CZECH TECHNICAL UNIVERSITY
IN PRAGUE

Faculty of Nuclear Sciences and Physical Engineering

DOCTORAL THESIS

On Integrability and T–duality of Principal Models

Prague 2002 Libor Šnobl

Acknowledgment

I would like to thank my supervisor professor Ladislav Hlavatý for numerous discussions, encouragement and other help during the whole PhD study, and my parents, Libuše Šnoblová and Zdeněk Šnobl for their support during my studies.

Contents

Chapter 1

Introduction and overview

My doctoral thesis is devoted to two areas of research, the integrability of principal chiral models and the classification of algebraic structures involved in the Poisson–Lie T–duality of σ–models, namely Manin triples and Drinfeld doubles. It consists of the papers

- L. Hlavatý, and L. Šnobl: Principal chiral models on non–semisimple groups, *J. Phys A* 34 (2001) 7795–7809.
- L. Šnobl, L. Hlavatý: Principal chiral models with non–constant metric, *Czech. J. of Phys.* 51 (2001) 1441–1446.
- L. Hlavatý, L. Šnobl: Poisson–Lie T–dual models with two–dimensional targets, *Mod. Phys. Lett. A* 17 (2002) 429–434.
- L. Šnobl, L. Hlavatý: Classification of 6–dimensional real Drinfeld doubles, accepted for publication in *Int. J. of Mod. Phys. A*.

and the preprint

- L. Hlavatý, L. Šnobl: Classification of 6–dimensional Manin triples. e–preprint math.QA/0202209.

In this chapter I provide a review of what is currently known about these subjects together with references to the literature and a brief summary of my results contained in the papers. At the end of the Chapter I recollect several open questions and propose possible directions for the future research.

The remaining chapters contain the papers, each preceded by a résumé. A bibliography[1], a list of my papers and preprints, a list of my conference contributions and a list of citations are given at the end.

[1]Please note that each paper has got its own list of references and the citations in it refer to that list.

1.1 σ–models and principal chiral models

σ–models are encountered quite often in modern theoretical physics, either as a corner–stone of a theory, e.g. the string theory or as toy models reproducing some properties of more realistic and more complex systems. They are in general field theoretical models on d–dimensional Minkowski spacetime M with values in D–dimensional target manifold T, whose action written in terms of fields

$$\phi^a : \; M \to T, \;\; a = 0, \ldots, D-1$$

is

$$S = \int d^d x \mathcal{L} = \int d^d x G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b, \tag{1.1}$$

G_{ab} is interpreted as a given metric on the target space T and that's why it is assumed to be nondegenerate and symmetric, greek indices are raised and lowered using the Minkowski metric $\eta = diag(+1, -1, \ldots, -1)$. One may also add an antisymmetric tensor times derivatives of the fields etc., this will be considered in Section 1.3. I will also consider only $d = 2$. The equations of motion are obtained as Euler–Lagrange equations by the variation of the action

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} - \frac{\partial \mathcal{L}}{\partial \phi^a} = 2\partial_\mu (G_{ab} \partial^\mu \phi^b) - \frac{\partial G_{bc}}{\partial \phi^a} \partial_\mu \phi^b \partial^\mu \phi^c =$$

$$= 2G_{ab} \left(\partial_\mu \partial^\mu \phi^b + (G^{-1})^{bc} \frac{\partial G_{cd}}{\partial \phi^e} \partial_\mu \phi^d \partial^\mu \phi^e - \frac{1}{2} (G^{-1})^{bc} \frac{\partial G_{de}}{\partial \phi^c} \partial_\mu \phi^d \partial^\mu \phi^e \right).$$

Defining the Levi–Civita connection

$$\Gamma^a_{bc} = \frac{1}{2} (G^{-1})^{ad} \left(\frac{\partial G_{dc}}{\partial \phi^b} + \frac{\partial G_{bd}}{\partial \phi^c} - \frac{\partial G_{bc}}{\partial \phi^d} \right)$$

and using the symmetry of $\partial_\mu \phi^a \partial^\mu \phi^b$ one can write the equations of motion in the form

$$\partial_\mu \partial^\mu \phi^a + \Gamma^a_{bc} \partial_\mu \phi^b \partial^\mu \phi^c = 0. \tag{1.2}$$

It is interesting to note that in $d = 2$ case, any left– or right–moving set of fields, i.e. $\phi^a(x^0 + x^1)$, $\forall a <$ D or $\phi^a(x^0 - x^1)$, $\forall a <$ D are solutions of equations of motion since $\partial_\mu \partial^\mu \phi^a(x^0 \pm x^1) = \phi''^a(x^0 \pm x^1) - (\pm 1)^2 \phi''^a(x^0 \pm x^1) = 0$ and $\Gamma^a_{bc} \partial_\mu \phi^b(x^0 \pm x^1) \partial^\mu \phi^c(x^0 \pm x^1) = \Gamma^a_{bc}(\phi'^b(x^0 \pm x^1)\phi'^c(x^0 \pm x^1) - (\pm 1)^2 \phi'^b(x^0 \pm x^1)\phi'^c(x^0 \pm x^1)) = 0$.

(Generalized) Principal chiral models or principal σ–models form a special subclass of σ–models, namely those with the target T not only a manifold but moreover a Lie group G. In this case one may write the action also in a slightly modified, probably more geometrical way. Instead of writing directly the derivatives of the fields $g \equiv \phi$ one can left translate them to the group unit and denote by the current

$$J^\mu = \mathrm{L}_{g^{-1}*} \partial^\mu g \equiv g^{-1} \partial^\mu g.$$

After choosing the basis $\{t_a\}$ of the Lie algebra $\mathcal{G}$ one may write the coordinates of the currents $J^\mu = J^{a,\mu} t_a$. The action is

$$S = \int d^d x L_{ab}(g) J^{a,\mu} J^b{}_{,\mu}$$

and the equations of motion are

$$\partial_\mu J^{a,\mu} + \Gamma^a_{bc} J^{\mu,b} J^c{}_{,\mu} = 0,$$

where Γ^a_{bc} is defined by relations $(7) - (11)$ in Chapter 2.

This notation is especially useful when the Lagrangian is left–invariant on the group, $L_{ab}(hg)(g^{-1}h^{-1}\partial^\mu hg)^a (g^{-1}h^{-1}\partial^\mu hg)^b = L_{ab}(g)(g^{-1}\partial^\mu g)^a (g^{-1}\partial^\mu g)^b$, i.e. $L_{ab}(hg) = L_{ab}(g)$, $\forall h \in G$ and L_{ab} are in this case just constants.

1.2 Integrability of principal chiral models

The simplest example of a principal chiral model is the model on the abelian group $G = (R^+, .)$ and $L_{ab} = const.$ The equation of motion when written in terms of θ, $g = \exp(\theta)$ is just the wave equation

$$\partial_\mu \partial^\mu \theta = 0$$

and the model is explicitly solvable

$$\theta = A(x^0 - x^1) + B(x^0 + x^1).$$

One may easily find other similarly trivial solvable principal chiral models, e.g. $G = (R^n, +)$, $L_{ab} = (const.)_{ab}$. A question naturally arises whether there are principal chiral models that are not of this trivial kind but are integrable using inverse spectral transformation. Since 1978 it is known that the answer is positive, the most famous example is the principal chiral model on semisimple group with the Killing form taken as the metric [1] but also other examples are known [2],[3].

The first step in the search for spectral and inverse spectral transformation is the reformulation of equations of motion as a condition of vanishing commutator of two so–called Lax operators. The form of Lax operators used in most known cases of integrable systems is

$$X_0 = \partial_0 + M(\phi, \lambda),\ X_1 = \partial_1 + L(\phi, \lambda)$$

where M, L are matrices (elements of some Lie algebra) depending on the fields ϕ and their derivatives and the spectral parameter λ. (This form was considered e.g. by Ablowitz, Kaup, Newell and Segur in their founding paper [4].) The Lax equation then reads

$$[\partial_0 + M(\phi, \lambda), \partial_1 + L(\phi, \lambda)] = \partial_0 L(\phi, \lambda) - \partial_1 M(\phi, \lambda) + [M(\phi, \lambda), L(\phi, \lambda)] = 0. \tag{1.3}$$

Since a Lie group G in principal chiral models is given from the beginning, one may assume that M, L have values in its Lie algebra $\mathcal{G}$.

It is not *apriori* clear of what kind the functional dependence of M, L on the field g might be. In order to be able to look for possible integrable principal chiral models, one is forced to make some assumptions (ansatz) about it. Since the equations of motion are linear in derivatives of $J^{a,\mu}$ and quadratic in $J^{a,\mu}$, the simplest reasonable ansatz proposed by N. Sochen in [5] for M, L is linear in currents $J^{a,\mu}$. Then the Lax equation might be equivalent to the equations of motion since it contains linearly derivatives of L and M and the commutator term, which is then quadratic in $J^{a,\mu}$. Using this ansatz N. Sochen reobtained the known Lax pairs for the principal chiral models studied in [1], [3], both of them have constant metric L_{ab}.

In [6], [7] using the same ansatz L. Hlavatý studied the existence of integrable principal chiral models with non–constant metric L_{ab} on the simplest groups available, namely on the simple group $SU(2)$ and on the solvable group of 1–dimensional affine transformations $Af(1)$. When the metric is non–constant, the coefficients of the currents $J^{a,\mu}$ in the ansatz for Lax operators might depend on g, but in the $SU(2)$ case the equivalence of Lax equations (1.3) with equations of motion enforces the constancy of the coefficients, in the $Af(1)$ case the consideration of non–constant coefficients leads to rather non–trivial partial differential equations for them (in addition to algebraic relations (16)–(21) of Chapter 2). A method of solution of this complicated system is unknown and consequently only the constant coefficients were investigated. In the $SU(2)$ case [6] L. Hlavatý obtained necessary conditions for integrability on the metric and tried to find some of their solutions in the case of diagonal metric (this was motivated by the fact that the known examples above can be written in diagonal form) but wasn't succesful; no example of integrable principal chiral models with non–constant metric was found. In the $Af(1)$ case [7] a nontrivial example of a model with non–constant metric seemingly allowing Lax formulation was presented. Later it turned out that it hadn't the spectral parameter since it could be trivially transformed away.

1.2.1 My results on integrability of principal chiral models

I concentrated on the search for principal chiral models with non–constant metric that allow Lax formulation of their equations of motion on low–dimensional Lie groups. If such nontrivial model had been found (unfortunately hadn't), a natural second step would be to use the inverse spectral transformation in order to find soliton solutions of its equations of motion.

In a continuation of the already mentioned works [6], [7] I and L. Hlavatý considered the $Af(1)$ case for non–diagonal metric and also investigated models on all non–semisimple 3–dimensional Lie groups (Chapter 2). We generalized the ansatz from the linear to affine one to get a nontrivial spectral parameter. At the end, all examples of principal chiral models on these groups that allow Lax formulation of the chosen form turned out to be equivalent to a sequence of linear partial differential equations, therefore unsuitable for solution using inverse spectral transformation. It is possible to use even more general ansatz for Lax operators, e.g. allow quadratic or higher order terms in the currents, but currently it appears that such generalization complicates the necessary conditions for integrability so that they cannot be solved even using the currently available computer algebra systems. I also found that in the $SU(2)$ case the diagonality of the metric together with the chosen ansatz for Lax operators immediately leads to the constancy of the metric. The investigation of the non–diagonal non–constant metric in the $SU(2)$ case again seems to be unmanageable at the time. The results were published in two papers and are presented in Chapters 2, 3.

1.3 T–duality

1.3.1 Dualities in general

The notion of duality is nowadays often used in physics, especially in connection with the superstring theory, like S– and T–duality, AdS/CFT duality etc. The general idea behind duality is quite simple: one assumes that there are two descriptions of the same physical situation and uses duality to translate the conclusions from one description into the other, e.g. the solutions of equations of motion. A simple example (examples in this section are taken from [8]) is provided in 4–dimensional Minkowski space by a massles scalar field $\Phi(x)$ with the action[2]

$$S_1 = \int d^4x F_\mu F^\mu = \int F \wedge *F, \quad F = d\Phi$$

and a massless antisymmetric second–rank tensor A

$$S_2 = \frac{1}{3!} \int d^4x \tilde{F}_{\mu\nu\rho} \tilde{F}^{\mu\nu\rho} = \int \tilde{F} \wedge *\tilde{F}, \quad \tilde{F} = dA$$

The equations of motion and Bianchi identities are in the first case

$$d * F = 0, \; dF = 0$$

and in the second case

$$d * \tilde{F} = 0, \; d\tilde{F} = 0,$$

the duality transformation is just the Hodge dual $\tilde{F} = *F$. Dualities are often of this kind, interchanging the rôles of equations of motion and Bianchi identities for respective theories. This duality can be derived from the so–called parent action for two independent fields $\tilde{F}, \phi$

$$S_P = \int \left(\tilde{F} \wedge *\tilde{F} + \phi d\tilde{F} \right)$$

The variation of S_P w.r.t. ϕ gives the Bianchi identity $d\tilde{F} = 0$, i.e. $\tilde{F}$ is locally exact, $\tilde{F} = dA$ and plugging this back into action one reobtains S_2. Similarly by the variation of S_P w.r.t. $\tilde{F}$ one finds $\tilde{F} = - * d\phi$ and after putting this into S_P one gets the action S_1 for ϕ. It is also worth mentioning that the duality connects algebraically field strengths, the connection between the original and dual fields is expressed as a differential equation

$$dA = - * d\Phi.$$

Such dualities might be almost trivial if one is able to describe the system completely and find explicit solutions of equations of motion but may be of

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[2] I use the conventions

$$F = \frac{1}{r!} F_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}, \; \forall F \in \Lambda^r(M),$$

$$*F = \frac{1}{(d-r)!} \frac{1}{r!} \epsilon_{\mu_1 \ldots \mu_{d-r} \nu_1 \ldots \nu_r} F^{\nu_1 \ldots \nu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{d-r}}, \; \forall F \in \Lambda^r(M)$$

and

$$\epsilon_{0123} = -1.$$

crucial importance if only a perturbative methods are available, especially in quantum theory. In such a case one might be able to connect through duality theories where perturbative computations appear to be convergent and theories in which they are strongly divergent (weak and strong coupling regimes); a typical example is electromagnetic duality replacing the electromagnetic tensor by its Hodge dual, i.e. exchanging the electric a magnetic field strengths and taking the coupling constant to its inverse.

1.3.2 Abelian T–duality

The T–duality, or the target space duality, denotes a special duality between σ–models, in general connecting σ–models with different target manifolds. One of the simplest examples of T–duality is the following. Let's assume that the metric G_{ab} in the model (1.1) has an isometry, i.e. exists a vector field $\mathbf{e} = e^a \frac{\partial}{\partial \phi^a}$ on the target manifold T such that

$$Lie_{\mathbf{e}} G_{ab} = G_{ac}\frac{\partial e^c}{\partial \phi^b} + G_{cb}\frac{\partial e^c}{\partial \phi^a} + \frac{\partial G_{ab}}{\partial \phi^c} e^c = 0,$$

i.e. $\mathbf{e}$ is a Killing vector field. Then the action is invariant under the infinitesimal transformation

$$\delta\phi^a = \epsilon e^a.$$

One may choose the system of coordinates so that $\mathbf{e} = \frac{\partial}{\partial \phi^0}$ $(a \in \{0, \dots D-1\})$ and write the parent action for the 1–form $V = V_\mu dx^\mu$, the original fields $\phi^i,\ i \geq 1$ and the 2-form $\Lambda = \frac{1}{2!}\Lambda_{\mu\nu} dx^\mu \wedge dx^\nu$ (Reminder: G_{ab} don't depend on ϕ^0, i.e. ϕ^0 doesn't appear in the action)

$$S_P = \int d^d x \left(G_{00} V^\mu V_\mu + 2G_{0i} V^\mu \partial_\mu \phi^i + G_{ij} \partial^\mu \phi^i \partial_\mu \phi^j + \Lambda^{\mu\nu} \partial_\mu V_\nu \right). \qquad (1.4)$$

By its variation with respect to Λ one finds that $\partial_\mu V_\nu - \partial_\nu V_\mu = 0$, i.e. $dV = 0$ and therefore locally exists ϕ^0 such that $V = d\phi^0$. Putting $V = d\phi^0$ into (1.4) one reobtains the original action. Variation of (1.4) with respect to V gives

$$2G_{00} V^\mu + 2G_{0i} \partial^\mu \phi^i - \partial_\nu \Lambda^{\nu\mu} = 0$$

and I can eliminate

$$V^\mu = \frac{1}{G_{00}} \left(\frac{1}{2} \partial_\nu \Lambda^{\nu\mu} - G_{oi} \partial^\mu \phi^i \right),$$

after substitution for V the parent action becomes (up to total divergences)

$$\tilde{S} = \int d^d x \left[\frac{1}{G_{00}} \left(-\frac{1}{4} \partial_\nu \Lambda^{\nu\mu} \partial_\rho \Lambda^\rho{}_\mu + G_{0i} \partial_\nu \Lambda^{\nu\mu} \partial_\mu \phi^i \right) + (G_{ij} - \frac{G_{0i} G_{0j}}{G_{00}}) \partial^\mu \phi^i \partial_\mu \phi^j \right]. \qquad (1.5)$$

One should note that there is still a relation between Bianchi identities of one model and the equations of motion in the other one. The equation derived by variation with respect to ϕ^0 in the first model (1.1) can be written

$$\partial_\mu(G_{00}V^\mu + G_{0i}\partial^\mu\phi^i) = 0, \text{ where } V^\mu = \partial^\mu\phi^0 \tag{1.6}$$

and the equation derived by variation with respect to Λ in the second model (1.5) is

$$\partial_\mu V_\nu - \partial_\nu V_\mu = 0, \text{ where } V^\mu = \frac{1}{G_{00}}\left(\frac{1}{2}\partial_\nu\Lambda^{\nu\mu} - G_{0i}\partial^\mu\phi^i\right) \tag{1.7}$$

The equation (1.7) follows directly as a kind of Bianchi identity from the definition of V^μ in (1.6) and vice versa the equation (1.6) is a Bianchi identity for V^μ defined in (1.7).

In the case $d = 2$ one may replace the 2–form Λ by a scalar $A = \frac{1}{2} * \Lambda$, $\Lambda_{\mu\nu} = 2\epsilon_{\mu\nu}A$. This substitution enables to write the first term in the action (1.5) as $\frac{1}{G_{00}}\partial^\mu A\partial_\mu A$, but the second term $2(G_{00})^{-1}(G_{0i}\epsilon^{\nu\mu}\partial_\nu A\partial_\mu\phi^i)$ still spoils the form of the σ–model action (1.1) since it contains the antisymmetric tensor $\epsilon^{\mu\nu}$ instead of the Minkowski metric $\eta^{\mu\nu}$ on M. In order to be able to rewrite the resulting model again as some kind of a σ–model, one is forced to introduce except the metric also antisymmetric terms into the action

$$S = \int d^2x\left(G_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b + \epsilon^{\mu\nu}B_{ab}(\phi)\partial_\mu\phi^a\partial^\nu\phi^b\right), \tag{1.8}$$

B_{ab} is assumed to be antisymmetric. Note: In the light–cone coordinates $z = x^0 - x^1$, $\bar{z} = x^0 + x^1$ one can unite G_{ab} and B_{ab} into $F_{ab} = G_{ab} + B_{ab}$ and write equivalently the action

$$S = \int dz d\bar{z} F_{ab}(\phi)\partial\phi^a\bar{\partial}\phi^b, \text{ where } \partial \equiv \frac{\partial}{\partial z},\ \bar{\partial} \equiv \frac{\partial}{\partial\bar{z}}. \tag{1.9}$$

Now it is possible to construct a T–duality transformation between two such models if one assumes that an isometry of both G_{ab} and B_{ab} exists, i.e. in suitable coordinates

$$Lie_{\frac{\partial}{\partial\phi^0}}G_{ab} = \frac{\partial G_{ab}}{\partial\phi^0} = 0,\ Lie_{\frac{\partial}{\partial\phi^0}}B_{ab} = \frac{\partial B_{ab}}{\partial\phi^0} = 0. \tag{1.10}$$

The dual model is given by Buscher's formulae [9], [10]

$$\tilde{G}_{00} = \frac{1}{G_{00}},\ \ \tilde{G}_{0i} = \frac{1}{G_{00}}B_{0i},\ \ \tilde{B}_{0i} = \frac{1}{G_{00}}G_{0i},$$

$$\tilde{G}_{ij} = G_{ij} - \frac{1}{G_{00}}(G_{0i}G_{0j} + B_{i0}B_{0j}),$$

$$\tilde{B}_{ij} = B_{ij} + \frac{1}{G_{00}}(G_{0i}B_{0j} + B_{i0}G_{0j}).$$

If in the original model is $B_{ab} = 0$ then one gets back to the action (1.5) with $\Lambda_{\mu\nu} = 2\epsilon_{\mu\nu}A$. (Note: in quantum version of these formulae encountered in the literature also a scalar (dilaton) field coupled to the Ricci scalar $R^{(2)}$ of the metric h on M appears, the action then reads (up to overall constant factors)

$$S = \int d^2x \left(\sqrt{-h}G_{ab}(\phi)h^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b + \sqrt{-h}R^2\psi(\phi) + \epsilon^{\mu\nu}B_{ab}(\phi)\partial_\mu\phi^a\partial^\nu\phi^b\right).$$

The dilaton field doesn't transform classically under T–duality, its transformation follows in path–integral formulation from requirement of maintaining the conformal invariance. Because I present only the classical description of T–duality, I have not included the dilaton into the action and the Weyl rescaling of the metric on M then allowed to choose the metric on M to be the flat Minkowski metric.)

