Math.* 53:249–315.

[5] N.Sochen. *Phys. Lett.* B 391:374–380, 1997.

[6] L. Hlavatý. Phys. Lett. A 271:207–212, 2000.

[7] L. Hlavatý. *Phys. Lett.* A 275:419–423, 2000.

[8] S. E. Hjelmeland, U. Lindström. Duality for the Non–Specialist, e–preprint hep-th/9705122

[9] T. Buscher. *Phys. Lett. B*, 201:466, 1988.

[10] T. Buscher. *Phys. Lett. B*, 194:59, 1987.

[11] X. C. de la Ossa, F. Quevedo. *Nucl.Phys. B* 403:377-394, 1993.

[12] C. Klimčík and P. Ševera. *Phys.Lett. B*, 351:455–462, 1995.

[13] C. Klimčík. *Nucl.Phys B (Proc.Suppl.)*, 46:116–121, 1996.

[14] V.G. Drinfeld. Quantum Groups. Proc. Int. Congr. Math. Berkeley, 798-820, 1986.

[15] M.A. Jafarizadeh and A. Rezaei-Aghdam. *Phys.Lett. B*, 458:470–490, 1999.

[16] X. Gomez. *J. Math. Phys.*, 41:4939, 2000.

[17] Rikard von Unge. J. High Energy Phys. JHEP 07 (2002) 014.

Resumé

Dizertace je věnována dvěma oblastem výzkumu, integrabilitě principálních chirálních modelů a klasifikaci algebraických struktur tvořících základ Poisson–Lie T–duality σ–modelů, jmenovitě Maninových trojic a Drinfeldových dvojic.

Skládá se z následujících publikovaných a k publikaci přijatých článků

- L. Hlavatý, and L. Šnobl: Principal chiral models on non–semisimple groups, *J. Phys A* 34 (2001) 7795–7809,
- L. Šnobl, L. Hlavatý: Principal chiral models with non–constant metric, *Czech. J. of Phys.* 51 (2001) 1441–1446,
- L. Hlavatý, L. Šnobl: Poisson–Lie T–dual models with two–dimensional targets, *Mod. Phys. Lett. A* 17 (2002) 429–434,
- L. Šnobl, L. Hlavatý: Classification of 6–dimensional real Drinfeld doubles, přijat k publikaci v *Int. J. of Mod. Phys. A*,

a z preprintu

- L. Hlavatý, L. Šnobl: Classification of 6–dimensional Manin triples. e–preprint math.QA/0202209.

V oblasti integrability principálních chirálních modelů jsem se soustředil na hledání modelů s nekonstantní metrikou připouštějících tzv. Laxovu formulaci v případě grup nízké dimenze. Protože tuto úlohu by bylo prakticky nemožné řešit zcela obecně, byl zvolen ansatz pro Laxovy operátory lineární v proudech s eventuelním konstantním členem. Za tohoto předpokladu bylo možné provést úplný rozbor modelů na vybraných grupách. Ukázal jsem, že na 2– a 3–rozměrných řešitelných grupách jsou takové modely vždy v jistém smyslu linearizovatelné. Naopak modely na grupě $SU(2)$ s diagonální metrikou jsou sice nelineární, ale podmínka existence Laxovy formulace implikuje konstantnost metriky. Žádné skutečně nelineární modely s nekonstantní metrikou s Laxovou formulací v těchto třídách za výše uvedeného předpokladu tedy neexistují.

Při studiu algebraických struktur Poisson–Lie T–duality se podařilo najít úplnou klasifikaci Maninových trojic (neboli Lieových bialgeber) a Drinfeldových dvojic v dimenzích 4 a 6, příslušné dvojice T–duálních modelů byly zkonstruovány v dimenzi 4. V dimenzi 6 bylo nalezeno ve zvolené parametrizaci 78 tříd neizomorfních Maninových trojic a bylo ukázáno, že tyto trojice definují celkem 22 tříd neizomorfních Drinfeldových dvojic. Již z těchto počtů je patrné, že v této dimenzi byla nalezena řada případů Drinfeldových dvojic připouštějících rozklad do několika Maninových trojic, v dimenzi 4 existuje jediná taková Drinfeldova dvojice. Studium odpovídajících duálních modelů může v budoucnu přispět k dalšímu prohloubení našich znalostí o T–dualitách a tedy k ještě lepšímu pochopení možností, které nabízejí duality jako nástroj neporuchových výpočtů v teorii superstrun.