It is easy to check that applying the Buscher's formulae twice one obtains back the original model. If several independent commuting vector fields $\mathbf{e_j}$ satisfying

$$(Lie_{\mathbf{e_j}}G)_{ab} = (Lie_{\mathbf{e_j}}B)_{ab} = 0$$

exist, i.e. there is an abelian algebra of isometries, then there are more mutually dual models. The dualization may proceed in several steps, using one vector field at each step for the dualization as prescribed by Buscher's formulae.

1.3.3 Non–abelian and Poisson–Lie T–duality

A question naturally arises what can be done for models with non–abelian algebra $\mathcal{G}$ of isometries and finally whether exists a generalization applicable also in the case with no isometries at all. If there are several isometries generated by non–commuting vector fields, one can of course choose just one of them and apply the procedure described in the previous section but cannot proceed further because of non–commutativity (since the vector fields don't commute, the coordinates in which they are all written as $\frac{\partial}{\partial\phi^i}$ don't exist).

Nevertheless also another approach, the so–called non–Abelian duality, exists if the group acts freely on T [11]. Then one may at least locally choose coordinates so that first r coordinates correspond to the group elements and the remaining ones parametrize different orbits of the action, and dualize all r coordinates at once. The approach is based on constructing the parent action by gauging the action e.g. in the light–cone coordinates (1.9), i.e. replacing

$\partial\phi^a$ and $\bar\partial\phi^a$ by $\mathcal{D}\phi^a = \partial\phi^a + A^b(T_b)^a{}_c\phi^c$ and $\bar{\mathcal{D}}\phi^a = \bar\partial\phi^a + \bar A^b(T_b)^a{}_c\phi^c$ where $a, b, c \in \{0, \ldots, r-1\}$, Ts form an r–dimensional representation of the Lie algebra $\mathcal{G}$, and adding a gauge fixing term, e.g. $\mathrm{Tr}(\Lambda^a T_a F)$, $F = dA + \frac{1}{2}[A, A]$ for $\mathcal{G}$ semisimple.

$$S_P = \int dz d\bar z \Big(F_{ab}(\phi)\mathcal{D}\phi^a\bar{\mathcal{D}}\phi^b + F_{ai}(\phi)\mathcal{D}\phi^a\bar\partial\phi^i + F_{ib}(\phi)\partial\phi^i\bar{\mathcal{D}}\phi^b +$$

$$+F_{ij}(\phi)\partial\phi^i\bar\partial\phi^j + \Lambda^a(\partial\bar A^e - \partial\bar A^e + f^e_{fg}A^f\bar A^g)(T_a)^b_c(T_e)^c_b\Big)\,,$$

where $a, \ldots, e < r$ and $r \le i, j \le D-1$, f^a_{bc} are structure constants of $\mathcal{G}$, $[T_b, T_c] = f^a_{bc}T_a$. By variation of the parent action with respect to Λ one finds that A is a pure gauge and by gauge fixing, i.e. by a suitable choice of the coordinate fields ϕ^a, one may put $A = 0$ and get back the original action. By variation of the parent action with respect to A^a and eliminating A^a from it one can find a dual action depending on Λ^a and $\phi^i, i \ge r$ only. The difficulty with this duality lies in the fact that the dual model might have no isometries at all and it is therefore not clear how to perform the inverse transformation.

Such considerations have led to the discovery of Poisson–Lie T–dual models by C. Klimčík and P. Ševera in [12]. For simplicity I will assume in the following that the group G acts on the target manifold not only freely but also transitively, i.e. $T \equiv G$. (Otherwise one may, as long as the action of the group is free, locally choose the coordinates as above and proceed in a similar fashion.) Let's start from the light–cone action

$$S = \int dz d\bar z F_{ij}(\phi)\partial\phi^i\bar\partial\phi^j \qquad (1.11)$$

and take as a basis of vector fields on G the left–invariant fields, i.e. elements of the Lie algebra, $\mathbf{e}_a = e^i_a\frac{\partial}{\partial\phi^i}$ and

$$[\mathbf{e}_a, \mathbf{e}_b] = f^c_{ab}\mathbf{e}_c.$$

A variation of ϕ can now be expressed as

$$\delta\phi^i = k^a(z, \bar z)e^i_a.$$

Putting this variation into the action (1.11) one finds[3]

$$\delta S = \int dz d\bar z \Big(k^a(Lie_{\mathbf{e}_a}F)_{ij}\partial\phi^i\bar\partial\phi^j + (\partial k^a)e^i_aF_{ij}\bar\partial\phi^j + (\bar\partial k^a)e^j_aF_{ij}\partial\phi^i\Big)\,.$$

[3]Up to total divergencies and using

$$(Lie_{\mathbf{e}_a}F)_{ij} = F_{ik}\frac{\partial e^k_a}{\partial\phi^j} + F_{kj}\frac{\partial e^k_a}{\partial\phi^i} + \frac{\partial F_{ij}}{\partial\phi^k}e^k_a.$$

Defining the current 1–forms on M

$$\mathcal{J}_a = J_a dz + \bar{J}_a d\bar{z},\ J_a = e_a^j F_{ij} \partial\phi^i,\ \bar{J}_a = e_a^i F_{ij} \bar{\partial}\phi^j, \tag{1.12}$$

one can write the equations of motion in the form

$$\partial\bar{J}_a + \bar{\partial} J_a = (Lie_{\mathbf{e}_a} F)_{ij} \partial\phi^i \bar{\partial}\phi^j. \tag{1.13}$$

It is clear that if the group G is a group of isometries of F, $Lie_{\mathbf{e}_a} F_{ij} = 0$, then the currents (1.12) are the corresponding Noether conserved currents $d{*}\mathcal{J}_a = 0$. A generalization of the isometry condition is obtained by demanding the ${*}\mathcal{J}_a$ to be not closed but to satisfy a Maurer–Cartan equation on some other group $\tilde{G}$ written in components

$$d * \mathcal{J}_a + \frac{1}{2} \tilde{f}_a^{bc} * \mathcal{J}_b \wedge *\mathcal{J}_c = (\partial\bar{J}_a + \bar{\partial} J_a - \tilde{f}_a^{bc} J_b \bar{J}_c) dz \wedge d\bar{z} = 0. \tag{1.14}$$

This assumption allows to express the currents as a "pure gauge" on the dual group $\tilde{G}$ with $\tilde{T}^a$ the generators of its algebra $[\tilde{T}^a, \tilde{T}^b] = \tilde{f}_c^{ab} \tilde{T}^c$

$$*\mathcal{J} = *\mathcal{J}_a \tilde{T}^a \Rightarrow *\mathcal{J} = \tilde{g}^{-1} d\tilde{g},\ \tilde{g} \in \tilde{G}.$$

The group elements $\tilde{g}$ written in some coordinates are then intepreted as the fields dual to original ϕs, the equation of motion of the original model (1.13) becomes now the Bianchi identity for $*\mathcal{J}$. Also one should note that as in the previous examples of duality, the original and dual fields are connected through a differential equation, e.g. if one locally parametrizes $\tilde{G}$ by $g = g_0 \exp(\tilde{\phi}^a \tilde{T}_a)$, the relation between ϕ^i and $\tilde{\phi}^a$ is

$$\partial\tilde{\phi}^a = \partial\phi^i F_{ij}(\phi) e_a^j,\ \bar{\partial}\tilde{\phi}^a = \bar{\partial}\phi^j F_{ij}(\phi) e_a^i.$$

It remains to investigate for which metrics F_{ij} the equivalence of (1.14) and (1.13) can be established. Obviously the condition is

$$(Lie_{\mathbf{e}_a} F)_{ij} \partial\phi^i \bar{\partial}\phi^j = \tilde{f}_a^{bc} J_b \bar{J}_c$$

i.e.

$$(Lie_{\mathbf{e}_a} F)_{ij} = \tilde{f}_a^{bc} F_{ik} e_b^k F_{lj} e_c^l. \tag{1.15}$$

Since Lie derivatives form a representation of the Lie algebra $\mathcal{G}$

$$[Lie_{\mathbf{e}_a}, Lie_{\mathbf{e}_b}] = f_{ab}^c Lie_{\mathbf{e}_c}$$

one can easily express a necessary condition as

$$([Lie_{\mathbf{e}_m}, Lie_{\mathbf{e}_i}] F)_{ab} = f_{mi}^c (Lie_{\mathbf{e}_c} F)_{ab},$$

expanding the commutator and using (1.15) one finds the following condition (times a common term $F_{ac}e^c_j e^d_k F_{db}$)

$$\tilde{f}^{jk}_l f^l_{mi} + \tilde{f}^{kl}_m f^j_{li} + \tilde{f}^{jl}_i f^k_{lm} + \tilde{f}^{jl}_m f^k_{il} + \tilde{f}^{kl}_i f^j_{lm} = 0. \tag{1.16}$$

The condition (1.16) is surprisingly just the condition on algebras $\mathcal{G}, \tilde{\mathcal{G}}$ for existence of a Manin triple on them, i.e. of a Lie algebra $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$ (as vector spaces) such that $\mathcal{G}, \tilde{\mathcal{G}}$ are its subalgebras maximally isotropic with respect to the form $\langle .,. \rangle$ defined by $\langle T_a, \tilde{T}^b \rangle = \delta^b_a$. Then (1.16) is just the Jacobi identity for one element from $\mathcal{G}$ and two elements from $\tilde{\mathcal{G}}$ or vice versa.

The metric of the dual model in the coordinates $\tilde{\phi}^a$ should satisfy a dual condition

$$(Lie_{\tilde{\mathbf{e}}^a} \tilde{F})^{ij} \partial\tilde{\phi}_i \bar{\partial}\tilde{\phi}_j = f^a_{bc} \tilde{J}^b \bar{\tilde{J}}^c,$$

which leads to the same necessary condition (1.16).

The appearance of Manin triples in the description of Poisson–Lie T–dual models indicates a connection between the T–duality and Drinfeld doubles; Drinfeld double D being a connected Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle .,. \rangle$ can be decomposed into a pair of subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ maximally isotropic with respect to $\langle .,. \rangle$ and $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$.

One may like to have not only some necessary conditions for existence of dual models but also a method of construction of such models, i.e. of their metrics, from some simpler structures, avoiding the need for solving the partial differential equations (1.15). Such a way of construction of Poisson–Lie T–dual models on Drinfeld doubles was presented by C. Klimčík and P. Ševera in [12] and [13]. The construction starts by postulating the equations of motion on the whole Drinfeld double, not depending on the choice of Manin triple:

$$\langle (\partial_\pm l) l^{-1}, \mathcal{E}^\pm \rangle = 0, \tag{1.17}$$

where subspaces

$$\mathcal{E}^+ = \mathrm{span}(X_i + E_{ij}(e)\tilde{X}^j),\ \mathcal{E}^- = \mathrm{span}(X_i - E_{ji}(e)\tilde{X}^j)$$

are orthogonal with respect to $\langle .,. \rangle$ and span the whole Lie algebra $\mathcal{D}$ and $\{X_i\}$, resp. $\{\tilde{X}^i\}$ form the bases of $\mathcal{G}$, resp. $\tilde{\mathcal{G}}$ such that

$$\langle X_i, X_j \rangle = 0,\ \langle \tilde{X}^i, \tilde{X}^j \rangle = 0,\ \langle X_i, \tilde{X}^j \rangle = \delta^j_i.$$

One writes $l = g.\tilde{h}$, $g \in G$, $\tilde{h} \in \tilde{G}$ (such decomposition of elements of the group D exists at least at the vicinity of the unit element according to [14]) and using the invariance of $\langle .,. \rangle$ arrives from (1.17) at equations

$$\langle g^{-1}(\partial_\pm g) + (\partial_\pm \tilde{h})\tilde{h}^{-1}, g^{-1}\mathcal{E}^\pm g \rangle = 0.$$

After defining $E_{ij}(g)$

$$g^{-1}\mathcal{E}^+g = \mathrm{span}(X_i + E_{ij}(g)\tilde{X}^j),\ ,g^{-1}\mathcal{E}^-g = \mathrm{span}(X_i - E_{ji}(g)\tilde{X}^j),$$

one finds (the upper and lower indices i,j denote the coordinates in the bases $\{X_i\}$ and $\{\tilde{X}^i\}$)

$$\Big((\partial_+\tilde{h})\tilde{h}^{-1}\Big)_i = -E_{ij}(g)(g^{-1}\partial_+g)^j \equiv A_{+i}(g),$$
$$\Big((\partial_-\tilde{h})\tilde{h}^{-1}\Big)_i = E_{ji}(g)(g^{-1}\partial_-g)^j \equiv A_{-i}(g).$$

Finally by further differentiation $\partial_-((\partial_+\tilde{h})\tilde{h}^{-1})_i$ and $\partial_+((\partial_-\tilde{h})\tilde{h}^{-1})_i$ and due to the interchangeability of the ordering of partial derivatives one eliminates $\tilde{h}$ from equations of motion

$$\partial_+A_{-i}(g) - \partial_-A_{+i}(g) + \tilde{f}_i^{kl}A_{+k}(g)A_{-l}(g) = 0. \tag{1.18}$$

The dual equation is found by writing $l = \tilde{g}.h$, $h \in G$, $\tilde{g} \in \tilde{G}$ and eliminating h from (1.17).

The resulting models have target spaces in the Lie groups G and $\tilde{G}$ and are equivalent to the models described by the Lagrangians

$$\mathcal{L} = E_{ij}(g)(g^{-1}\partial_-g)^i(g^{-1}\partial_+g)^j, \tag{1.19}$$
$$\tilde{\mathcal{L}} = \tilde{E}^{ij}(\tilde{g})(\tilde{g}^{-1}\partial_-\tilde{g})_i(\tilde{g}^{-1}\partial_+\tilde{g})_j, \tag{1.20}$$

where

$$E(g) = (a(g) + E(e)b(g))^{-1}E(e)d(g), \tag{1.21}$$

$E(e)$ is a constant matrix and $a(g), b(g), d(g)$ are submatrices of the adjoint representation of the group G on $\mathcal{D}$ in the basis $(X_i, \tilde{X}^j)$

$$Ad(g)^T = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}. \tag{1.22}$$

The matrix $\tilde{E}(\tilde{g})$ is constructed analogously with

$$Ad(\tilde{g})^T = \begin{pmatrix} \tilde{d}(\tilde{g}) & \tilde{b}(\tilde{g}) \\ 0 & \tilde{a}(\tilde{g}) \end{pmatrix},\ \tilde{E}(\tilde{e}) = E(e)^{-1}. \tag{1.23}$$

This construction not only shows that examples of Poisson–Lie T–duality without isometries of the target, i.e. with both G and $\tilde{G}$ nonabelian, exist but also explains that one may consider dual all models arising from the same Drinfeld double (with a given constant matrix $E^{ij}(e)$), not only the pairs of models on one Manin triple; all such models share the same original equations on the double (1.17).

1.3.4 My results on T–duality and Drinfeld doubles

I and L. Hlavatý have concentrated on investigation and classification of Drinfeld doubles in the lowest nontrivial dimensions 4 and 6. The classification of Drinfeld doubles in dimension 4 and the corresponding pairs of T–dual models are presented in Chapter 4.[4] The main conclusion of this paper is that even in this case a Drinfeld double with several non–isomorphic Manin triples exists and provides a motivation for investigation of higher–dimensional cases. In the dimension 6 we found all non–isomorphic Manin triples and wrote them in 78 classes (Chapter 5) and then investigated which of these Manin triples define isomorphic Drinfeld doubles (Chapter 6). It turned out that in the chosen parametrization there are 22 classes of non–isomorphic 6–dimensional real Drinfeld doubles. One of interesting conclusions of this paper is that not only rather different Manin triples might lead to the same Drinfeld double, but also the same underlying Lie algebra $\mathcal{D}$ may be equipped with different bilinear forms and define different Drinfeld doubles.

1.4 Conclusions and future prospects

Concerning the integrability of principal σ–models we have found no interesting, truly nonlinear integrable model on any 2– and 3– dimensional non–semisimple Lie group, although given the ansatz for Lax operators the investigation seems to be complete (up to completness of results obtained using computer algebra systems, as mentioned in Chapter 2). I don't know whether it is due to our linear ansatz for Lax operators or whether it is a general property of principal models on non–semisimple groups. One may imagine a generalization of the ansatz but it appears that the arising conditions for the Lax operators would be too complicated to solve.

The classification of principal models on the group $SU(2)$ with diagonal

[4] This work was originally inspired by a question raised by S. Majid at the Bialowieža workshop: whether the model with Lax pair found in Chapter 2 has something to do with the Poisson–Lie T–dual models. The answer is unfortunately negative, the Lagrangian of our model is much simpler than the Lagrangians of the Poisson–Lie T–dual models and also it depends on the coordinate θ_1 only in the parametrization of the group

$$g = \begin{pmatrix} \exp(\theta_1) & \theta_2 \\ 0 & 1 \end{pmatrix}$$

whereas Poisson–Lie T–dual Lagrangians depend only on θ_2. Beside that as V. Kavka showed in his Diploma Thesis, the T–dual models constructed in Chapter 4 don't pass the Painlevé test and are therefore not integrable. Nevertheless I would like to express my gratitude to S. Majid because his remark has led me to the study of T–duality.

metric and Lax pair linear in currents can be now considered complete: only models with constant metric allow Lax pair and all models with constant metric have a Lax pair. Unfortunately a classification of integrable principal chiral models with nonconstant non-diagonal metric on $SU(2)$ seems to be technically unfeasible in the present time.

Concerning the Drinfeld doubles and Poisson–Lie T–duality we have constructed a complete list of six-dimensional real Drinfeld doubles up to their isomorphisms i.e. maps preserving both the Lie structure and an ad-invariant symmetric bilinear form $\langle .,. \rangle$ that define the double. In our parametrization there are just 22 classes of the non-isomorphic Drinfeld doubles. One can see that for many Drinfeld doubles there are several decompositions into Manin triples. We can in principle construct the explicit Lagrangians of the pairs of Poisson–Lie T–dual models for every Manin triple but given the large number of Manin triples this does not make much sense unless a concrete physical motivation picks up some of them. The investigation of properties of quantum analogs of different models on the same Drinfeld double and whether the connection between them survives the quantization, might be of interest in the superstring theory. As far as I know no explicit examples of such models were studied before, only recently a paper by R. von Unge on this subject appeared [17].

An important point that follows from the classification of Drinfeld doubles is that there are several different Drinfeld doubles corresponding to Lie algebras $so(1,3)$, $sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R})$, $sl(2,\mathbf{R}) \triangleright \mathbf{R}^3$ whereas on solvable Lie algebras the Drinfeld double is unique (in some cases up to the sign of the bilinear form). It might be interesting to know whether such behaviour holds in any dimension or is just a low–dimensional artifact.

On the other hand there are Manin triples with one isotropic subalgebra abelian that are equivalent as Drinfeld doubles even though the other subalgebras are different (see $(6_0|1)$ and $(5|1)$ or $(6_a|1)$ and $(6_{\frac{1}{a}}|1)$). That's why it was necessary to investigate the (non)equivalence of the Manin triples of this form. Moreover the above given examples indicate the diversity of Drinfeld double structures one may encounter in higher dimensions. Beside that from the present classification procedure one can find whether a given six-dimensional Lie algebra can be equipped by a suitable ad-invariant bilinear form and turned into a Drinfeld double (and how many such forms exist). The investigation then can be reduced to a direct check of equivalence with a particular six-dimensional Lie algebra. For example, one can see that there is no Drinfeld double on $SO(4)$.

Let me note that the complete sets of equivalent σ–models for a fixed Drinfeld double are given by the so–called modular space of the double. The construction of all non–isomorphic Manin triples for the double is the first step in the construction of the modular spaces.

Chapter 2

Principal chiral models on non–semisimple groups

Journal of Physics A 34 (2001) 7795–7809.

In this paper we investigated principal chiral models on solvable 2– and 3–dimensional Lie groups. We were trying to find models whose equations of motion can be formally rewritten as a Lax pair. Lax operators were assumed for simplicity to be in linear form in currents (with possible constant term).

It turned out that equations of such models are in most cases equivalent to a sequence of linear partial differential equations, e.g. a wave equation for one field and two linear partial differential equations for the remaining fields depending nonlinearly on the solution of the wave equation. Only one of the models appeared to be truly nonlinear, but further considerations (see Chapter 3) showed that it can be also brought to a similar form of sequence of linear equations.

Possible generalizations of the ansatz for Lax pair, e.g. non–constant coefficients of the currents in Lax operators, quadratic ansatz for Lax operators etc., were not considered because of rapidly growing complexity of computations.

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 7795–7809 PII: S0305-4470(01)22233-1

Principal chiral models on non-semisimple groups

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Received 21 February 2001
Published 14 September 2001
Online at stacks.iop.org/JPhysA/34/7795

Abstract

Generalized principal models on non-semisimple groups are defined. An ansatz for the Lax form of the equations of motion is chosen and models on two- and three-dimensional non-semisimple groups that admit this Lax formulation are classified. Only one of these models has truly nonlinear equations of motion, and the Lax pair is explicitly given. The equations of motion of all the other models can be brought to linear partial differential equations.

PACS numbers: 02.20.1a, 02.30.lk, 02.30.Jr, 02.30.Zz, 02.40.−k, 11.30.Rd

1. Introduction

Integrable models in two dimensions are important theoretical laboratories for investigating possible phenomena of nonlinear theories in higher dimensions. Principal chiral models are examples of nonlinear relativistic field theories on the group manifolds. It is well known that they are classically integrable in 1 + 1 dimensions (see [1]).

Until now, the principal models were investigated mainly on semisimple groups because the bilinear forms used for the construction of the field actions were actually taken as the ad-invariant Killing metrics on the corresponding Lie algebras. As the forms should be non-degenerate, these models were defined on semisimple groups only. A few years ago Sochen [2] suggested a generalization of the principal models for metrics that are not ad-invariant (see also [3] and [4] for a different point of view). It opened up the possibility of defining the principal models on non-semisimple groups as well. An example of such model, including the Lax formulation of equations of motion was formulated in [5] but the parameters in the Lax pair could be transformed off, so that there remained no free spectral parameters, necessary for the inverse spectral method. We use a more general ansatz for the Lax operators in this paper that enables us to introduce such a free parameter.

The main topic of this paper is classification of models on the two- and three-dimensional non-semisimple groups that admit Lax formulation of a form given below.

2. Generalized principal models

Generalized principal chiral models [2] are given by the action

$$I[g] = \int d^2x\, \eta^{\mu\nu} L_{ab}(g) J_\mu^a J_\nu^b \tag{1}$$

where G is a Lie group, $\mathcal{L}(G)$ its Lie algebra,

$$J_\mu = \left(g^{-1}\partial_\mu g\right) \in \mathcal{L}(G) \tag{2}$$

g: $\mathbf{R^2} \to G$, $\mu, \nu \in \{0, 1\}$, $\eta = \mathrm{diag}(1, -1)$, L is a G-dependent symmetric non-degenerate bilinear form. We consider the bilinear form L as a metric on the group manifold and the generalization of principal models from ad-invariant Killing form on $\mathcal{L}(G)$ to more general case enables us to introduce the principal models on non-semisimple groups also.

Lie products of elements on the basis of $\mathcal{L}(G)$ define the structure coefficients

$$[t_a, t_b] = c_{ab}{}^c t_c \tag{3}$$

and on the same basis we define the coordinates of the field J_ν

$$J_\nu = g^{-1}\partial_\nu g = J_\nu^b t_b. \tag{4}$$

Fields automatically satisfy the Bianchi identities

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0. \tag{5}$$

Varying the action (1) we obtain the equations of motion for the generalized principal chiral models

$$\partial_\mu J^{\mu,a} + \Gamma^a_{bc} J^b_\mu J^{\mu,c} = 0 \tag{6}$$

where the connection Γ is a sum of two parts

$$\Gamma^a_{bc} = S^a_{bc} + \gamma^a_{bc}. \tag{7}$$

S^a_{bc} is a so-called flat connection

$$S^a_{bc} = \frac{1}{2}\left(C^a_{bc} + C^a_{cb}\right) \qquad C^a_{bc} = (L^{-1})^{ap} c^q_{pb} L_{qc} \tag{8}$$

and γ^a_{bc} are Christoffel symbols for the metric $L_{ab}(g)$

$$\gamma^a_{bc} = \frac{1}{2}(L^{-1})^{ad}(U_b L_{cd} + U_c L_{bd} - U_d L_{bc}). \tag{9}$$

The vector fields U_a are defined in the local group coordinates θ_i as

$$U_a = U^i_a(\theta)\frac{\partial}{\partial\theta_i} \tag{10}$$

where the matrix U is the inverse of the matrix V of vielbein coordinates

$$U^i_a(\theta) = (V^{-1})^i_a(\theta) \qquad V^a_i = \left(g^{-1}\frac{\partial g}{\partial\theta_i}\right)^a. \tag{11}$$

Note that the connection (7) is symmetric in the lower indices

$$\Gamma^a_{bc} = \Gamma^a_{cb}. \tag{12}$$

2.1. Lax pairs

It is evident from (6) and (2) that the equations of motion of generalized principal chiral models may form highly nonlinear systems of PDEs. One of the most powerful method for solving nonlinear PDEs is the so-called inverse scattering method that transforms the PDEs to solvable system of ODEs. The inverse transform requires solving the Riemann–Hilbert problem of determining a complex function from their values at a curve (for a detailed explanation, see, e.g., [6]).

The first step of the method consists in writing the system of PDEs in terms of a commutator of two differential operators X_0, X_1 containing a free parameter that is later used as the independent variable in the associated Riemann–Hilbert problem. These operators are called Lax pair and serve to define an associated linear spectral problem defining the (direct) transform. Finding such a Lax pair for a given system of PDEs is a rather nontrivial problem.

The ansatz that we are going to use for the Lax operators X_0, X_1 of the generalized principal chiral models is

$$X_0 = \partial_0 + P_{ab} J_0^b t_a + Q_{ab} J_1^b t_a + A_a t_a \tag{13}$$

$$X_1 = \partial_1 + \tilde{Q}_{ab} J_0^b t_a + \tilde{P}_{ab} J_1^b t_a + B_a t_a \tag{14}$$

where $P, Q, \tilde{P}, \tilde{Q}$ are four arbitrary constant $\dim G \times \dim G$ matrices and A, B are two arbitrary constant vectors.

By explicit evaluation of the zero curvature condition

$$[X_0, X_1] = 0 \tag{15}$$

using the equations of motion (6) and Bianchi identities (5), and equating the coefficients of different powers and derivatives of J_μ^a, one finds the following necessary conditions that the operators X_0, X_1 must satisfy in order to form a Lax pair:

$$\tilde{P} = P, \tilde{Q} = Q \tag{16}$$

$$(P_{bp} P_{cq} - Q_{bp} Q_{cq}) c_{bc}^a = P_{ab} c_{pq}^b \tag{17}$$

$$\tfrac{1}{2} c_{cd}^a (P_{cp} Q_{dq} + P_{cq} Q_{dp}) = Q_{ab} \Gamma_{pq}^b \tag{18}$$

$$c_{cd}^a (P_{cp} B_d + A_c Q_{dp}) = 0 \tag{19}$$

$$c_{cd}^a (Q_{cp} B_d + A_c P_{dp}) = 0 \tag{20}$$

$$c_{cd}^a A_c B_d = 0. \tag{21}$$

Equation (16) is the reason for originally counterintuitive notation in equations (13) and (14). In the following we always immediately replace $\tilde{P}$ by P and $\tilde{Q}$ by Q.

In order to guarantee the equivalence between equation (15) and the equations of motion (6) one needs further restrictions on P, Q, A, B (otherwise consider e.g. $P = Q = A = B = 0$). Such a condition can be found quite easily by rewriting the left-hand side of the equation (15) and using equations (16)–(21) and the Bianchi identities (5), one gets

$$[X_0, X_1] = Q_{ab} \left(\partial_\mu J^{\mu,b} + \Gamma_{pq}^b J_\mu^p J^{\mu,q} \right). \tag{22}$$

It is now clear that equation (15) is equivalent to the equations of motion (6) if and only if the matrix Q is invertible.

To sum up, the Lax formulation (13)–(15) *is equivalent to the equations of motion if and only if the equations* (16)–(21) *hold and* Q *is invertible*.

Moreover, the previous equations impose a condition on Γ. Hence, we can express (18) in an equivalent form

$$\frac{1}{2} (Q^{-1})^{ba} c_{cd}^a (P_{cp} Q_{dq} + P_{cq} Q_{dp}) = \Gamma_{pq}^b \tag{23}$$

and conclude that *only the generalized principle models with the constant connection* Γ *admit the Lax formulation* (13)–(15) because the left-hand side of the previous equation is constant.

3. Abelian groups

The case of the principal models on Abelian groups having the Lax formulation (15) can be investigated rather quickly by the following method. Because in this case $c_{ab}^{c} = 0$, equation (17) is satisfied identically and equation (18) simplifies to $0 = Q_{ab}\Gamma^b_{pq}$. This equation represents for any given pair p, q a set of $\dim G$ linear equations for $\dim G$ variables Γ^b_{pq} with an invertible matrix of coefficients ($=Q$); therefore only the trivial solution $\Gamma^b_{pq} = 0$ is possible, leading to the model

$$\partial_\mu J^{\mu,a} = 0. \tag{24}$$

Because we may choose coordinates $\theta\colon g(\theta) = \exp\left(\sum_{i=1}^n \theta_i t_i\right)$, and the corresponding expression for the fields is $J^a_\mu = \partial_\mu \theta_a$, we have in such coordinates a free model

$$\partial_\mu \partial^\mu \theta_i = 0. \tag{25}$$

In the following we shall systematically explore the generalized principal models on non-semisimple two- and three-dimensional Lie groups.

4. Two-dimensional solvable group

Every non-Abelian two-dimensional connected Lie group is isomorphic to the group of affine transformations of real line. Let us denote it by $Af(1)$. This group can be conveniently realized as a matrix group consisting of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad a > 0. \tag{26}$$

A suitable parametrization of this group is

$$g(\theta_1, \theta_2) = \begin{pmatrix} \exp(\theta_1) & \theta_2 \\ 0 & 1 \end{pmatrix} \tag{27}$$

where $\theta_1, \theta_2 \in \mathbf{R}$. The basis of the corresponding Lie algebra can be chosen from

$$t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{28}$$

The nonzero structure coefficients for this choice of basis are

$$c^2_{12} = 1 \qquad c^2_{21} = -1. \tag{29}$$

The coordinates of vector fields J_μ in this basis are $(\partial_\mu\theta_1, \mathrm{e}^{-\theta_1}\partial_\mu\theta_2)$. The differential operators U_a in this case are

$$U_1 = \frac{\partial}{\partial\theta_1} \qquad U_2 = \mathrm{e}^{\theta_1}\frac{\partial}{\partial\theta_2}. \tag{30}$$

The equations of motion are

$$0 = \frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} + \Gamma^1_{11}\left(\left(\frac{\partial\theta_1}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right) + 2\Gamma^1_{12}\mathrm{e}^{-\theta_1}\left(\frac{\partial\theta_1}{\partial x_0}\frac{\partial\theta_2}{\partial x_0} - \frac{\partial\theta_1}{\partial x_1}\frac{\partial\theta_2}{\partial x_1}\right) + \Gamma^1_{22}\mathrm{e}^{-2\theta_1}\left(\left(\frac{\partial\theta_2}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_2}{\partial x_1}\right)^2\right) \tag{31}$$

$$0 = e^{-\theta_1}\left(\frac{\partial^2\theta_2}{\partial x_0^2} - \frac{\partial^2\theta_2}{\partial x_1^2} - \frac{\partial\theta_1}{\partial x_0}\frac{\partial\theta_2}{\partial x_0} + \frac{\partial\theta_1}{\partial x_1}\frac{\partial\theta_2}{\partial x_1}\right) + \Gamma^2_{11}\left(\left(\frac{\partial\theta_1}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right) + 2\Gamma^2_{12}e^{-\theta_1}$$
$$\times\left(\frac{\partial\theta_1}{\partial x_0}\frac{\partial\theta_2}{\partial x_0} - \frac{\partial\theta_1}{\partial x_1}\frac{\partial\theta_2}{\partial x_1}\right) + \Gamma^2_{22}e^{-2\theta_1}\left(\left(\frac{\partial\theta_2}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_2}{\partial x_1}\right)^2\right). \tag{32}$$

These equations were already investigated in [5] but only for diagonal metric L and less general ansatz of Lax operators.

To find models admitting Lax formulation one can proceed in the following way. Equation (17) is equivalent to two equations

$$P_{12} = 0 \qquad P_{22}(P_{11} - 1) = Q_{11}Q_{22} - Q_{21}Q_{12}\ (= \det\ Q). \tag{33}$$

Because the matrix Q is invertible, i.e. $\det\ Q \neq 0$, the second condition can be rewritten

$$P_{22} = \frac{\det\ Q}{P_{11} - 1}. \tag{34}$$

Equation (18) should be considered first for $a = 1$. Then the left-hand side of (18) vanishes and one finds

$$0 = Q_{11}\Gamma^1_{pq} + Q_{12}\Gamma^2_{pq} \qquad \forall p, q. \tag{35}$$

We divide our investigation into two cases depending on the value of Q_{12}.

4.1. Case $Q_{12} \neq 0$

In this case we immediately find that $\Gamma^2_{pq} = K\Gamma^1_{pq}, \forall p, q$ (where $K = -Q_{11}/Q_{12}$) and the defining equations (7) for Γ can be rewritten in an equivalent form

$$\frac{\partial L_{11}}{\partial\theta_1} = 2\Gamma^1_{11}(L_{11} + KL_{12}) \tag{36}$$
$$\frac{\partial L_{11}}{\partial\theta_2} = e^{-\theta 1}\left(2\Gamma^1_{12}L_{11} + 2K\Gamma^1_{12}L_{12} - L_{12}\right) \tag{37}$$
$$\frac{\partial L_{12}}{\partial\theta_1} = \Gamma^1_{12}L_{11} + K\Gamma^1_{11}L_{22} + \Gamma^1_{11}L_{11} + K\Gamma^1_{11}L_{12} + \frac{1}{2}L_{12} \tag{38}$$
$$\frac{\partial L_{12}}{\partial\theta_2} = \frac{e^{-\theta 1}}{2}\left(2\Gamma^1_{22}L_{11} + 2\Gamma^1_{12}(KL_{22} + L_{12}) + 2K\Gamma^1_{22}L_{12} - L_{22}\right) \tag{39}$$
$$\frac{\partial L_{22}}{\partial\theta_1} = 2\Gamma^1_{12}L_{12} + L_{22} + 2K\Gamma^1_{12}L_{22} \tag{40}$$
$$\frac{\partial L_{22}}{\partial\theta_2} = 2e^{-\theta_1}\Gamma^1_{22}(L_{12} + KL_{22}) \tag{41}$$

where K and Γ's are constants. From these equations it is rather easy to calculate following necessary conditions for the existence of the metric L. Using the equations (36)–(41) one evaluates the difference of second derivatives

$$\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}L_{ij} - \frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1}L_{ij} \tag{42}$$

in terms of L_{kl}. Since this difference must be zero, one obtains a set of three (for $(i, j) = (1, 1), (1, 2), (2, 2)$) linear equations for L_{kl}. In order to have a nontrivial (nonzero) solution, the matrix of this set of equations must have a zero determinant, i.e.,

$$-\frac{1}{2}e^{-3\theta_1}\left(K\Gamma^1_{22} + \Gamma^1_{12}\right)\Big[4K^2\Gamma^1_{11}\left(\Gamma^1_{22}\right)^2 - 4K^2\left(\Gamma^1_{12}\right)^2\Gamma^1_{22} + 8K\Gamma^1_{11}\Gamma^1_{12}\Gamma^1_{22} - 8K\left(\Gamma^1_{12}\right)^2$$
$$- 3\Gamma^1_{22} + 4\left(\Gamma^1_{11}\right)^2\Gamma^1_{22} - \Gamma^1_{11}\Gamma^1_{22} - 4\Gamma^1_{11}\left(\Gamma^1_{12}\right)^2 + 4\left(\Gamma^1_{12}\right)^2\Big] = 0.$$

As this must hold for all θ_1, we get either

$$\Gamma^1_{12} = -K\Gamma^1_{22} \tag{43}$$

or

$$4K^2\Gamma^1_{11}\left(\Gamma^1_{22}\right)^2 - 4K^2\left(\Gamma^1_{12}\right)^2\Gamma^1_{22} + 8K\Gamma^1_{11}\Gamma^1_{12}\Gamma^1_{22} - 8K\left(\Gamma^1_{12}\right)^2 - 3\Gamma^1_{22} + 4\left(\Gamma^1_{11}\right)^2\Gamma^1_{22} - \Gamma^1_{11}\Gamma^1_{22} - 4\Gamma^1_{11}\left(\Gamma^1_{12}\right)^2 + 4\left(\Gamma^1_{12}\right)^2 = 0. \tag{44}$$

It can be found by careful investigation that if (44) holds, then the only possible metrics L are singular (for all values of θ_1, θ_2).

The remaining possibility is that (43) holds. In this case, from (42) one obtains further conditions on L's and Γ's, namely that either

$$\Gamma^1_{11} = -\frac{1}{2} \qquad \Gamma^1_{12} = \Gamma^1_{22} = 0 \tag{45}$$

or

$$L_{12} = -KL_{22}, L_{11} = \frac{\left(-2K^2\Gamma^1_{11}\Gamma^1_{22} + 2\Gamma^1_{11} + 4K^4\Gamma^1_{22} + 1 + 4K^2\Gamma^1_{22}\right)L_{22}}{2\Gamma^1_{22}\left(2K^2\Gamma^1_{22} - 2\Gamma^1_{11} + 3\right)}. \tag{46}$$

Otherwise the metric L is singular.

In case (45) one can compute the metric L and the connection Γ from the equations (36)–(41) and the matrices P, Q from equations (18)–(21), but the resulting matrix Q is not invertible, i.e. the Lax formulation (15) is not equivalent to the equations of motion (6).

In case (46) we can again solve equations (36)–(41). The resulting connection and the metric are

$$\Gamma^1_{11} = \frac{1}{2} + K^2\Gamma^1_{22} \qquad \Gamma^1_{22} = -K\Gamma^1_{22} \tag{47}$$

$$L_{11} = \frac{\Gamma^1_{11}}{\Gamma^1_{22}}\alpha e^{\theta_1} \qquad L_{12} = -K\,\alpha e^{\theta_1} \qquad L_{22} = \alpha e^{\theta_1} \tag{48}$$

where $\Gamma^1_{22}, \alpha \in \mathbf{R}\backslash\{0\},\ K \in \mathbf{R}$ are parameters of the model. In the following we will denote $\Gamma^1_{22} = -\kappa^2/2$.

The resulting equations of motion can be found substituting the above given Γ into equations (31) and (32). To get the Lax operators for this model we still have to solve equations (18)–(21). The solutions in this case depend on three arbitrary parameters λ, ρ, σ where $\sigma \neq 0$

$$P = \begin{pmatrix} \frac{1}{2} & 0 \\ \epsilon_1(\kappa K\sigma + \rho) & -\epsilon_1\sigma\kappa \end{pmatrix} \tag{49}$$

$$Q = \begin{pmatrix} \frac{\epsilon_1 K}{2}\kappa & -\frac{\epsilon_1}{2}\kappa \\ \sigma + K\kappa\rho & -\kappa\rho \end{pmatrix} \qquad \epsilon_1 = \pm 1 \tag{50}$$

$$A = (\lambda, 2\lambda(\epsilon_1\rho - \epsilon_2\sigma)) \tag{51}$$

$$B = (\epsilon_2\lambda, 2\lambda(\epsilon_1\epsilon_2\rho - \sigma)) \qquad \epsilon_2 = \pm 1. \tag{52}$$

The Lax operator X_0 then reads (for simplicity we set $\epsilon_1 = \epsilon_2 = +1$)

$$X_0 = \begin{pmatrix} \partial_0 + \frac{1}{2}Y_0 + \lambda, & \sigma Y_1 + \rho Y_0 + 2\lambda(\rho - \sigma) \\ 0 & \partial_0 \end{pmatrix} \tag{53}$$

where Y_μ are linear functions of the fields J^a_μ,

$$Y_0 = J^1_0 - \kappa J^2_1 + K\kappa J^1_1 \tag{54}$$

$$Y_1 = J^1_1 - \kappa J^2_0 + K\kappa J^1_0. \tag{55}$$

The expression for X_1 can be obtained from (53) by an interchange of indices 0, 1 in (53).

We can transform the Lax operators to the form with one parameter only by the similarity transform $\tilde{X}_\mu = T X_\mu T^{-1}$ with

$$T = \begin{pmatrix} 1 & 2\rho \\ 0 & \sigma \end{pmatrix}. \tag{56}$$

The transformed Lax operator then is of the form

$$\tilde{X}_0 = \begin{pmatrix} \partial_0 + \frac{1}{2}Y_0 + \lambda, & Y_1 - 2\lambda \\ 0 & \partial_0 \end{pmatrix}. \tag{57}$$

An analogous expression is obtained for $\tilde{X}_1$.

From the formula (57) it is clear that if the ansatz (13) and (14) is chosen without the constant terms $A_a t_a$, $B_a t_a$ (cf. [5]) then $\lambda = 0$ and there is no free parameter for the inverse scattering method.

4.2. *Case* $Q_{12} = 0$

If $Q_{12} = 0$, then the invertibility of Q leads to $Q_{11} \neq 0$ and equation (35) simplifies to

$$0 = \Gamma^1_{pq}. \tag{58}$$

It is clear that equation (31) is then just the wave equation

$$\frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} = 0. \tag{59}$$

Using the same approach as before for $Q_{12} \neq 0$, one finds that the defining relation of Γ (7) can be reformulated in the following way

$$\frac{\partial L_{11}}{\partial\theta_1} = 2\Gamma^2_{11} L_{12} \tag{60}$$

$$\frac{\partial L_{11}}{\partial\theta_2} = \mathrm{e}^{-\theta_1}\left(-1 + 2\Gamma^2_{12}\right) L_{12} \tag{61}$$

$$\frac{\partial L_{12}}{\partial\theta_1} = \frac{1}{2} L_{12} + \Gamma^2_{11} L_{22} + \Gamma^2_{12} L_{12} \tag{62}$$

$$\frac{\partial L_{12}}{\partial\theta_2} = \frac{1}{2}\mathrm{e}^{-\theta_1}\left(2\Gamma^2_{22} L_{12} + 2\Gamma^2_{12} L_{22} - L_{22}\right) \tag{63}$$

$$\frac{\partial L_{22}}{\partial\theta_1} = \left(1 + 2\Gamma^2_{12}\right) L_{22} \tag{64}$$

$$\frac{\partial L_{22}}{\partial\theta_2} = 2\mathrm{e}^{-\theta_1}\Gamma^2_{22} L_{22}. \tag{65}$$

The interchangeability of the ordering of partial derivatives of θ_i (equation (42)) leads to conditions

$$\Gamma^2_{12} = -\frac{1}{2} \qquad \Gamma^2_{22} = 0. \tag{66}$$

By solving (60)–(65) one finds all possible metrics in the form

$$L_{11} = \mathrm{e}^{2\theta_1}\left(\Gamma^2_{11}\right)^2 \alpha + 2\mathrm{e}^{\theta_1}\Gamma^2_{11}\beta + \gamma \tag{67}$$

$$L_{12} = \Gamma^2_{11}\alpha\mathrm{e}^{2\theta_1} + \beta\mathrm{e}^{\theta_1} \qquad L_{22} = \alpha\mathrm{e}^{2\theta_1}$$

where $\alpha, \beta, \gamma, \Gamma_{11}^2 \in \mathbf{R}$ are parameters such that $\det L \neq 0$. By evaluation of Γ we arrive at the explicit form of equations of motion

$$\frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} = 0 \qquad \mathrm{e}^{-\theta_1}\left(\frac{\partial^2\theta_2}{\partial x_0^2} - \frac{\partial^2\theta_2}{\partial x_1^2}\right) + \Gamma_{11}^2\left(\left(\frac{\partial\theta_1}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right) = 0. \tag{68}$$

The equations of motion are in fact two coupled linear wave equations; the first one is homogeneous, i.e. explicitly solvable ($\theta_1 = F(x_0 - x_1) + G(x_0 + x_1)$), and the second one contains nonlinear terms in already known θ_1 only, it is therefore just the inhomogeneous wave equation. That is why the application of the inverse spectral method is questionable in this case.

5. Three-dimensional solvable Lie groups

Structure of all three-dimensional solvable real Lie algebras can be written in the following form (see e.g. [7]):

$$\begin{aligned} [t_2, t_3] &= b_{11}t_1 + b_{12}t_2 \\ [t_3, t_1] &= b_{21}t_1 + b_{22}t_2 \\ [t_1, t_2] &= 0 \end{aligned} \tag{69}$$

where the 2×2 matrix $B = (b_{ij})$ is one of the following

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 1 \\ -1 & \sigma \end{pmatrix}, \quad \begin{pmatrix} \sigma & 1 \\ -1 & -\sigma \end{pmatrix}$$

σ is a positive real number. Each of these algebras can be realized as a matrix algebra in the form

$$\left\{ \begin{pmatrix} (1+b_{21})z & -b_{11}z & x \\ b_{22}z & (1-b_{12})z & y \\ 0 & 0 & z \end{pmatrix} \middle| \, x, y, z \in \mathbf{R} \right\}. \tag{70}$$

For convenience, in the following we will denote by capital letters indices going from 1 to 2 only (e.g. $A \in \{1, 2\}$), other index conventions remain unchanged.

The structure coefficients are

$$c_{12}{}^q = 0 \qquad c_{A3}{}^3 = 0 \tag{71}$$
$$c_{31}{}^1 = b_{21} \qquad c_{31}{}^2 = b_{22} \qquad c_{23}{}^1 = b_{11} \qquad c_{23}{}^2 = b_{12}. \tag{72}$$

Considering the equations (17) for $a = 3$ one finds $P_{3B}c_{pq}{}^B = 0$, i.e. for $(p, q) = (3, 1)$ and $(p, q) = (2, 3)$

$$P_{31}b_{21} + P_{32}b_{22} = 0 \qquad P_{31}b_{11} + P_{32}b_{12} = 0. \tag{73}$$

We divide our investigation into two possibilities, first $\det B \neq 0$ ($\Rightarrow P_{31} = P_{32} = 0$) and second $\det B = 0$.

5.1. Three–dimensional solvable groups with det $B \neq 0$

Considering the case of $\det B \neq 0$ (i.e. all cases in the previous classification except the Abelian algebra ($B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$), the nilpotent Heisenberg algebra ($B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$) and the case $B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$), we can conclude that

$$P_{31} = P_{32} = 0. \tag{74}$$

Furthermore, considering equations (17) for $(p, q) = (1, 2)$ one finds

$$c^A_{13}(Q_{31}Q_{12} - Q_{11}Q_{32}) + c^A_{23}(Q_{31}Q_{22} - Q_{21}Q_{32}) = 0. \tag{75}$$

Using the relations between the structure constants and the matrix B, one finds that the linear set of equations (75) for $Q_{31}Q_{12} - Q_{11}Q_{32}$ and $Q_{31}Q_{22} - Q_{21}Q_{32}$ has only trivial solution (for $\det B \neq 0$), i.e.

$$Q_{11}Q_{32} = Q_{31}Q_{12} \qquad Q_{21}Q_{32} = Q_{31}Q_{22}. \tag{76}$$

It follows that $\det Q = Q_{33}(Q_{11}Q_{22} - Q_{12}Q_{21})$ and if $Q_{31} \neq 0$ or $Q_{32} \neq 0$ then $Q_{11}Q_{22} = Q_{21}Q_{12}$ and $\det Q = 0$.

We are therefore led to

$$Q_{31} = Q_{32} = 0. \tag{77}$$

Using equations (18) one finds for $a = 3$ that

$$0 = Q_{3b}\Gamma^b_{pq} = Q_{33}\Gamma^3_{pq} \qquad \text{i.e.} \quad \Gamma^3_{pq} = 0 \tag{78}$$

and for $p = A, q = B$

$$0 = \frac{1}{2}c^a_{CD}(P_{CA}Q_{DB} + P_{CB}Q_{DA}) = Q_{ab}\Gamma^b_{AB} \tag{79}$$

leading together with (78) to

$$\begin{aligned} 0 &= Q_{11}\Gamma^1_{AB} + Q_{12}\Gamma^2_{AB} \\ 0 &= Q_{21}\Gamma^1_{AB} + Q_{22}\Gamma^2_{AB}. \end{aligned} \tag{80}$$

Since $\det Q = Q_{33}(Q_{11}Q_{22} - Q_{12}Q_{21}) \neq 0$, we have

$$\Gamma^1_{AB} = \Gamma^2_{AB} = 0. \tag{81}$$

The corresponding equations of motion (6) have the following form

$$\partial_\mu J^{\mu,A} + 2\Gamma^A_{B3}J^B_\mu J^{\mu,3} + \Gamma^A_{33}J^3_\mu J^{\mu,3} = 0 \tag{82}$$

$$\partial_\mu J^{\mu,3} = 0. \tag{83}$$

To gain more insight into these equations one should explicitly write the fields $J^{\mu,a}$. First, one needs a suitable realization of the Lie group G. It can be obtained by exponentiation of the elements of the algebra (70)

$$g(x, y, z) = \exp\left(\begin{pmatrix} (1 + b_{21})z & -b_{11}z & x \\ b_{22}z & (1 - b_{12})z & y \\ 0 & 0 & z \end{pmatrix}\right). \tag{84}$$

After a reparametrization we can write a general group element in the form

$$g(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} D(\theta_3) & & \theta_1 \\ & & \theta_2 \\ 0 & 0 & \exp(\theta_3) \end{pmatrix} \qquad \theta_i \in \mathbf{R} \tag{85}$$

where $D(\theta_3) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \exp\begin{pmatrix} (1+b_{21})\theta_3 & -b_{11}\theta_3 \\ b_{22}\theta_3 & (1-b_{12})\theta_3 \end{pmatrix}$.

Then

$$g^{-1}(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} D^{-1}(\theta_3) & & \exp(-\theta_3)(\det\, D^{-1})(d_{12}\theta_2 - d_{22}\theta_1) \\ & & \exp(-\theta_3)(\det\, D^{-1})(d_{11}\theta_2 - d_{21}\theta_1) \\ 0 & 0 & \exp(-\theta_3) \end{pmatrix} \tag{86}$$

and

$$\partial_\mu g(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} \partial_\mu D(\theta_3) & & \partial_\mu\theta_1 \\ & & \partial_\mu\theta_2 \\ 0 & 0 & (\partial_\mu\theta_3)\exp(\theta_3) \end{pmatrix}. \tag{87}$$

The fields $J_\mu = g^{-1}\partial_\mu g$ can then be computed

$$J_\mu = \begin{pmatrix} F(\theta_3,\partial_\mu\theta_3) & & (\det\, D)^{-1}(d_{22}\partial_\mu\theta_1 - d_{12}\partial_\mu\theta_2 + \mathrm{e}^{-z}(\partial_\mu\theta_3)(d_{12}\theta_2 - d_{22}\theta_1)) \\ & & (\det\, D)^{-1}(d_{21}\partial_\mu\theta_1 - d_{11}\partial_\mu\theta_2 + \mathrm{e}^{-z}(\partial_\mu\theta_3)(d_{11}\theta_2 - d_{21}\theta_1)) \\ 0 & 0 & \partial_\mu\theta_3 \end{pmatrix} \tag{88}$$

where $F(\theta_3,\partial_\mu\theta_3) = D^{-1}(\theta_3)\partial_\mu D(\theta_3)$. Reading off the coordinates of the fields J_μ in the basis (t_1,t_2,t_3) one concludes that

(i) $J_\mu^3 = \partial_\mu\theta_3$, i.e. the equation of motion (83) for θ_3 is just the wave equation $\partial_\mu\partial^\mu\theta_3 = 0$ and

(ii) J_μ^1, J_μ^2 are linear in θ_1,θ_2 and their derivatives, i.e. the equations of motion (82) for $\theta_{1,2}$ after substitution of the explicit form of θ_3 turn out to be a system of two coupled linear PDEs for unknown θ_1,θ_2.

Because inverse scattering method is usually not applied to linear PDEs, we do not study this case further.

5.2. Three–dimensional solvable groups with $\det B = 0$

The condition $\det B = 0$ allows three possibilities:

(i) three-dimensional nilpotent group, i.e. Heisenberg group,

(ii) centrally extended $Af(1)$ group ($B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$) and

(iii) three-dimensional Abelian group (already considered, see section 3).

5.2.1. Heisenberg group.

The Heisenberg group is a nilpotent three-dimensional group. It can be realized as a matrix group of upper triangular 3×3 matrices with unit diagonal. We choose parametrization

$$g(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} 1 & \theta_1 & \theta_3 + \frac{\theta_1\theta_2}{2} \\ 0 & 1 & \theta_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{89}$$

The basis of the corresponding Lie algebra is then

$$t_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{90}$$

and the nonzero structure coefficients are

$$c_{12}{}^3 = 1 \qquad c_{21}{}^3 = -1. \tag{91}$$

The coordinates of the vector fields J_μ evaluated in the given basis are

$$J_\mu = \left(\partial_\mu \theta_1, \partial_\mu \theta_2, \partial_\mu \theta_3 + \frac{\theta_2}{2} \partial_\mu \theta_1 - \frac{\theta_1}{2} \partial_\mu \theta_2 \right). \tag{92}$$

The differential operators U_a are in this case

$$U_1 = \frac{\partial}{\partial \theta_1} - \frac{\theta_2}{2} \frac{\partial}{\partial \theta_3} \qquad U_2 = \frac{\partial}{\partial \theta_2} + \frac{\theta_1}{2} \frac{\partial}{\partial \theta_3} \qquad U_3 = \frac{\partial}{\partial \theta_3}. \tag{93}$$

Equation (17) is in this case equivalent to a set of equations

$$\begin{aligned} &P_{13} = P_{23} = 0 \qquad Q_{13} = Q_{23} = 0 \\ &P_{11} P_{22} - Q_{11} Q_{22} = Q_{12} Q_{21} - P_{12} P_{21} = P_{33}. \end{aligned} \tag{94}$$

Equation (18) for $a = 1, 2$ is

$$Q_{11} \Gamma^1_{pq} + Q_{12} \Gamma^2_{pq} = 0 \qquad Q_{21} \Gamma^1_{pq} + Q_{22} \Gamma^2_{pq} = 0. \tag{95}$$

Invertibility of Q together with $Q_{13} = Q_{23} = 0$ implies $\det \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \neq 0$ and consequently

$$\Gamma^1_{pq} = \Gamma^2_{pq} = 0. \tag{96}$$

Also it follows that $Q_{1b} \Gamma^b_{3q} = 0$, $Q_{2b} \Gamma^b_{3q} = 0$. Similarly, equation (18) for $a = 3, p = 3$ leads to $Q_{3b} \Gamma^b_{3q} = 0$. All these equations imply

$$\Gamma^b_{3q} = 0 \tag{97}$$

and

$$\tilde{\Gamma}_{d3c} \equiv L_{db} \Gamma^b_{3c} = 0. \tag{98}$$

After expressing this equality in coordinates using definition of Γ (7) and some simple algebra, one finds

$$U_3 L_{ij} = 0 \tag{99}$$

$$U_1 L_{33} = 0 \tag{100}$$

$$U_2 L_{33} = 0 \tag{101}$$

$$L_{33} = L_{23,1} - L_{13,2} = U_1 L_{23} - U_2 L_{13}. \tag{102}$$

Similarly, for $b, c \neq 3$ we find using (96)

$$\tilde{\Gamma}_{jbc} \equiv L_{jd} \Gamma^d_{bc} = L_{j3} \Gamma^3_{bc} \tag{103}$$

leading to

$$U_1 L_{11} = L_{13}\Gamma^3_{11} \quad (104)$$

$$L_{13} + U_2 L_{11} = 2L_{13}\Gamma^3_{12} \quad (105)$$

$$2L_{23} + 2U_2 L_{12} - U_1 L_{22} = 2L_{13}\Gamma^3_{22} \quad (106)$$

$$-2L_{13} + 2U_1 L_{12} - U_2 L_{11} = 2L_{23}\Gamma^3_{11} \quad (107)$$

$$-L_{23} + U_1 L_{22} = 2L_{23}\Gamma^3_{12} \quad (108)$$

$$U_2 L_{22} = 2L_{23}\Gamma^3_{22} \quad (109)$$

$$2U_1 L_{13} = 2L_{33}\Gamma^3_{11} \quad (110)$$

$$U_1 L_{23} + U_2 L_{13} = 2L_{33}\Gamma^3_{12} \quad (111)$$

$$2U_2 L_{23} = 2L_{33}\Gamma^3_{22}. \quad (112)$$

Using the last three equations together with (99)–(102) to express Γ^3_{ij} and substituting it into the remaining equations one finds a set of coupled first-order differential equations. Using the fact that nothing depends on θ_3 (see (99), $U_3 = \frac{\partial}{\partial\theta_3}$) one can solve these equations:

$$L_{13} = \alpha\theta_1 + \beta\theta_2 + K_{13} \quad (113)$$

$$L_{23} = \gamma\theta_1 + \delta\theta_2 + K_{23} \quad (114)$$

$$L_{33} = \gamma - \beta \quad (115)$$

$$L_{11} = \frac{L_{13}^2}{L_{33}} + K_{11} \quad (116)$$

$$L_{12} = \frac{L_{13}L_{23}}{L_{33}} + K_{12} \quad (117)$$

$$L_{22} = \frac{L_{23}^2}{L_{33}} + K_{22} \quad (118)$$

where $\alpha, \beta, \gamma, \delta, K_{i,j} \in \mathbf{R}, \beta \neq \gamma$ are parameters such that $K_{11}K_{22} - K_{12}^2 \neq 0$. The corresponding equations of motion are of the following form

$$\frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} = 0 \quad (119)$$

$$\frac{\partial^2\theta_2}{\partial x_0^2} - \frac{\partial^2\theta_2}{\partial x_1^2} = 0 \quad (120)$$

$$\frac{\partial^2\theta_3}{\partial x_0^2} - \frac{\partial^2\theta_3}{\partial x_1^2} + F(\theta_1, \theta_2) = 0 \quad (121)$$

where $F(\theta_1, \theta_2)$ is a certain function of θ_1, θ_2 and their derivatives. It is clear that the homogeneous wave equations (119) and (120) can be solved explicitly and then (121) is an inhomogeneous wave equation. A corresponding Lax pair can be found by solving equations (18)–(21). To sum up, the only possible generalized principal chiral model for the Heisenberg group expressible by the Lax operators (13) and (14) is again equivalent to the inhomogeneous wave equation.

5.2.2. Centrally extended $Af(1)$ group. We consider a 2×2 matrix realization of this group with the parametrization

$$g(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \exp\left(\frac{\theta_1-\theta_3}{2}\right) & \exp\left(\frac{\theta_1}{2}\right)\frac{\theta_2}{2} \\ 0 & \exp\left(\frac{\theta_1+\theta_3}{2}\right) \end{pmatrix}. \quad (122)$$

In the corresponding Lie algebra we choose a basis

$$t_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \qquad t_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{123}$$

The nonzero structure coefficients for this choice of basis are

$$c_{23}{}^2 = 1 \qquad c_{32}{}^2 = -1. \tag{124}$$

The coordinates of the vector fields J_μ in this basis are

$$J_\mu = \left(\partial_\mu\theta_1, \mathrm{e}^{-\frac{\theta_3}{2}}\left(\partial_\mu\theta_2 - \frac{1}{2}\theta_2\partial_\mu\theta_3\right), \partial_\mu\theta_3\right). \tag{125}$$

The differential operators U_a are in this case

$$U_1 = \frac{\partial}{\partial\theta_1} \qquad U_2 = \mathrm{e}^{(-\theta_3/2)}\frac{\partial}{\partial\theta_2} \qquad U_3 = \frac{1}{2}\theta_2\frac{\partial}{\partial\theta_2} + \frac{\partial}{\partial\theta_3}. \tag{126}$$

We have used an approach rather similar to the one used in the case of $Af(1)$. First we have used equation (18) with $a = 1, 3$ to derive a linear relation between Γ^i_{jk}:

$$0 = Q_{1b}\Gamma^b_{pq} \qquad 0 = Q_{3b}\Gamma^b_{pq} \tag{127}$$

reducing (together with the invertibility of Q) the possible values of Γ to

$$\Gamma^i_{pq} = \kappa^i\Delta_{pq} \tag{128}$$

where $\kappa^i, \Delta_{pq} = \Delta_{qp} \in \mathbf{R}$.

In the next step we put the above given expressions for Γ into the definition of the connection (7) and solve it with respect to derivatives of the metric L.

Using those PDEs for L we calculate a necessary condition for the existence of a Lax pair. We evaluate the difference of second derivatives

$$\frac{\partial}{\partial\theta_a}\frac{\partial}{\partial\theta_b}L_{ij} - \frac{\partial}{\partial\theta_b}\frac{\partial}{\partial\theta_a}L_{ij} \tag{129}$$

in terms of L_{kl}. Since this difference must be zero, we obtain a set of 18 (for $i, j \in \{1, 2, 3\}$ and $(a, b) = (1, 2), (1, 3), (2, 3)$) equations for six components of the metric L_{kl} and nine constants κ^j, Δ_{pq}.

We have solved equations (129) using Maple V computer algebra system only, neither Mathematica 4 nor Reduce 3.6 were able to solve it. Therefore we have to rely on the results of Maple and we are not able to independently check the completeness of the solution.

All possible connections allowing invertible metric L and Lax formulation of equations of motion (13)–(15) appear to be of one of the following two forms (we recall that $J^2_\mu = \mathrm{e}^{-\frac{\theta_3}{2}}(\partial_\mu\theta_2 - \frac{1}{2}\theta_2\partial_\mu\theta_3)$):

(i) $\Gamma^1_{pq} = \Gamma^3_{pq} = 0, \Gamma^2_{1q} = \Gamma^2_{22} = 0, \Gamma^2_{23} = -\frac{1}{2}, \Gamma^2_{33} \in \mathbf{R}$. The corresponding equations of motion are

$$\partial_\mu\partial^\mu\theta_1 = 0 \qquad \partial_\mu J^{\mu,2} - \frac{1}{2}J^{\mu,2}\,\partial_\mu\theta_3 + \Gamma^2_{33}\partial_\mu\theta_3\partial^\mu\theta_3 = 0 \qquad \partial_\mu\partial^\mu\theta_3 = 0 \tag{130}$$

i.e., the equations of motion for θ_1, θ_3 are the wave equations and can be solved explicitly. The equation of motion for θ_2 is linear after substitution of the solution θ_3 because $J^{\mu,2}$ is linear in θ_2.

(ii) $\Gamma^3_{pq} = \Gamma^2_{pq} = 0$, $\Gamma^1_{1j} = 0$, $\Gamma^1_{22}, \Gamma^1_{23}, \Gamma^1_{33} \in \mathbf{R}$. The equations of motion are

$$\partial_\mu \partial^\mu \theta_1 + F(\theta_2, \theta_3) = 0 \qquad \partial_\mu J^{\mu,2} = 0 \qquad \partial_\mu \partial^\mu \theta_3 = 0 \tag{131}$$

where $F(\theta_2, \theta_3)$ is a certain given function of θ_2, θ_3 and their derivatives. In this case we have again the wave equation for θ_3. After substituting the solution of this equation into $\partial_\mu J^{\mu,2} = 0$ we have a linear PDE for θ_2 and finally substituting both θ_2, θ_3 into an equation of motion for θ_1 we have an inhomogeneous wave equation for θ_1:

$$\partial_\mu \partial^\mu \theta_1 + F(\theta_2, \theta_3) = 0.$$

This case also includes the model with $\Gamma^a_{pq} = 0$.

To conclude, in the case of centrally extended group $Af(1)$ we have found no intrinsically nonlinear model but one should be aware that the completeness of this result relies on the computation done only in one computer algebra system.

6. Conclusions

We have analysed generalized principal chiral models given by the action of the form (1) where the target manifold of the fields are two- and three-dimensional connected and simply connected non-semisimple Lie groups. We have found that in these cases all but one equations of motions admitting Lax formulation (13)–(15) can be brought to linear PDEs.

The only truly nonlinear system of equations comes from the generalized principal model on the two-dimensional solvable group with the non-constant metric

$$L(\theta_1, \theta_2) = \begin{pmatrix} \frac{-1+K^2\kappa^2}{\kappa^2}\alpha \mathrm{e}^{\theta_1} & -K\alpha \mathrm{e}^{\theta_1} \\ -K\alpha \mathrm{e}^{\theta_1} & \alpha \mathrm{e}^{\theta_1} \end{pmatrix} \tag{132}$$

where $K \in \mathbf{R}$, $\alpha, \kappa \in \mathbf{R}\backslash\{0\}$. Its equations of motion read

$$\partial_\nu \partial^\nu \theta_1 + \frac{1}{2}\partial_\nu \theta_1 \partial^\nu \theta_1 - \frac{1}{2}\kappa^2 \left(K^2 \partial_\nu \theta_1 \partial^\nu \theta_1 - 2K \mathrm{e}^{-\theta_1} \partial_\nu \theta_1 \partial^\nu \theta_2 + \mathrm{e}^{-2\theta_1} \partial_\nu \theta_2 \partial^\nu \theta_2\right) = 0 \tag{133}$$

$$\partial_\nu \partial^\nu \theta_2 - K \mathrm{e}^{\theta_1} \partial_\nu \partial^\nu \theta_1 = \partial_\nu \theta_1 \partial^\nu \theta_2 \tag{134}$$

and the Lax pair is

$$X_0 = \begin{pmatrix} \partial_0 + \lambda + \frac{1}{2}\left(J_0^1 + K\kappa J_1^1 - \kappa J_1^2\right), & -2\lambda + J_1^1 - \kappa J_0^2 + K\kappa J_0^1 \\ 0 & \partial_0 \end{pmatrix} \tag{135}$$

$$X_1 = \begin{pmatrix} \partial_1 + \lambda + \frac{1}{2}\left(J_1^1 + K\kappa J_0^1 - \kappa J_0^2\right), & -2\lambda + J_0^1 - \kappa J_1^2 + K\kappa J_1^1 \\ 0 & \partial_1 \end{pmatrix} \tag{136}$$

where $J^1_\mu = \partial_\mu \theta_1$, $J^2_\mu = \mathrm{e}^{-\theta_1}\partial_\mu \theta_2$ and λ is a free parameter that can be used for the inverse scattering method.

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Chapter 3

Principal chiral models with non–constant metric

Czechoslovak Journal of Physics 51 (2001) 1441–1446.

This article is a summary of the talk at the 10th International Colloquium "Quantum Groups and Integrable Systems". It contains a brief review of results of previous paper and two new results.

Firstly, the Lax pair of the 2–dimensional nonlinear system (two second order partial differential equations) obtained before (see Chapter 2) hinted a possible reformulation after a change of variables in terms of linear differential equations, namely as a wave equation and two first order equations depending nonlinearly on the solution of the wave equation.

Secondly, we reconsidered an older work by L. Hlavatý [6] concerning the principal models on $SU(2)$ and we proved that the technical assumption of the diagonal form of the metric, simplifying the computation to a manageable state, leads immediately to constancy of the metric. The consideration of non–diagonal metrics seemed far too complicated and was not pursued further.

Principal chiral models with non–constant metric*)

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Received 22 August 2001

Field equations for generalized principal chiral models with non–constant metric and their possible Lax formulation are considered. Ansatz for Lax operators is taken linear in currents. Results of a complete investigation of models allowing Lax formulation with linear ansatz for Lax operators on solvable 2– and 3–dimensional groups are given; all such models appear to be almost linear. Also models on simple group SU(2) with diagonal metric are considered; it turns out that Lax formulation exists in this case for constant metrics only.

PACS: 02.30.Ik,02.30.Jr
Key words: principal models, chiral models, sigma models, Lax pair, integrable models, SU(2)

1 Introduction

We have investigated the generalisation of principal chiral models [1] to models with non–constant metric (this idea was suggested by Sochen in [2]) This approach allowed to study also models on non–semisimple groups. We have studied the case of 2– and 3– dimensional groups, both solvable and simple. In this article we provide a brief overview and extension of our results, submitted for publication elsewhere (see [3], [6] and [5]).

Generalised principal chiral models [2] are given by the action

$$I[g] = \int d^2x \eta^{\mu\nu} L_{ab}(g) J^a_\mu J^b_\nu \tag{1}$$

where G is a Lie group, $\mathcal{L}(G)$ its Lie algebra,

$$J_\mu = (g^{-1}\partial_\mu g) \in \mathcal{L}(G), \tag{2}$$

$g : \mathbf{R^2} \to G,\ \mu,\nu \in \{0,1\},\ \eta = diag(1,-1)$, L is a G-dependent symmetric nondegenerate bilinear form. We consider the bilinear form L as a metric on the group manifold and the generalization of principal models from ad-invariant Killing form on $\mathcal{L}(G)$ to more general case enables us to introduce the principal models also on non–semisimple groups.

Lie products of elements of the basis of $\mathcal{L}(G)$ define the structure coefficients

$$[t_a, t_b] = c_{ab}{}^c t_c \tag{3}$$

*) Presented at the 10th International Colloquium on Quantum Groups: "Quantum Groups and Integrable Systems", Prague, 21–23 June 2001.

and in the same basis we define the coordinates of the field J_ν

$$J_\nu = g^{-1}\partial_\nu g = J^b_\nu t_b. \tag{4}$$

Fields automatically satisfy Bianchi identities

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0. \tag{5}$$

Varying the action (1) we obtain the equations of motion for the generalized principal chiral models

$$\partial_\mu J^{\mu,a} + \Gamma^a_{bc} J^b_\mu J^{\mu,c} = 0 \tag{6}$$

where the connection Γ is defined by

$$\Gamma^a_{bc} = \frac{1}{2}(L^{-1})^{ad}(c_{db}{}^q L_{qc} + c_{dc}{}^q L_{qb} + U_b L_{cd} + U_c L_{bd} - U_d L_{bc}). \tag{7}$$

The vector fields U_a are defined in the local group coordinates θ_i as

$$U_a = U^i_a(\theta)\frac{\partial}{\partial\theta_i} \tag{8}$$

where the matrix U is the inverse of the matrix V of vielbein coordinates

$$U^i_a(\theta) = (V^{-1})^i_a(\theta),\; V^a_i = (g^{-1}\frac{\partial g}{\partial\theta_i})^a. \tag{9}$$

Note that the connection (7) is symmetric in the lower indices, $\Gamma^a_{bc} = \Gamma^a_{cb}$.

1.1 Lax pairs

The ansatz that we are going to use for the Lax operators X_0, X_1 of the generalized principal chiral models is

$$X_0 = \partial_0 + P_{ab}J^b_0 t_a + Q_{ab}J^b_1 t_a + A_a t_a, \tag{10}$$

$$X_1 = \partial_1 + \tilde{Q}_{ab}J^b_0 t_a + \tilde{P}_{ab}J^b_1 t_a + B_a t_a, \tag{11}$$

where $P, Q, \tilde{P}, \tilde{Q}$ are four arbitrary constant $dimG \times dimG$ matrices and A, B are two arbitrary constant vectors.

By explicit evaluation of the zero curvature condition

$$[X_0, X_1] = 0, \tag{12}$$

using the equations of motions (6) and Bianchi identities (5) and equating the coefficients of different powers and derivatives of J^a_μ one finds following necessary and sufficient conditions that the operators X_0, X_1 form a Lax pair:

$$\tilde{P} = P, \quad \tilde{Q} = Q, \quad \exists\, Q^{-1} \tag{13}$$

$$(P_{bp}P_{cq} - Q_{bp}Q_{cq})c_{bc}{}^a = P_{ab}c_{pq}{}^b, \tag{14}$$

$$\frac{1}{2}c_{cd}{}^a(P_{cp}Q_{dq} + P_{cq}Q_{dp}) = Q_{ab}\Gamma^b_{pq}, \tag{15}$$

$$c_{cd}{}^a(P_{cp}B_d + A_c Q_{dp}) = 0, \tag{16}$$

$$c_{cd}{}^a(Q_{cp}B_d + A_c P_{dp}) = 0, \tag{17}$$

$$c_{cd}{}^a A_c B_d = 0. \tag{18}$$

Moreover, the previous equations impose a condition on Γ. Namely, we can express (15) in an equivalent form

$$\frac{1}{2}(Q^{-1})^{ba}{c_{cd}}^{a}(P_{cp}Q_{dq}+P_{cq}Q_{dp})=\Gamma^{b}_{pq} \tag{19}$$

and conclude that **only the generalized principal models with the constant connection Γ admit the Lax formulation** (10)–(12) because the left–hand side of the previous equation is constant.

2 2–dimensional solvable group

Every non-Abelian two–dimensional connected Lie group is isomorphic to the group of affine transformations of real line. Let us denote it by $Af(1)$. We have used its matrix realisation with the following parametrisation (θ_1, $\theta_2 \in \mathbf{R}$)

$$g(\theta_1,\theta_2)=\begin{pmatrix}\exp(\theta_1) & \theta_2\\ 0 & 1\end{pmatrix} \tag{20}$$

There exist for the group $Af(1)$ two classes of metrics allowing Lax pair of the form considered:

1. One class of metrics with Lax formulation, leading to equations of motion

$$\partial_\mu\partial^\mu\theta_1=0 \quad , \quad \partial_\mu\partial^\mu\theta_2+Ke^{\theta_1}\left[\left(\frac{\partial\theta_1}{\partial x_0}\right)^2-\left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right]=0 \tag{21}$$

   The first equation is just the wave equation, its general solution has the well–known form $\theta_1=F(x_0-x_1)+G(x_0+x_1)$. We can then substitute this solution into the second equation and find a linear equation for θ_2.

2. The class of metrics of the form

$$L(\theta_1,\theta_2)=\alpha e^{\theta_1}\begin{pmatrix}\frac{-1+K^2\kappa^2}{\kappa^2} & -K\\ -K & 1\end{pmatrix} \tag{22}$$

   where $K\in\mathbf{R}$, $\alpha,\kappa\in\mathbf{R}\setminus\{0\}$. Its equations of motion read

$$\begin{aligned}\partial_\nu\partial^\nu\theta_1+\frac{1}{2}\partial_\nu\theta_1\partial^\nu\theta_1-\frac{1}{2}\kappa^2(K^2\partial_\nu\theta_1\partial^\nu\theta_1&\\ -2Ke^{-\theta_1}\partial_\nu\theta_1\partial^\nu\theta_2+e^{-2\theta_1}\partial_\nu\theta_2\partial^\nu\theta_2)&=0, \qquad (23)\\ \partial_\nu\partial^\nu\theta_2-Ke^{\theta_1}\partial_\nu\partial^\nu\theta_1&=\partial_\nu\theta_1\partial^\nu\theta_2 \qquad (24)\end{aligned}$$

   and the Lax pair reads

$$X_0=\begin{pmatrix}\partial_0+\frac{1}{2}Y_0+\lambda, & Y_1-2\lambda\\ 0 & \partial_0\end{pmatrix},\ X_1=\begin{pmatrix}\partial_1+\frac{1}{2}Y_1+\lambda, & Y_0-2\lambda\\ 0 & \partial_1\end{pmatrix} \tag{25}$$

   where $Y_0=J_0^1-\kappa J_1^2+K\kappa J_1^1$, $Y_1=J_1^1-\kappa J_0^2+K\kappa J_0^1$ and $J_\mu^1=\partial_\mu\theta_1$, $J_\mu^2=e^{-\theta_1}\partial_\mu\theta_2$, λ may be interpreted as a spectral parameter.

In order to get deeper understanding of the model considered we explicitly evaluate corresponding Lax equation (12) and find

$$\partial_0 Y_1 - \partial_1 Y_0 = 0, \qquad \partial_0 Y_0 - \partial_1 Y_1 + \frac{1}{2}(Y_0 Y_0 - Y_1 Y_1) = 0. \tag{26}$$

We may consider the first of these equations (26) a condition for existence of ϕ such that

$$Y_\mu = 2\frac{\partial_\mu \phi}{\phi} \tag{27}$$

and the second equation (26) becomes

$$\partial_\mu \partial^\mu \phi = 0. \tag{28}$$

Once one has a solution ϕ of the wave equation (28), he may substitute this solution into (27). After explicit calculation of Y_μ in this model one finds

$$\partial_\mu \theta_1 - \kappa \mathrm{e}^{-\theta_1} \partial_{\bar{\mu}} \theta_2 + K\kappa \partial_{\bar{\mu}} \theta_1 = 2\frac{\partial_\mu \phi}{\phi}, \tag{29}$$

where $\bar{\mu}$ is defined $\bar{1} = 0, \bar{0} = 1$.

After substitution $\mathrm{e}^{\theta_1} = \rho$, $\kappa\theta_2 = W$ we finally obtain a set of linear partial differential equations for ρ, W

$$\partial_0 \rho - \partial_1 W + K\kappa \partial_1 \rho = 2\frac{\partial_0 \phi}{\phi}\rho, \ \partial_1 \rho - \partial_0 W + K\kappa \partial_0 \rho = 2\frac{\partial_1 \phi}{\phi}\rho. \tag{30}$$

We have thus transformed the original nonlinear problem into several steps, each containing linear equations only. This approach can be used to find some simple solutions of the principal chiral model (23–24), but it is probably impossible to write explicitly ρ, W (and consequently θ_1, θ_2) for a general solution ϕ of the wave equation (27). We also see that we have linearized the equations (23–24) without inverse spectral transform. On the other hand, the Lax pair (25) proved to be useful for guessing the linearizing transformation (27).

3 3–dimensional solvable Lie groups

Models on 3-dimensional solvable Lie groups were investigated in [6]. It was shown that most of such groups allow models of the following form only:

$$\begin{aligned} \partial_\mu J^{\mu,A} + 2\Gamma^A_{B3} J^B_\mu J^{\mu,3} + \Gamma^A_{33} J^3_\mu J^{\mu,3} &= 0 \\ \partial_\mu J^{\mu,3} &= 0 \end{aligned}$$

where $J^3_\mu = \partial_\mu \theta_3$, J^A_μ are linear in $\partial_\mu \theta_B$ and θ_B (and nonlinear in θ_3); i.e. the equation of motion for θ_3 is just the wave equation $\partial_\mu \partial^\mu \theta_3 = 0$ and J^1_μ, J^2_μ are linear in θ_1, θ_2 and their derivatives and consequently the equations of motion for

$\theta_{1,2}$ after substitution of the explicit form of θ_3 turn out to be a system of two coupled linear partial differential equations for unknown θ_1, θ_2.

The only exceptions, allowing equations of motion of a different form, are 3-dimensional nilpotent group, i.e. Heisenberg group, and centrally extended $Af(1)$ group. These cases were considered separately.

The Heisenberg group leads to models that can be written again in terms of linear equations (although not of the form given above). The case of centrally extended $Af(1)$ group was investigated using computer algebra system and we have also found no intrinsically nonlinear model, i.e. the results are again similar to the previous one.

4 Generalized principal chiral models on $SU(2)$

As mentioned in the introduction to the first chapter, chiral models on simple groups were the original ones considered because of nondegeneracy of their Killing form. Results concerning the case of models with such ad-invariant metrics and corresponding inverse scattering method were published firstly in [1]. The generalization by Sochen [2] allowed to consider also the case with nonconstant metric. In the paper [5] one of us has tried to construct such model on $SU(2)$ group for diagonal metric, but has not found any.

In the following we present a simple explanation why there is no such model with diagonal nonconstant metric on $SU(2)$. We use the usual basis of $su(2)$ with the structure coefficients $c_{ab}^c = i\epsilon_{abc}$. As was mentioned in [5], the connection Γ in the case of diagonal metric on $SU(2)$ has a following form (no sums over repeated indices):

$$i\Gamma^a_{bc} = \epsilon_{abc}\frac{L_{bb} - L_{cc}}{2L_{aa}}, \ \forall a \neq b, a \neq c, c \neq b, \tag{31}$$

$$i\Gamma^a_{bb} = -\frac{U_a L_{bb}}{2L_{aa}}, \ \forall a \neq b, \tag{32}$$

$$i\Gamma^a_{ab} = i\Gamma^a_{ba} = \frac{U_b L_{aa}}{2L_{aa}}, \tag{33}$$

If we write explicitly the equations (31) for different choices of indices, we find

$$L_{22} = 2i\Gamma^1_{23}L_{11} + L_{33}, \tag{34}$$

$$L_{33} = 2i\Gamma^2_{31}L_{22} + L_{11}, \qquad L_{11} = 2i\Gamma^3_{12}L_{33} + L_{22}. \tag{35}$$

We eliminate from (35) L_{22} using the equation (34) and find

$$(1 - 2i\Gamma^1_{23})L_{11} = (2i\Gamma^3_{12} + 1)L_{33}, \tag{36}$$

$$(4i\Gamma^1_{23}i\Gamma^2_{31} + 1)L_{11} = (1 - 2i\Gamma^2_{31})L_{33}. \tag{37}$$

Since Γs are constant due to (19), we find that L_{11} is a constant multiple of L_{33} (otherwise the nonsingularity of the metric L would require all coefficients in the equations (36–37) be zero, i.e. $\Gamma^1_{23} = -\frac{i}{2}, \Gamma^3_{12} = \frac{i}{2}, \Gamma^2_{31} = -\frac{i}{2}$ and $-4\Gamma^1_{23}\Gamma^2_{31} + 1 = 2 = 0$ leading to a contradiction).

Together with (34) we have found that L_{11} and L_{22} are constant multiples of L_{33}. Using the relation

$$\det L = \text{const.} \tag{38}$$

proven in [5] we find $\det L = K(L_{33})^3 =$ const. (where $K \neq 0$ is a certain constant), i.e. L_{33} = Const.. Therefore, **the only diagonal metrics L admitting the Lax formulation in the form considered (i.e. linear in currents) are the constant ones.** It was shown in [5] that constant diagonal metrics always allow such a Lax pair.

5 Conclusion

We have found no interesting, truly nonlinear integrable model on any 2– and 3– dimensional non–semisimple Lie group. We don't know whether it is due to our ansatz (10–11) for Lax operators or whether it is a general property of principal models on non–semisimple groups.

The investigation of principal models on the group $SU(2)$ with diagonal metric and Lax pair linear in currents can be considered complete: only models with constant metric allow Lax pair and all models with constant metric have a Lax pair.

Classification of principal chiral models with nonconstant non-diagonal metric on $SU(2)$ seems to be technically unfeasible in the present time.

This work was partially supported by grant No. 1929/2001 of Czech Council of Universities.

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Chapter 4

Classification of Poisson–Lie T–dual models with two–dimensional targets

Modern Physics Letters A17 (2002) 429–434.

In this paper we constructed all 4–dimensional Manin triples and the corresponding Drinfeld doubles. Quite surprisingly, even in this low dimensional case exists a Drinfeld double possessing decompositions into non–isomorphic Manin triples, i.e. into non–isomorphic pairs of maximal isotropic subalgebras. Corresponding Poisson–Lie T–dual models on the Drinfeld doubles were explicitly constructed. This is the simplest example of such Drinfeld double, several 6–dimensional examples were found in [15] and a complete investigation of 6–dimensional case is contained in the next article (Chapter 6).

The knowledge of explicit examples of Drinfeld doubles decomposable into non–isomorphic Manin triples might be of interest in the research of Poisson–Lie T–duality, especially it might be possible to check whether the duality between models corresponding to different Manin triples survives also in quantum theory; our considerations were only on the classical level.

Modern Physics Letters A, Vol. 17, No. 7 (2002) 429–434

CLASSIFICATION OF POISSON–LIE T–DUAL MODELS WITH TWO DIMENSIONAL TARGETS

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Received 20 December 2001
Revised 3 March 2002

Four–dimensional Manin triples and Drinfeld doubles are classified and corresponding two–dimensional Poisson–Lie T–dual sigma models on them are constructed. The simplest example of a Drinfeld double allowing decomposition into two nontrivially different Manin triples is presented.

Keywords: Poisson-Lie T-duality, sigma models, Drinfeld doubles, Manin triples, string theory

PACS Nos.: 02.20.Qs, 02.20.Sv, 11.25.Hf

1. Introduction

A very important symmetry of string theories, or more specifically, two–dimensional sigma models is the T–duality. In the pioneering work,[1] Klimčík and Ševera introduced its nonabelian version – the Poisson–Lie T–duality and showed that the dual sigma models can be formulated on Drinfeld doubles. The explicit form of dual models on the nonabelian double $GL(2|\mathbb{R})$ was presented in the following work.[2] Other dual models were given in a series of forthcoming papers, see e.g. Refs. 3–5. An attempt to classify all dual principal sigma models with three–dimensional target space[6] made us to revisit the models with the two–dimensional targets and classify them. In the following we classify all four–dimensional Drinfeld doubles and the Poisson–Lie T–dual models on them.

2. Classification of four–dimensional Drinfeld doubles

The Drinfeld double D is defined as a Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad–invariant nondegenerate bilinear form $\langle .,. \rangle$ can be decomposed into a pair of maximally isotropic subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ such that $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. Any such decomposition written as an ordered set

$(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ is called a Manin triple. It is clear that to any Drinfeld double exist at least two Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$, $(\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$. Later we show an example of Drinfeld double with more than two possible decomposition into Manin triples.

One can see that the dimensions of the subalgebras are equal and that bases $\{T_i\}, \{\tilde{T}^i\}$ in the subalgebras can be chosen so that

$$\langle T_i, T_j \rangle = 0, \qquad \langle T_i, \tilde{T}^j \rangle = \langle \tilde{T}^j, T_i \rangle = \delta_i^j, \qquad \langle \tilde{T}^i, \tilde{T}^j \rangle = 0. \tag{1}$$

This canonical form of the bracket is invariant with respect to the transformations

$$T'_i = T_k A_i^k, \qquad \tilde{T}'^j = (A^{-1})^j_k \tilde{T}^k. \tag{2}$$

Due to the ad-invariance of $\langle ., . \rangle$ the algebraic structure of $\mathcal{D}$ is

$$[T_i, T_j] = f_{ij}{}^k T_k, \qquad [\tilde{T}^i, \tilde{T}^j] = \tilde{f^{ij}}{}_k \tilde{T}^k,$$
$$[T_i, \tilde{T}^j] = f_{ki}{}^j \tilde{T}^k + \tilde{f^{jk}}{}_i T_k. \tag{3}$$

From the above given facts it is clear that the subalgebras $\mathcal{G}$,$\tilde{\mathcal{G}}$ of the four–dimensional Drinfeld double are two–dimensional and surprisingly the Jacobi identities do not impose any condition on coefficients $f_{ij}{}^k, \tilde{f^{ij}}{}_k$ in this case. Each of the subalgebras is solvable and due to the invariance of (1) w.r.t. (2), the basis $\{T_1, T_2\}$ can be chosen so that the nontrivial Lie bracket in the first subalgebra is

$$[T_1, T_2] = n T_2 \tag{4}$$

where $n = 0$ or 1. However, the Lie bracket in the second subalgebra in general cannot be written in a similar way without breaking the canonical form (1) of the bracket $\langle , \rangle$ or the canonical form (4) of the subalgebra $\mathcal{G}$. Nevertheless, we can use the transformations (2) with

$$A = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}, \tag{5}$$

that preserve (4) to bring the Lie bracket of the second subalgebra to one of the following form

$$[\tilde{T}^1, \tilde{T}^2] = \beta \tilde{T}^2, \qquad \beta \in \mathbb{R} \qquad \text{or } [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^1. \tag{6}$$

In summary, *there are just four types of nonisomorphic four-dimensional Manin triples.*

Abelian Manin triple:

$$[T_i, T_j] = 0, \qquad [\tilde{T}^i, \tilde{T}^j] = 0, \qquad [T_i, \tilde{T}^j] = 0, \qquad i, j = 1, 2. \tag{7}$$

Semiabelian Manin triple (only nontrivial brackets are displayed):

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \qquad [T_2, \tilde{T}^1] = T_2, \qquad [T_2, \tilde{T}^2] = -T_1. \tag{8}$$

Type A nonabelian Manin triple ($\beta \neq 0$):

$$[T_1, T_2] = T_2, \qquad [\tilde{T}^1, \tilde{T}^2] = \beta \tilde{T}^2,$$
$$[T_1, \tilde{T}^2] = -\tilde{T}^2, \qquad [T_2, \tilde{T}^1] = \beta T_2, \qquad [T_2, \tilde{T}^2] = -\beta T_1 + \tilde{T}^1. \tag{9}$$

Type B nonabelian Manin triple:

$$[T_1, T_2] = T_2, \qquad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^1,$$
$$[T_1, \tilde{T}^1] = T_2, \qquad [T_1, \tilde{T}^2] = -T_1 - \tilde{T}^2, \qquad [T_2, \tilde{T}^2] = \tilde{T}^1. \tag{10}$$

An interesting fact is that Drinfeld doubles corresponding to semiabelian Manin triple (8) and type B nonabelian Manin triple (10) are the same, i.e. *these Manin triples are different decomposition into maximally isotropic subalgebras of the same Lie algebra with the same invariant form.* The transformation of the dual basis between these decompositions is

$$\begin{array}{rcll} X_1 = & -\tilde{T}^1 + \tilde{T}^2, & X_2 = T_1 + T_2, \\ \tilde{X}^1 = & T_2, & \tilde{X}^2 = \tilde{T}^1, \end{array} \tag{11}$$

where $(X_i, \tilde{X}^j)$ denote the dual basis in the type B nonabelian Manin triple and $(T_i, \tilde{T}^j)$ is the basis in the semiabelian Manin triple. The other Manin triples specify the algebra of the Drinfeld double uniquely, i.e. there is one connected and simply connected Drinfeld double to each of these Manin triples.

3. Dual sigma models

Having all four–dimensional Drinfeld doubles we can construct the two–dimensional Poisson–Lie T–dual sigma models on them. The construction of the models is described in the papers.[1,2] The models have target spaces in the Lie groups G and $\tilde{G}$ and are defined by the Lagrangians

$$\mathcal{L} = E_{ij}(g)(g^{-1}\partial_- g)^i (g^{-1}\partial_+ g)^j, \tag{12}$$

$$\tilde{\mathcal{L}} = \tilde{E}_{ij}(\tilde{g})(\tilde{g}^{-1}\partial_- \tilde{g})^i (\tilde{g}^{-1}\partial_+ \tilde{g})^j, \tag{13}$$

where

$$E(g) = (a(g) + E(e)b(g))^{-1} E(e) d(g), \tag{14}$$

$E(e)$ is a constant matrix and $a(g), b(g), d(g)$ are 2×2 submatrices of the adjoint representation of the group G on $\mathcal{D}$ in the basis $(T_i, \tilde{T}^j)$

$$Ad(g)^T = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}. \tag{15}$$

The matrix $\tilde{E}(\tilde{g})$ is constructed analogously with

$$Ad(\tilde{g})^T = \begin{pmatrix} \tilde{d}(\tilde{g}) & \tilde{b}(\tilde{g}) \\ 0 & \tilde{a}(\tilde{g}) \end{pmatrix}, \qquad \tilde{E}(\tilde{e}) = E(e)^{-1} = \begin{pmatrix} x & y \\ u & v \end{pmatrix}. \tag{16}$$

Both equations of motion of the above given lagrangian systems can be reduced from equation of motion on the whole Drinfeld double, not depending on the choice of Manin triple:

$$\langle(\partial_\pm l)l^{-1}, \mathcal{E}^\pm\rangle = 0, \tag{17}$$

where subspaces $\mathcal{E}^+ = \mathrm{span}(T^i + E^{ij}(e)\tilde{T}_j)$, $\mathcal{E}^- = \mathrm{span}(T^i - E^{ji}(e)\tilde{T}_j)$ are orthogonal w.r.t. $\langle,\rangle$ and span the whole Lie algebra D. One writes $l = g.\tilde{h}$, $g \in G$, $\tilde{h} \in \tilde{G}$ (such decomposition of group elements exists at least at the vicinity of the unit element) and eliminates $\tilde{h}$ from (17), respectively $l = \tilde{g}.h$, $h \in G$, $\tilde{g} \in \tilde{G}$ and eliminates h from (17). The resulting equations of motion for g, resp. $\tilde{g}$ are the equations of motion of the corresponding lagrangian system (see Ref. 1).

The corresponding models for the Manin triples (7)–(10) are the following.

Abelian double: The adjoint representations of the groups $G, \tilde{G}$ are trivial so that

$$\tilde{E}(\tilde{g}) = \tilde{E}(e) = E(g)^{-1} = E(e)^{-1}, \tag{18}$$

and the Lagrangians of the dual models are

$$\mathcal{L} = (xv - uy)^{-1}\,(v\,\partial_-\chi\partial_+\chi - y\,\partial_-\chi\partial_+\theta - u\,\partial_-\theta\partial_+\chi + x\,\partial_-\theta\partial_+\theta)\,, \tag{19}$$

$$\tilde{\mathcal{L}} = x\,\partial_-\sigma\partial_+\sigma + y\,\partial_-\sigma\partial_+\rho + u\,\partial_-\rho\partial_+\sigma + v\,\partial_-\rho\partial_+\rho. \tag{20}$$

Semiabelian double: The adjoint representations of the groups $G, \tilde{G}$ are

$$Ad(g)^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\theta & 1 & 0 \\ \theta & 0 & 0 & 1 \end{pmatrix}, \qquad Ad(\tilde{g})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\rho & e^\sigma & 0 & 0 \\ 0 & 0 & 1 & \rho e^{-\sigma} \\ 0 & 0 & 0 & e^{-\sigma} \end{pmatrix}$$

where (χ, θ) and (σ, ρ) are group coordinates of G and $\tilde{G}$. The Lagrangians of the dual models are

$$\begin{aligned} \mathcal{L} &= \left(v\,x - u\,y + u\,\theta - y\,\theta + \theta^2\right)^{-1}[v\,\partial_-\chi\partial_+\chi + (\theta - y)\,\partial_-\chi\partial_+\theta \\ &\quad -(\theta + u)\,\partial_-\theta\partial_+\chi + x\,\partial_-\theta\partial_+\theta]\,, \qquad (21) \\ \tilde{\mathcal{L}} &= \left(x - u\,\rho - y\,\rho + v\rho^2\right)\partial_-\sigma\partial_+\sigma + (y - v\rho)\,\partial_-\sigma\partial_+\rho \\ &\quad +(u - v\rho)\,\partial_-\rho\partial_+\sigma + v\,\partial_-\rho\partial_+\rho. \qquad (22) \end{aligned}$$

Similarly one may use the other possible decomposition of the double into maximally isotropic subalgebras, i.e. **type B nonabelian Manin triple**. In this case the adjoint representations of the groups $G, \tilde{G}$ are

$$Ad(g)^T = \begin{pmatrix} 1 & \theta e^{-\chi} & 0 & 0 \\ 0 & e^{-\chi} & 0 & 0 \\ 0 & -1 + e^{-\chi} & 1 & 0 \\ -1 + e^{\chi} & \theta - \theta e^{-\chi} & -\theta & e^{\chi} \end{pmatrix},$$

$$Ad(\tilde{g})^T = \begin{pmatrix} e^{-\rho} & -\sigma & \sigma - e^{\rho}\,\sigma & -1+e^{-\rho} \\ 0 & 1 & -1+e^{\rho} & 0 \\ 0 & 0 & e^{\rho} & 0 \\ 0 & 0 & e^{\rho}\,\sigma & 1 \end{pmatrix}$$

and the Lagrangians of the dual models are

$$\begin{aligned} \mathcal{L} &= [v\,x + (e^{\chi}-1-y)(e^{\chi}-1+u)]^{-1}\left[(v+u\,\theta+y\,\theta+x\,\theta^2)\partial_-\chi\partial_+\chi\right. \\ &\quad + (-1+e^{\chi}-y-x\,\theta)\,\partial_-\chi\partial_+\theta - (-1+e^{\chi}+u+x\,\theta)\,\partial_-\theta\partial_+\chi + x\partial_-\theta\partial_+\theta] \,, \end{aligned} \tag{23}$$

$$\begin{aligned} \tilde{\mathcal{L}} &= \left[v\,x - u\,y + e^{\rho}\,(u - 2\,v\,x - y + 2\,u\,y) + e^{2\,\rho}\,(1+v\,x+y-u\,(1+y))\right]^{-1} \\ &\quad \left[x\,\partial_-\sigma\partial_+\sigma + \left(v\,x - e^{-\rho}\,v\,x + y + e^{-\rho}\,u\,y - u\,y - x\,\sigma\right)\,\partial_-\sigma\partial_+\rho\right. \\ &\quad - \left(v\,x - e^{-\rho}\,v\,x - u + e^{-\rho}\,u\,y - u\,y + x\,\sigma\right)\,\partial_-\rho\partial_+\sigma \\ &\quad \left.- \left(u\,\sigma + y\,\sigma - v - x\,\sigma^2\right)\,\partial_-\rho\partial_+\rho\right] . \end{aligned} \tag{24}$$

This model has the same equations of motion in the double (17) as the previous one (up to transformation of matrix $E(e)$ induced by the change of basis of algebra) and in this sense is equivalent to it.

Type A nonabelian doubles: The adjoint representations of the groups $G, \tilde{G}$ are

$$Ad(g)^T = \begin{pmatrix} 1 & \theta e^{-\chi} & 0 & 0 \\ 0 & e^{-\chi} & 0 & 0 \\ 0 & -\beta\,\theta e^{-\chi} & 1 & 0 \\ \beta\,\theta & \beta\,\theta^2 e^{-\chi} & -\theta & e^{\chi} \end{pmatrix},$$

$$Ad(\tilde{g})^T = \begin{pmatrix} 1 & 0 & 0 & -\beta^{-1}\rho e^{-\sigma} \\ -\rho & e^{\sigma} & \beta^{-1}\rho & \beta^{-1}\rho^2 e^{-\sigma} \\ 0 & 0 & 1 & \rho e^{-\sigma} \\ 0 & 0 & 0 & e^{-\sigma} \end{pmatrix}$$

where β parametrizes different Drinfeld doubles. The Lagrangians of the dual models are

$$\begin{aligned} \mathcal{L} &= \left(v\,x - u\,y + u\,\beta\,\theta - y\,\beta\,\theta + \beta^2\,\theta^2\right)^{-1}\left[(v+u\,\theta+y\,\theta+x\,\theta^2)\partial_-\chi\partial_+\chi\right. \\ &- \quad (y + x\,\theta - \beta\,\theta)\,\partial_-\chi\partial_+\theta - (u + x\,\theta + \beta\,\theta)\,\partial_-\theta\partial_+\chi + x\partial_-\theta\partial_+\theta] \,, \end{aligned} \tag{25}$$

$$\begin{aligned} \tilde{\mathcal{L}} &= \left(\beta^2 - u\,\beta\,\rho + y\,\beta\,\rho + v\,x\,\rho^2 - u\,y\,\rho^2\right)^{-1} \\ &\quad \left[\beta^2\left(x - u\,\rho - y\,\rho + v\,\rho^2\right)\,\partial_-\sigma\partial_+\sigma\right. \\ &\quad +\beta\,(y\,\beta + v\,x\,\rho - u\,y\,\rho - v\,\beta\,\rho)\,\partial_-\sigma\partial_+\rho \\ &\quad \left.-\beta\,(-u\,\beta + v\,x\,\rho - u\,y\,\rho + v\,\beta\,\rho)\,\partial_-\rho\partial_+\sigma + v\,\beta^2\,\partial_-\rho\partial_+\rho\right] . \end{aligned} \tag{26}$$

By rescaling $E(e) \mapsto E(e)/\beta,\ \mathcal{L} \mapsto \mathcal{L}\beta, \tilde{\mathcal{L}} \mapsto \tilde{\mathcal{L}}/\beta$ we obtain the $GL(2|\mathbb{R})$ model found in the work.[2] It means that even though we have a one-parametric class of nonisomorphic Drinfeld doubles of type A the corresponding dual models are equivalent.

4. Conclusions

We have classified the four–dimensional Drinfeld doubles and constructed the Poisson–Lie T–dual models on them. The investigation of the Drinfeld doubles showed explicitly that neither the subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ *per se* specify the Drinfeld double completely (viz. (9) vs. (10)) nor the Drinfeld double fixes the subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ uniquely (viz. (8) and (10)). It turned out that besides the pair of dual models on $GL(2|\mathbb{R})$ presented in Ref. 2 and the trivial abelian models, there exist two pairs of dual models (21), (22) and (23), (24) on the semiabelian double (8). This is the simplest (and the only one known to the authors) example of nontrivial modular space of σ-models mutually connected by Poisson–Lie T–duality transformation.

It would be very interesting to find whether any of the semiabelian or nonabelian models is integrable.

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Chapter 5

Classification of 6–dimensional real Manin triples

preprint math.QA/0202209.

In the following preprint we presented the list of all 6–dimensional Manin triples. It is a significant generalization of the paper [15] of M.A. Jafarizadeh and A. Rezaei-Aghdam who have assumed both isotropic subalgebras in the Bianchi form and consequently missed quite a lot of other cases. After its submission to electronic archive, we found that an equivalent classification of 3–dimensional Lie bialgebras was already published by X. Gomez in [16], albeit using different classification of 3-dimensional Lie algebras as a starting point.

This work therefore provides an independent check of the Gomez's work. We compared the results and found that after translating the notations they are equivalent. Consequently, our work was not published.

Nevertheless there are two reasons for including it into this thesis. Firstly, we have used it as a starting point for investigation of Drinfeld doubles (Chapter 6) and secondly the classification was performed using different, more straightforward, if more computationally demanding, method.

Our method basically starts from the first subalgebra of the Manin triple written in a basis with fixed commutation relations (the so–called Bianchi form), then gradually solving the Jacobi identities between the subalgebras and in the second subalgebra itself and finally using the remaining freedom in the choice of Bianchi basis of the first subalgebra to write the second subalgebra in the simplest possible form. (Gomez had used more theoretical approach, using the notion of twisting etc.) The intermediate results therefore contain also general commutation relations of any basis of the second subalgebra dual to the original Bianchi-type one. These intermediate results might be of interest in some applications.

Classification of 6–dimensional real Manin triples

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March 27, 2002

Abstract

We present a complete list of 6–dimensional Manin triples or, equivalently, of 3–dimensional Lie bialgebras. We start from the well known classification of 3–dimensional real Lie algebras and assume the canonical bilinear form on the 6-dimensional Drinfeld double. Then we solve the Jacobi identities for the dual algebras. Finally we find mutually non–isomorphic Manin triples. The complete list consists of 78 classes of Manin triples, or 44 Lie bialgebras if one considers dual bialgebras equivalent.

PACS codes: 02.20.Qs, 02.20.Sv, 11.25.Hf

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1 Introduction

In recent years, the study of T–duality in string theory has led to discovery of Poisson–Lie T–dual sigma models. Klimčík and Ševera have found a procedure allowing to construct such models from a given Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$, i.e. a decomposition of a Lie algebra $\mathcal{D}$ into two maximally isotropic subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$. The construction of the Poisson–Lie T–dual sigma models is described in [1] and [2]. The models have target spaces in the Lie groups G and $\tilde{G}$ and are defined by the Lagrangians

$$\mathcal{L} = E_{ij}(g)(g^{-1}\partial_- g)^i (g^{-1}\partial_+ g)^j \tag{1}$$

$$\tilde{\mathcal{L}} = \tilde{E}_{ij}(\tilde{g})(\tilde{g}^{-1}\partial_- \tilde{g})^i (\tilde{g}^{-1}\partial_+ \tilde{g})^j \tag{2}$$

where the matrices $E(g)$ and $\tilde{E}(\tilde{g})$ are constructed from a constant invertible matrix $E(e)$ by virtue of the adjoint representation of the group G resp $\tilde{G}$ on $\mathcal{D}$. It implies that any pair of Poisson–Lie T–dual sigma models is given (up to the constant matrix E) by the corresponding Manin triple and that's why it is interesting and useful to classify these structures.

One can easily see that the dimension of the Lie algebra $\mathcal{D}$ must be even. In the dimension two $\mathcal{D}$ must be abelian and there is just one Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}}) \equiv (\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ where $dim\mathcal{G} = dim\tilde{\mathcal{G}} = 1$. The classification of Manin triples for the four–dimensional Lie algebras together with the pairs of dual models was given in [3]. In this paper we are going to classify the Manin triples of the 6–dimensional real Lie algebras.

Important steps in this direction were made in [4] where a list of possible maximally isotropic subalgebras of the 6–dimensional Lie algebras can be found. It turns out that the subalgebras don't specify the Manin triple completely. For certain algebras there exist several rather different possible pairings, allowing to construct different Manin triples. In the present paper, we present a complete list of real 6–dimensional Manin triples, i.e. we give not only the possible subalgebras, but also the corresponding ad–invariant form (i.e we write dual bases of the algebras with respect to this form and their Lie brackets). The complex solvable Manin triples were classified in [5].

As we shall see Manin triples are equivalent to Lie bialgebras and the classification of the three–dimensional Lie bialgebras (i.e. six–dimensional Manin triples) was given in [6]. Our classification was done independently without knowledge of [6]. The consequent comparison proved that the results are identical even though we have started from a different description of the three–dimensional algebras and used a completely different method. It means that the present work can be considered as an independent check of [6] with the results expressed in a different form, namely as Manin triples.

In the following sections, we firstly recall the definitions of Manin triple, Drinfeld double and Lie bialgebra, then briefly explain the approach we have used to find all algebras of 6–dimensional Drinfeld doubles, and finally give a complete list of all 6–dimensional Manin triples.

2 Manin triples, Drinfeld doubles, Lie bialgebras

The Drinfeld double D is defined as a Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad–invariant nondegenerate bilinear form $\langle .,. \rangle$ can be decomposed into a pair of maximally isotropic subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ such that $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. This ordered triple of algebras $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ is called Manin triple.

One can see that the dimensions of the subalgebras are equal and that bases $\{X_i\}, \{\tilde{X}_i\}$ in the subalgebras can be chosen so that

$$\langle X_i, X_j \rangle = 0,\ \langle X_i, \tilde{X}^j \rangle = \langle \tilde{X}^j, X_i \rangle = \delta_i^j,\ \langle \tilde{X}^i, \tilde{X}^j \rangle = 0. \tag{3}$$

This canonical form of the bracket is invariant with respect to the transformations

$$X_i' = X_k A_i^k,\ \tilde{X}'^j = (A^{-1})_k^j \tilde{X}^k. \tag{4}$$

Due to the ad-invariance of $\langle .,. \rangle$ the algebraic structure of $\mathcal{D}$ is

$$[X_i, X_j] = f_{ij}{}^k X_k,\ [\tilde{X}^i, \tilde{X}^j] = \tilde{f^{ij}}{}_k \tilde{X}^k,$$
$$[X_i, \tilde{X}^j] = f_{ki}{}^j \tilde{X}^k + \tilde{f^{jk}}{}_i X_k. \tag{5}$$

It is clear that to any Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ one can construct the dual one by interchanging $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$, i.e. interchanging the structure constants $f_{ij}{}^k \leftrightarrow \tilde{f^{ij}}{}_k$. All properties of Lie algebras (the nontrivial being the Jacobi identities) remain to be satisfied. On the other hand for given Drinfeld double more than two Manin triples can exist.

One can rewrite the structure of a Manin triple also in another, equivalent, but for certain considerations more suitable, form of Lie bialgebra.

A Lie bialgebra is a Lie algebra g equipped also by a Lie cobracket[1] $\delta : g \to g \otimes g : \delta(x) = \sum x_{[1]} \otimes x_{[2]}$ such that

$$\sum x_{[1]} \otimes x_{[2]} \quad = \quad -\sum x_{[2]} \otimes x_{[1]}, \tag{6}$$

[1]Summation index is suppressed

$$\begin{aligned}(id\otimes\delta)\circ\delta(x) &+ \text{cyclic permutations of tensor indices} = 0, \quad (7)\\ \delta([x,y]) &= \sum [x,y_{[1]}]\otimes y_{[2]} + y_{[1]}\otimes[x,y_{[2]}] - \\ &- [y,x_{[1]}]\otimes x_{[2]} - x_{[1]}\otimes[y,x_{[2]}] \quad (8)\end{aligned}$$

(for detailed account on Lie bialgebras see e.g. [7] or [8], Chapter 8).

The correspondence between a Manin triple and a Lie bialgebra can now be formulated in the following way. Because both subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ of the Manin triple are of the same dimension and are connected by nondegenerate pairing, it is natural to consider $\tilde{\mathcal{G}}$ as a dual $\mathcal{G}^*$ to $\mathcal{G}$ and to use the Lie bracket in $\tilde{\mathcal{G}}$ to define the Lie cobracket in $\mathcal{G}$; $\delta(x)$ is given by $\langle\delta(x),\tilde{y}\otimes\tilde{z}\rangle = \langle x,[\tilde{y},\tilde{z}]\rangle$, $\forall\tilde{y},\tilde{z}\in\mathcal{G}^*$, i.e. $\delta(X_i) = \tilde{f}_i^{jk}X_j\otimes X_k$. The Jacobi identities in $\tilde{\mathcal{G}}$

$$\tilde{f}^{kl}_m\tilde{f}^{ij}_l + \tilde{f}^{il}_m\tilde{f}^{jk}_l + \tilde{f}^{jl}_m\tilde{f}^{ki}_l = 0 \quad (9)$$

are then equivalent to the property of cobracket (7) and the $\tilde{\mathcal{G}}$–component of the mixed Jacobi identities [2]

$$\tilde{f^{jk}}_l f_{mi}{}^l + \tilde{f^{kl}}_m f_{li}{}^j + \tilde{f^{jl}}_i f_{lm}{}^k + \tilde{f^{jl}}_m f_{il}{}^k + \tilde{f^{kl}}_i f_{lm}{}^j = 0 \quad (10)$$

are equivalent to (8).

From now on, we will use the formulation in terms of Manin triples, Lie bialgebra formulation of all results can be easily derived from it. We also consider only algebraic structure, the Drinfeld doubles as the Lie groups can be obtained in principle by means of exponential map and usual theorems about relation between Lie groups and Lie algebras apply, e.g. there is a one to one correspondence between (finite–dimensional) Lie algebras and connected and simply connected Lie groups. The group structure of the Drinfeld double can be deduced e.g. by taking matrix exponential of adjoint representation of its algebra.

3 Method of classification

In this section we present the approach we have used to find all 6–dimensional Manin triples, i.e. 3–dimensional Lie bialgebras.

Starting point for our computations is the well known classification of 3–dimensional real Lie algebras (see e.g. [9] or [4]). Non–isomorphic Lie algebras are written in 11 classes, traditionally known as Bianchi algebras. Their commutation relations are:

$$[X_1,X_2] = -aX_2 + n_3X_3,\ [X_2,X_3] = n_1X_1,\ [X_3,X_1] = n_2X_2 + aX_3, \quad (11)$$

[2]The Jacobi identities $[X_i,[\tilde{X}^j,\tilde{X}^k]]+\text{cyclic} = 0$ lead to both (10) (terms proportional to $\tilde{X}^l$) and (9) (terms proportional to X_l).

where the parameters a, n_1, n_2, n_3 have the following values

Class	a	n_1	n_2	n_3
I	0	0	0	0
II	0	1	0	0
VII_0	0	1	1	0
VI_0	0	1	-1	0
IX	0	1	1	1
$VIII$	0	1	1	-1
V	1	0	0	0
IV	1	0	0	1
$VII_a\,(a > 0)$	a	0	1	1
III	1	0	1	-1
$VI_a\,(a > 0, a \neq 1)$	a	0	1	-1

Therefore the 1st subalgebra $\mathcal{G}$ of the Manin triple $\mathcal{D}$ must be one of the Bianchi algebras given above and we can choose its basis so that the Lie brackets are of the form (11). In the 2nd subalgebra $\tilde{\mathcal{G}}$ we choose the dual basis $\tilde{X}^i$ so that (3) holds, and treat nine independent components of structure coefficients $\tilde{f}^{ij}_k$ of the 2nd subalgebra $\tilde{\mathcal{G}}$ in the basis $\tilde{X}^i$ as unknowns. We cannot assume that the $\tilde{f}^{ij}_k$ are of the form (11) as well because it can be incompatible with (3). Then we solve the mixed Jacobi identities (10) (these relations form a system of linear equations in $\tilde{f}^{ij}_k$) and the Jacobi identities for the dual algebra (9) (i.e. quadratic in $\tilde{f}^{ij}_k$).

As a result, we have found all structure coefficients of $\tilde{\mathcal{G}}$ consistent with the definition of Manin triple and the next step was to determine the Bianchi classes of obtained algebras $\tilde{\mathcal{G}}$. Finally we have found the the non–isomorphic Manin triples by considering Manin triples connected by the transformations (4) (i.e. change of basis in $\mathcal{G}$ accompanied by the dual change of basis in $\tilde{\mathcal{G}}$ with respect to $\langle , \rangle$) as equivalent and choosing one representant in each equivalence class.

In computations computer algebra systems Maple V and Mathematica 4 were independently used for manipulating expressions and solving sets of linear and quadratic equations, their results were checked one against the other.

Before listing our results, we shall give an example showing the progress of computation in some detail.

Example: Let us consider the algebra $VIII$, i.e. $\mathcal{G} = sl(2, \Re)$.

$$[X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

When one solves the mixed Jacobi identities (10), he finds that the 2nd subalgebra must have the form

$$[\tilde{X}^1, \tilde{X}^2] = -\alpha\tilde{X}^1 + \beta\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \gamma\tilde{X}^2 + \alpha\tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\gamma\tilde{X}^1 - \beta\tilde{X}^3.$$

The Jacobi identities in the 2nd subalgebra (9) in this case don't impose any further condition on the structure constants $\tilde{f^{ij}}_k$, i.e. we have already found the structure of all possible 2nd subalgebras $\tilde{\mathcal{G}}$ in the Manin triple.

Next we find the Bianchi forms of $\tilde{\mathcal{G}}$. It turns out that the 2nd algebra is of the Bianchi type I ($\tilde{f^{ij}_k} = 0$) if $\alpha = \beta = \gamma = 0$ and of type V otherwise.

Then we find values of $\tilde{f^{ij}}_k$ that allow transformation (4) leading to the rescaled Bianchi form V of the 2nd subalgebra $\tilde{\mathcal{G}}$ and leaving the Bianchi form of the 1st subalgebra $sl(2,\Re)$ invariant. This is possible only for

$$\alpha^2 + \beta^2 - \gamma^2 > 0$$

(for $\alpha^2 + \beta^2 - \gamma^2 < 0$ the transformation matrix would be complex, not real, for $\alpha^2 + \beta^2 - \gamma^2 = 0$ it would be singular). Therefore we have in the case $\alpha^2 + \beta^2 - \gamma^2 > 0$ a one–parametric set of non–equivalent Manin triples

$$[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0$$

and we must find representants of remaining classes of possible Manin triples. We choose the forms

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1,\ b > 0$$

for $\alpha^2 + \beta^2 - \gamma^2 < 0$ and

$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -(\tilde{X}^1 + \tilde{X}^3)$$

for $\alpha^2 + \beta^2 - \gamma^2 = 0$, $\alpha \neq 0 \vee \beta \neq 0 \vee \gamma \neq 0$ and easily verify that every possible 2nd subalgebra $\tilde{\mathcal{G}}$ can be taken to one of the given forms by transformation (4) which doesn't change the structure constants of the 1st subalgebra $\mathcal{G} = sl(2,\Re)$.

Details of computations for each Bianchi algebra are given in the Appendix.

4 Results: 6–dimensional Manin triples

The forms of non–equivalent Manin triples were choosen according to the following criteria: The 1st subalgebra is in the Bianchi form, the 2nd is in the form closest to Bianchi, i.e. Bianchi form if possible, or the structure constants are multiple of the Bianchi ones, or form a permution of the Bianchi ones, or, if neither is possible, are choosen to be as many zeros and small integers as possible.

In order to shorten the list, we have not explicitly written out the structure of algebras that can be found by the duality transform $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$ from the ones given in the list.

1. Dual algebras to Bianchi algebra IX:

   $$[X_1, X_2] = X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

   Dual algebras:

   (a) Bianchi algebra I

   $$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

   (b) Bianchi algebra V

   $$[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0.$$

2. Dual algebras to Bianchi algebra $VIII$:

   $$[X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

   Dual algebras:

   (a) Bianchi algebra I

   $$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

   (b) Bianchi algebra V
      i. $[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0.$
      ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1,\ b > 0.$
      iii. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -(\tilde{X}^1 + \tilde{X}^3).$

3. Dual algebras to Bianchi algebra VII_a:

   $$[X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_2 + aX_3,\ a > 0.$$

   Dual algebras:

   (a) Bianchi algebra I

   $$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

   (b) Bianchi algebra II
      i. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$
      ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

(c) Bianchi algebra $VII_{1/a}$

$[\tilde{X}^1,\tilde{X}^2]=b(-\frac{1}{a}\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3]=0,$
$[\tilde{X}^3,\tilde{X}^1]=b(\tilde{X}^2+\frac{1}{a}\tilde{X}^3),\ b\in\Re-\{0\}.$

4. Dual algebras to Bianchi algebra VII_0:

$$[X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$

ii. $[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$

(c) Bianchi algebra IV

$$[\tilde{X}^1,\tilde{X}^2]=b(-\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=b\tilde{X}^3,\ b\in\Re-\{0\}.$$

(d) Bianchi algebra V

i. $[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^2,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=\tilde{X}^3,.$

ii. $[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=b\tilde{X}^2,\ [\tilde{X}^3,\tilde{X}^1]=-b\tilde{X}^1,\ b>0.$

5. Dual algebras to Bianchi algebra VI_a:

$$[X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0,\ [X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$$

(b) Bianchi algebra II

$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=\tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^1]=0.$$

(c) Bianchi algebra $VI_{1/a}$

i. $[\tilde{X}^1,\tilde{X}^2]=-b(\frac{1}{a}\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=b(\tilde{X}^2+\frac{1}{a}\tilde{X}^3),\ b\in\Re-\{0\}.$

ii. $[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^3]=\frac{a+1}{a-1}(\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^3,\tilde{X}^1]=\tilde{X}^1.$

iii. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1,\ [\tilde{X}^2, \tilde{X}^3] = \frac{a-1}{a+1}(-\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^1.$

6. Dual algebras to Bianchi algebra VI_0:

$$[X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(c) Bianchi algebra IV

i. $[\tilde{X}^1, \tilde{X}^2] = b(-\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b \in \Re - \{0\}.$

ii. $[\tilde{X}^1, \tilde{X}^2] = (-\tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3.$

(d) Bianchi algebra V

i. $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^3.$

ii. $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^1 + \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3.$

iii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^1.$

7. Dual algebras to Bianchi algebra V:

$$[X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

ii. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

and dual algebras ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to algebras given above for VI_0, VII_0, $VIII$, IX.

8. Dual algebras to Bianchi algebra IV:

$$[X_1, X_2] = -X_2 + X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

iii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^2,\ b \in \Re - \{0\}.$

and dual algebras ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to algebras given above for VI_0, VII_0.

9. Dual algebras to Bianchi algebra III:

$$[X_1, X_2] = -X_2 - X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_2 + X_3.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(c) Bianchi algebra III

i. $[\tilde{X}^1, \tilde{X}^2] = -b(\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \tilde{X}^3),\ b \in \Re - \{0\}.$

ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2 + \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

iii. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^1.$

10. Dual algebras to Bianchi algebra II:

$$[X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = 0.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

ii. $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

and dual algebras ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to algebras given above for III, IV, VI_0, VI_a, VII_0, VII_a.

11. Dual algebras to Bianchi algebra I:

$$[X_1, X_2] = 0,\ [X_2, X_3] = 0,\ [X_3, X_1] = 0.$$

Dual algebras: all Bianchi algebras (in their Bianchi forms)

5 Conclusions

We have classified 6–dimensional Manin triples or, equivalently, 3–dimensional Lie bialgebras. In computations computer algebra systems Maple V and Mathematica 4 were used for solving the sets of linear and quadratic equations that follow from the Jacobi identities and similarity transformations. The results were calculated independently in both systems and afterwards were checked one against the other. The complete list consists of 78 classes of Manin triples (if one considers dual Lie bialgebras equivalent, then the count is 44). An open problem that remains is detecting the Manin triples that belong to the same Drinfeld double or, in other words, the classification of the 6-dimensional Drinfeld doubles.

One of interesting results is the number of possible Lie bialgebra structures for the algebra $VIII$, i.e. $sl(2,\Re)$. In this case there are up to rescaling 3 non–equivalent Manin triples. As mentioned in the Introduction, to every Manin triple correspond a pair of Poisson–Lie T–dual models. Therefore, there should exist 3 different pairs of non-abelian Poisson-Lie T-dual models for $sl(2,\Re)$. Only one of them appeared in the literature so far [10]. There is a natural question whether these models are equivalent (i.e. whether they correspond to the decomposition of one Drinfeld double) and if they lead after quantisation to the same quantum model.

Appendix: Most general form of $\tilde{\mathcal{G}}$ of Manin triple with given $\mathcal{G}$

In this Appendix we present our computations in some detail. For each Bianchi algebra we give solutions of the mixed Jacobi identities (10), i.e. linear equations in $\tilde{f}$, the remaining non–trivial Jacobi identities in $\tilde{\mathcal{G}}$ (9), i.e. in general quadratic equations in $\tilde{f}$ and their solutions, in general depending on several parameters $\alpha, \beta, \ldots$ Finally we specify values of parameters allowing transformation (4) of $\tilde{\mathcal{G}}$ into forms of $\tilde{\mathcal{G}}$ given in the list of non–isomorphic Manin triples.

- $\mathcal{G} = IX$

  The mixed Jacobi identities (10) imply

  $$\tilde{f^{23}}_3 = -\tilde{f^{12}}_1, \tilde{f^{23}}_2 = \tilde{f^{13}}_1, \tilde{f^{13}}_3 = \tilde{f^{12}}_2, \tilde{f^{23}}_1 = 0, \tilde{f^{12}}_3 = 0, \tilde{f^{13}}_2 = 0.$$

  The Jacobi identities in $\tilde{\mathcal{G}}$ (9) in this case don't impose any new condition. The general form of $\tilde{\mathcal{G}}$ is therefore

  $$[\tilde{X}^1, \tilde{X}^2] = \alpha\tilde{X}^1 + \beta\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \gamma\tilde{X}^2 - \alpha\tilde{X}^3,$$

$$[\tilde{X}^3, \tilde{X}^1] = -\gamma\tilde{X}^1 - \beta\tilde{X}^3$$

$\tilde{\mathcal{G}}$ can be transformed into

  - Bianchi algebra I in the standard form IX (a) if $\alpha = \beta = \gamma = 0$,
  - Bianchi algebra V in the rescaled standard form IX (b) with $b = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$ otherwise.

- $\mathcal{G} = VIII$
  The mixed Jacobi identities (10) imply

  $$\tilde{f^{12}}_1 = -\tilde{f^{23}}_3, \tilde{f^{13}}_1 = \tilde{f^{23}}_2, \tilde{f^{12}}_2 = \tilde{f^{13}}_3, \tilde{f^{23}}_1 = 0, \tilde{f^{13}}_2 = 0, \tilde{f^{12}}_3 = 0.$$

  The Jacobi identities in $\tilde{\mathcal{G}}$ (9) in this case don't impose any new condition. The general form of $\tilde{\mathcal{G}}$ is therefore

  $$[\tilde{X}^1, \tilde{X}^2] = -\alpha\tilde{X}^1 + \beta\tilde{X}^2, \ [\tilde{X}^2, \tilde{X}^3] = \gamma\tilde{X}^2 + \alpha\tilde{X}^3,$$
  $$[\tilde{X}^3, \tilde{X}^1] = -\gamma\tilde{X}^1 - \beta\tilde{X}^3.$$

  $\tilde{\mathcal{G}}$ can be transformed into

  - Bianchi algebra I in the standard form $VIII$ (a) if $\alpha = \beta = \gamma = 0$,
  - Bianchi algebra V
    * in the rescaled standard form $VIII$ (b) i. with $b = \sqrt{\alpha^2 + \beta^2 - \gamma^2}$ if $\alpha^2 + \beta^2 - \gamma^2 > 0$,
    * in the form $VIII$ (b) ii. with $b = \sqrt{-(\alpha^2 + \beta^2 - \gamma^2)}$ if $\alpha^2 + \beta^2 - \gamma^2 < 0$,
    * in the form $VIII$ (b) iii. if $\alpha^2 + \beta^2 - \gamma^2 = 0$, and $\alpha \neq 0 \vee \beta \neq 0 \vee \gamma \neq 0$[3].

- $\mathcal{G} = VII_a$
  The mixed Jacobi identities (10) imply

  $$\tilde{f^{13}}_2 = a\tilde{f^{13}}_3, \tilde{f^{12}}_2 = \tilde{f^{13}}_3, \tilde{f^{23}}_3 = -\frac{a^2\tilde{f^{23}}_2 + a^2\tilde{f^{13}}_1 - \tilde{f^{23}}_2 + \tilde{f^{13}}_1}{2a},$$
  $$\tilde{f^{12}}_1 = -\frac{a^2\tilde{f^{23}}_2 + a^2\tilde{f^{13}}_1 + \tilde{f^{23}}_2 - \tilde{f^{13}}_1}{2a}, \tilde{f^{12}}_3 = -a\tilde{f^{13}}_3.$$

  The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

  $$4a\tilde{f^{23}}_1\tilde{f^{13}}_3 + (a\tilde{f^{23}}_2)^2 + 2a^2\tilde{f^{23}}_2\tilde{f^{13}}_1 + (\tilde{f^{23}}_2)^2 - 2\tilde{f^{23}}_2\tilde{f^{13}}_1 +$$
  $$+ (a\tilde{f^{13}}_1)^2 + (\tilde{f^{13}}_1)^2 = 0.$$

  The solutions of this equation give the following general forms of $\tilde{\mathcal{G}}$:

[3] In order to avoid abundant parentheses, logical conjuctions written in terms of symbols are considered with higher priority than that written by words and, or.

1.

$$
\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\frac{1}{2a}(a^2\alpha + \beta a^2 + \alpha - \beta)\tilde{X}^1 + \gamma\tilde{X}^2 - \gamma a\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= -\frac{1}{4\gamma a}(a^2\alpha^2 + 2\alpha\beta a^2 + \alpha^2 - 2\alpha\beta + \beta^2 a^2 + \beta^2)\tilde{X}^1 \\
&\quad +\alpha\tilde{X}^2 - \frac{1}{2a}(a^2\alpha + \beta a^2 - \alpha + \beta)\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^1 - \gamma a\tilde{X}^2 - \gamma\tilde{X}^3.
\end{aligned}
$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra $VII_{1/a}$ in the rescaled standard form VII_a (c) with $b = -a\gamma$.

2. $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = \alpha\tilde{X}^1$, $[\tilde{X}^3, \tilde{X}^1] = 0$.
$\tilde{\mathcal{G}}$ can be transformed into
   - Bianchi algebra I in the standard form VII_a (a) if $\alpha = 0$,
   - Bianchi algebra II
     * in the standard form VII_a (b) i. if $\alpha > 0$,
     * in the form VII_a (b) ii. if $\alpha < 0$.

- $\mathcal{G} = VII_0$
The mixed Jacobi identities (10) imply

$$\tilde{f^{12}}_1 = -\tilde{f^{23}}_3, \tilde{f^{12}}_2 = \tilde{f^{13}}_3, \tilde{f^{23}}_2 = \tilde{f^{13}}_1, \tilde{f^{13}}_2 = 0, \tilde{f^{23}}_1 = 0.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$\tilde{f^{12}}_3\tilde{f^{13}}_1 = 0.$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$
\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha\tilde{X}^1 + \beta\tilde{X}^2 + \gamma\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^3.
\end{aligned}
$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VII_0 (a) if $\gamma = \beta = \alpha = 0$,
- Bianchi algebra II
  * in the form VII_0 (b) i. if $\gamma > 0$ and $\beta = \alpha = 0$,
  * in the form VII_0 (b) ii. if $\gamma < 0$ and $\beta = \alpha = 0$,

- Bianchi algebra IV in the rescaled standard form VII_0 (c) with $b = -\frac{\beta^2+\alpha^2}{\gamma}$ if $\gamma \neq 0$ and $\beta \neq 0 \vee \alpha \neq 0$,
- Bianchi algebra V in the standard form VII_0 (d) i. with if $\gamma = 0$ and $\beta \neq 0 \vee \alpha \neq 0$.

2.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha\tilde{X}^1 + \beta\tilde{X}^2, \\
[\tilde{X}^2, \tilde{X}^3] &= \gamma\tilde{X}^2 + \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\gamma\tilde{X}^1 - \beta\tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VII_0 (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra V
  - in the standard form VII_0 (d) i. if $\gamma = 0$,
  - in the form VII_0 (d) ii. with $b = |\gamma|$ if $\gamma \neq 0$.

- $\mathcal{G} = VI_a$

The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_1 = -\frac{-a^2\tilde{f^{12}}_1 + a^2\tilde{f^{23}}_3 - \tilde{f^{23}}_3 - \tilde{f^{12}}_1}{2a}, \tilde{f^{12}}_3 = a\tilde{f^{12}}_2, \tilde{f^{13}}_3 = \tilde{f^{12}}_2,$$

$$\tilde{f^{13}}_2 = a\tilde{f^{12}}_2, \tilde{f^{23}}_2 = \frac{-a^2\tilde{f^{12}}_1 + a^2\tilde{f^{23}}_3 + \tilde{f^{23}}_3 + \tilde{f^{12}}_1}{2a}.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$4a\tilde{f^{23}}_1\tilde{f^{12}}_2 + (a\tilde{f^{12}}_1)^2 - 2a^2\tilde{f^{12}}_1\tilde{f^{23}}_3 - 2\tilde{f^{12}}_1\tilde{f^{23}}_3 - (\tilde{f^{12}}_1)^2 +$$
$$+(a\tilde{f^{23}}_3)^2 - (\tilde{f^{23}}_3)^2 = 0.$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha\tilde{X}^1 + \beta\tilde{X}^2 + a\beta\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= -\frac{a^2\alpha^2 - 2\alpha\gamma a^2 - 2\alpha\gamma - \alpha^2 + \gamma^2 a^2 - \gamma^2}{4a\beta}\tilde{X}^1 \\
&\quad + \frac{(-a^2\alpha + \gamma a^2 + \gamma + \alpha)}{2a}\tilde{X}^2 + \gamma\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= \frac{-a^2\alpha + \gamma a^2 - \gamma - \alpha}{2a}\tilde{X}^1 - a\beta\tilde{X}^2 - \beta\tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra $VI_{1/a}$ in the rescaled standard form VI_a (c) with $b = -a\beta$.

2.
$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha \tilde{X}^1, \\
[\tilde{X}^2, \tilde{X}^3] &= \beta \tilde{X}^1 + \alpha \frac{a+1}{a-1} \tilde{X}^2 + \alpha \frac{a+1}{a-1} \tilde{X}^1, \\
[\tilde{X}^3, \tilde{X}^1] &= \alpha \tilde{X}^1.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VI_a (a) if $\alpha = \beta = 0$,
- Bianchi algebra II in the standard form VI_a (b) if $\alpha = 0$ and $\beta \neq 0$.
- Bianchi algebra $VI_{1/a}$ in the form VI_a (c) ii. if $\alpha \neq 0$.

3.
$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha \tilde{X}^1, \\
[\tilde{X}^2, \tilde{X}^3] &= \beta \tilde{X}^1 - \alpha \frac{a-1}{a+1} \tilde{X}^2 + \alpha \frac{a-1}{a+1} \tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\alpha \tilde{X}^1.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VI_a (a) if $\alpha = \beta = 0$,
- Bianchi algebra II in the standard form VI_a (b) if $\alpha = 0$ and $\beta \neq 0$.
- Bianchi algebra $VI_{1/a}$ in the form VI_a (c) ii. if $\alpha \neq 0$.

- $\mathcal{G} = VI_0$

The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_3 = \tilde{f^{12}}_2, \tilde{f^{13}}_1 = \tilde{f^{23}}_2, \tilde{f^{12}}_1 = -\tilde{f^{23}}_3, \tilde{f^{13}}_2 = 0, \tilde{f^{23}}_1 = 0.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$\tilde{f^{12}}_3 \tilde{f^{23}}_2 = 0.$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.
$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha \tilde{X}^1 + \beta \tilde{X}^2 + \gamma \tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha \tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta \tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VI_0 (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra II in the form VI_0 (b) if $\gamma \neq 0$ and $\alpha = \beta = 0$,
- Bianchi algebra IV
  - in the rescaled standard form VI_0 (c) i. with $b = \frac{\alpha^2-\beta^2}{\gamma}$ if $\gamma \neq 0$ and $\alpha^2 \neq \beta^2$,
  - in the form VI_0 (c) ii. with if $\gamma \neq 0$ and $\alpha^2 = \beta^2 \neq 0$,
- Bianchi algebra V
  - in the standard form VI_0 (d) i. if $\gamma = 0$ and $\alpha^2 \neq \beta^2$,
  - in the form VI_0 (d) ii. if $\gamma = 0$ and $\alpha^2 = \beta^2 \neq 0$,

2.

$$\begin{array}{rcl}[\tilde{X}^1, \tilde{X}^2] &=& -\alpha\tilde{X}^1 + \beta\tilde{X}^2, \\ [\tilde{X}^2, \tilde{X}^3] &=& \gamma\tilde{X}^2 + \alpha\tilde{X}^3, \\ [\tilde{X}^3, \tilde{X}^1] &=& -\gamma\tilde{X}^1 - \beta\tilde{X}^3.\end{array}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VI_0 (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra V
  - in the form VI_0 (d) i. if $\gamma = 0$ and $\alpha^2 \neq \beta^2$,
  - in the form VI_0 (d) ii. if $\gamma = 0$ and $\alpha^2 = \beta^2$,
  - in the form VI_0 (d) iii. with $b = |\gamma|$ if $\gamma \neq 0$.

- $\mathcal{G} = V$

The mixed Jacobi identities (10) imply

$$\tilde{f^{12}}_1 = \tilde{f^{23}}_3, \tilde{f^{13}}_3 = -\tilde{f^{12}}_2, \tilde{f^{23}}_2 = -\tilde{f^{13}}_1.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) in this case don't impose any new condition. The general form of $\tilde{\mathcal{G}}$ is therefore

$$\begin{array}{rcl}[\tilde{X}^1, \tilde{X}^2] &=& \alpha\tilde{X}^1 + \beta\tilde{X}^2 + \gamma\tilde{X}^3, \\ [\tilde{X}^2, \tilde{X}^3] &=& \delta\tilde{X}^1 - \epsilon\tilde{X}^2 + \alpha\tilde{X}^3, \\ [\tilde{X}^3, \tilde{X}^1] &=& -\epsilon\tilde{X}^1 - \zeta\tilde{X}^2 + \beta\tilde{X}^3.\end{array}$$

Finding the Bianchi forms of this algebra for all values of parameters seems to be rather complicated, because this case contains also all 2nd subalgebras of duals of Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ with $\tilde{\mathcal{G}} \equiv V$ given above. Therefore we compute only the values of parameters for which $\tilde{\mathcal{G}}$ is isomorphic to $I, \ldots, V$. We find that $\tilde{\mathcal{G}}$ can be transformed into[4]

- Bianchi algebra I in the standard form V (a) if $\alpha = \beta = \gamma = \delta = \epsilon = \zeta = 0$,
- Bianchi algebra II
  - in the form V (b) i. if
    - $\exists x, y$ s.t. $\alpha = x\gamma, \beta = y\gamma, \epsilon = -xy\gamma, \zeta = -y^2\gamma, \delta = x^2\gamma, \gamma \neq 0$
    - or $\alpha = \beta = \gamma = 0$ and $\exists x$ s.t. $\epsilon = -x\delta, \zeta = -x^2\delta, x \neq 0, \delta \neq 0$
    - or $\alpha = \beta = \gamma = \delta = \epsilon = 0, \zeta \neq 0$,
  - in the form V (b) ii. if $\alpha = \beta = \gamma = \epsilon = \zeta = 0$ and $\delta \neq 0$.

- $\mathcal{G} = IV$
  The mixed Jacobi identities (10) imply

$$\tilde{f^{12}}_3 = 0, \tilde{f^{12}}_2 = 0, \tilde{f^{23}}_2 = -\tilde{f^{13}}_1 - 2\tilde{f^{12}}_1, \tilde{f^{23}}_3 = \tilde{f^{12}}_1, \tilde{f^{13}}_3 = 0.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$(\tilde{f^{12}}_1)^2 = 0.$$

The solution of this equation gives the most general form of $\tilde{\mathcal{G}}$:

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= 0, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^1 - \beta\tilde{X}^2, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^1 - \gamma\tilde{X}^2.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form IV (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra II
  - in the standard form IV (b) i. if $\gamma = \beta = 0$ and $\alpha > 0$,
  - in the form IV (b) ii. if $\gamma = \beta = 0$ and $\alpha < 0$,
  - in the form IV (b) iii. with $b = -\gamma$ if $\gamma \neq 0$ and $\beta^2 + \alpha\gamma = 0$,
- Bianchi algebra VI_0
  - in the rescaled standard form with $b = \gamma$ if $\gamma \neq 0$ and $\beta^2 + \alpha\gamma > 0$. The corresponding Manin triple is dual to the triple VI_0 (c) i.

[4]It is helpful to exploit the fact that the commutant of II is one–dimensional, i.e. suitably written matrix of structure coefficients has rank 1.

* in the form

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^1$$

if $\gamma = 0$ and $\beta \neq 0$. The corresponding Manin triple is dual to the triple VI_0 (c) ii.

- Bianchi algebra VII_0 in the rescaled standard form with $b = \gamma$ if $\gamma \neq 0$ and $\beta^2 + \alpha\gamma < 0$. The corresponding Manin triple is dual to the triple VII_0 (c) i.

- $\mathcal{G} = III$
The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_3 = \tilde{f^{12}}_2, \tilde{f^{12}}_1 = \tilde{f^{13}}_1, \tilde{f^{12}}_3 = \tilde{f^{12}}_2, \tilde{f^{13}}_2 = \tilde{f^{12}}_2, \tilde{f^{23}}_3 = \tilde{f^{23}}_2.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$\tilde{f^{23}}_1 \tilde{f^{12}}_2 - \tilde{f^{13}}_1 \tilde{f^{23}}_3 = 0.$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$\begin{aligned} [\tilde{X}^1, \tilde{X}^2] &= \alpha\tilde{X}^1 + \beta\tilde{X}^2 + \beta\tilde{X}^3, \\ [\tilde{X}^2, \tilde{X}^3] &= \frac{\alpha\gamma}{\beta}\tilde{X}^1 + \gamma\tilde{X}^2 + \gamma\tilde{X}^3, \\ [\tilde{X}^3, \tilde{X}^1] &= -\alpha\tilde{X}^1 - \beta\tilde{X}^2 - \beta\tilde{X}^3. \end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra III in the rescaled standard form III (c) i. with $b = 1/\beta$.

2.

$$\begin{aligned} [\tilde{X}^1, \tilde{X}^2] &= 0, \\ [\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^1 + \beta\tilde{X}^2 + \beta\tilde{X}^3, \\ [\tilde{X}^3, \tilde{X}^1] &= 0. \end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form III (a) if $\alpha = \beta = 0$,
- Bianchi algebra II in the form III (b) i. if $\beta = 0$ and $\alpha \neq 0$,
- Bianchi algebra III in the form III (c) ii. if $\beta \neq 0$.

3.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha \tilde{X}^1, \\
[\tilde{X}^2, \tilde{X}^3] &= \beta \tilde{X}^1, \\
[\tilde{X}^3, \tilde{X}^1] &= -\alpha \tilde{X}^1.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form III (a) if $\alpha = \beta = 0$,
- Bianchi algebra II in the form III (b) i. if $\alpha = 0$ and $\beta \neq 0$,
- Bianchi algebra III in the form III (c) iii. if $\alpha \neq 0$.

- $\mathcal{G} = II$

Finding the Bianchi forms of the 2nd algebra for all values of parameters again seems to be rather complicated, because it contains also all 2nd subalgebras of duals of Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ with $\tilde{\mathcal{G}} \equiv II$ given above. Therefore we compute only the values of parameters for which possible $\tilde{\mathcal{G}}$s are isomorphic to I, II.

The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_1 = \tilde{f^{23}}_2, \tilde{f^{23}}_1 = 0, \tilde{f^{12}}_1 = -\tilde{f^{23}}_3.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$-\tilde{f^{13}}_3 \tilde{f^{23}}_3 + \tilde{f^{23}}_3 \tilde{f^{12}}_2 - 2\tilde{f^{12}}_3 \tilde{f^{23}}_2 = 0,$$

$$-2\tilde{f^{13}}_2 \tilde{f^{23}}_3 - \tilde{f^{12}}_2 \tilde{f^{23}}_2 + \tilde{f^{23}}_2 \tilde{f^{13}}_3 = 0.$$

The solutions of these equations give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha \tilde{X}^1 - \frac{2\beta\alpha - \gamma\delta}{\gamma} \tilde{X}^2 - \frac{\alpha^2 \beta}{\gamma^2} \tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= \gamma \tilde{X}^2 + \alpha \tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\gamma \tilde{X}^1 - \beta \tilde{X}^2 - \delta \tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ of this form represents Bianchi algebras IV, V only.

2.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha \tilde{X}^2 + \beta \tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= 0, \\
[\tilde{X}^3, \tilde{X}^1] &= -\gamma \tilde{X}^2 - \delta \tilde{X}^3.
\end{aligned}$$

- Bianchi algebra I in the standard form V (a) if $\alpha = \beta = \gamma = \delta = 0$,
- Bianchi algebra II
  * in the form II (b) i. if $\exists x : \gamma = -x^2\beta, \delta = -x\beta, \alpha = x\beta, \beta > 0$ or $\delta = \alpha = \beta = 0, \gamma < 0$,
  * in the form II (b) ii. if $\exists x : \gamma = -x^2\beta, \delta = -x\beta, \alpha = x\beta, \beta < 0$ or $\delta = \alpha = \beta = 0, \gamma > 0$,

Bianchi algebras $III, IV, V, VI_a, VI_0, VII_a, VII_0$ otherwise.

3.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha\tilde{X}^1 + \beta\tilde{X}^2 + \gamma\tilde{X}^1, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^3.
\end{aligned}$$

- Bianchi algebra I in the standard form V (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra II
  * in the form II (b) i. if $\alpha = \beta = 0, \gamma > 0$,
  * in the form II (b) ii. if $\alpha = \beta = 0, \gamma < 0$,

Bianchi algebras IV, V otherwise.

- $\mathcal{G} = I$
  $\tilde{\mathcal{G}}$ might be any 3–dimensional Lie algebra, it can be brought to its Bianchi form by the transformation (4).

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Chapter 6

Classification of 6–dimensional real Drinfeld doubles

accepted for publication in International Journal of Modern Physics A.

In this paper we classified all 6–dimensional real Drinfeld doubles, i.e. we identified Manin triples giving rise to the same Drinfeld double.

Our investigation is based on the invariants of the underlying Lie algebra $\mathcal{D}$, i.e. rank of its Killing form and dimensions of derived subalgebras $\mathcal{D}^i$, $\mathcal{D}_i$. These invariants give a coarse sorting into classes of Manin triples that might possibly lead to the same Drinfeld doubles. Manin triples in each of these classes are then studied and the (non)equivalence of the corresponding Drinfeld doubles is rigorously proven.

Interesting conclusion is that not only rather different Manin triples might lead to the same Drinfeld double, but also the same underlying Lie algebra $\mathcal{D}$ may be equipped, e.g. for $\mathcal{D}$ semisimple, with different bilinear forms and define different Drinfeld doubles. [1]

[1] This paper is presented in the form of the proof before publication, the final version is not yet available. Therefore, in case of any doubts please consult the list of corrections at the end.

International Journal of Modern Physics A, Vol. 00, No. 00 (2002) 1–25

CLASSIFICATION OF SIX-DIMENSIONAL REAL DRINFELD DOUBLES

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Received 26 March 2002

Starting from the classification of real Manin triples we look for those that are isomorphic as six-dimensional Drinfeld doubles i.e. Lie algebras with the ad-invariant form used for construction of the Manin triples. We use several invariants of the Lie algebras to distinguish the nonisomorphic structures and give the explicit form of maps between Manin triples that are decompositions of isomorphic Drinfeld doubles. The result is a complete list of six-dimensional real Drinfeld doubles. It consists of 22 classes of nonisomorphic Drinfeld doubles.

Keywords: Drinfeld doubles; Manin triples; Lie algebras; Lie bialgebras; T-duality.

1. Introduction

In recent years, the study of T-duality in string theory has led to the discovery of Poisson–Lie T-dual sigma models. Klimčík and Ševera have found a procedure allowing us to construct the dual models from Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$, i.e. a decompositions of a Lie algebra $\mathcal{D}$ (it must be even-dimensional) into two maximally isotropic subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ w.r.t. a bilinear form. The construction of the Poisson–Lie T-dual sigma models is described in Refs. 1 and 2.

The Lie group possessing a Lie algebra that can be written as a Manin triple is called the Drinfeld double. The classification of the two-dimensional Drinfeld doubles is trivial and the four-dimensional Drinfeld doubles can be found e.g. in the paper Ref. 3 together with the corresponding two-dimensional T-dual models. Examples of six-dimensional Drinfeld doubles and three-dimensional dual models were given e.g. in Refs. 4–6. There was an attempt to classify the six-dimensional Drinfeld doubles by the Bianchi forms of their three-dimensional isotropic subalgebras in Ref. 6 but it is not sufficient for the specification of the Drinfeld double.

As we shall see Manin triples are equivalent to Lie bialgebras and the classification of the three-dimensional Lie bialgebras (i.e. six-dimensional Manin triples) was given in Ref. 7. Without knowledge of this this work we have performed a classification of the six-dimensional Manin triples in Ref. 8. The consequent comparison proved that the results are identical even though we have started from a different description of the three-dimensional algebras and used a completely different method. It means that in Ref. 8 we have done an independent check of Ref. 7 and on the other hand, expressed the results in a different form, namely as Manin triples.

The goal of this paper is to find which of the Manin triples represent decomposition of the same (or more precisely isomorphic) Drinfeld doubles. We use the notation of Ref. 8 because the less compact sorting of the triples into parametrized classes turned out more appropriate for the classification. The result is a complete list of the real nonisomorphic six-dimensional Drinfeld doubles. Let us note that the Drinfeld double is defined not only by its Lie structure but also by a bilinear form. There are e.g. two classes of Drinfeld doubles for so$(1,3)$ as we shall see.

In the following sections, we first recall the definitions of Manin triple, Lie bialgebra and Drinfeld double, then briefly explain the approach we have used to distinguish the nonisomorphic structures. The main result of the paper is the classification theorem in Sec. 3. Explicit forms of maps between Manin triples that are decompositions of the isomorphic Drinfeld doubles are contained in the proof of the theorem.

2. Manin Triples, Lie Bialgebras, Drinfeld Doubles

The Drinfeld double D is defined as a connected Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ can be decomposed into a pair of subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ maximally isotropic w.r.t. $\langle\cdot,\cdot\rangle$ and $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. This ordered triple of algebras $(\mathcal{D},\mathcal{G},\tilde{\mathcal{G}})$ is called Manin triple.

One can see that the dimensions of the subalgebras are equal and that bases $\{X_i\}$, $\{\tilde{X}^i\}$, $i=1,2,3$ in the subalgebras can be chosen so that

$$\langle X_i, X_j\rangle = 0\,, \quad \langle X_i, \tilde{X}^j\rangle = \langle \tilde{X}^j, X_i\rangle = \delta_i^j\,, \quad \langle \tilde{X}^i, \tilde{X}^j\rangle = 0\,. \tag{1}$$

This canonical form of the bracket is invariant with respect to the transformations

$$X'_i = X_k A_i^k\,, \qquad \tilde{X}'^j = (A^{-1})_k^j \tilde{X}^k\,. \tag{2}$$

The Manin triples that are related by the transformation (2) are considered isomorphic. Due to the ad-invariance of $\langle\cdot,\cdot\rangle$ the algebraic structure of $\mathcal{D}$ is determined by the structure of the maximally isotropic subalgebras because in the basis $\{X_i\}$, $\{\tilde{X}^i\}$ the Lie product is given by

$$\begin{aligned}
[X_i, X_j] &= f_{ij}{}^k X_k\,,\\
[\tilde{X}^i, \tilde{X}^j] &= \tilde{f^{ij}}{}_k \tilde{X}^k\,,\\
[X_i, \tilde{X}^j] &= f_{ki}{}^j \tilde{X}^k + \tilde{f^{jk}}{}_i X_k\,.
\end{aligned} \tag{3}$$

It is clear that to any Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ one can construct the dual one by interchanging $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$, i.e. interchanging the structure coefficients ${f_{ij}}^k \leftrightarrow \tilde{f^{ij}}_k$. All properties of Lie algebras (the nontrivial being the Jacobi identities) remain to be satisfied. On the other hand for given Drinfeld double more than two Manin triples can exist and we shall see many examples of that.

One can rewrite the structure of a Manin triple also in another, equivalent, but for certain considerations more suitable, form of a Lie bialgebra defined as a Lie algebra g equipped also by a Lie cobracket[a] $\delta : g \to g \otimes g : \delta(x) = \sum x_{[1]} \otimes x_{[2]}$ such that

$$\sum x_{[1]} \otimes x_{[2]} = -\sum x_{[2]} \otimes x_{[1]} \,, \tag{4}$$

$$(id \otimes \delta) \circ \delta(x) + \text{cyclic permutations of tensor indices} = 0 \,, \tag{5}$$

$$\begin{aligned} \delta([x,y]) = \sum [x, y_{[1]}] \otimes y_{[2]} + y_{[1]} \otimes [x, y_{[2]}] \\ - [y, x_{[1]}] \otimes x_{[2]} - x_{[1]} \otimes [y, x_{[2]}] \end{aligned} \tag{6}$$

(for detailed account on Lie bialgebras see e.g. Ref. 9 or 10, Chapter 8).

The correspondence between a Manin triple and a Lie bialgebra can now be formulated in the following way. Because both subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ of the Manin triple are of the same dimension and are connected by nondegenerate pairing, it is natural to consider $\tilde{\mathcal{G}}$ as a dual $\mathcal{G}^*$ to $\mathcal{G}$ and to use the Lie bracket in $\tilde{\mathcal{G}}$ to define the Lie cobracket in $\mathcal{G}$; $\delta(x)$ is given by $\langle \delta(x), \tilde{y} \otimes \tilde{z} \rangle = \langle x, [\tilde{y}, \tilde{z}] \rangle$, $\forall \tilde{y},\ \tilde{z} \in \mathcal{G}^*$, i.e. $\delta(X_i) = \widetilde{f_i^{jk}} X_j \otimes X_k$. The Jacobi identities in $\tilde{\mathcal{G}}$

$$\widetilde{f_m^{kl}}\widetilde{f_l^{ij}} + \widetilde{f_m^{il}}\widetilde{f_l^{jk}} + \widetilde{f_m^{jl}}\widetilde{f_l^{ki}} = 0 \tag{7}$$

are then equivalent to the property of cobracket (5) and the $\tilde{\mathcal{G}}$-component of the mixed Jacobi identities[b]

$$\widetilde{{f^{jk}}_l}{f_{mi}}^l + \widetilde{{f^{kl}}_m}{f_{li}}^j + \widetilde{{f^{jl}}_i}{f_{lm}}^k + \widetilde{{f^{jl}}_m}{f_{il}}^k + \widetilde{{f^{kl}}_i}{f_{lm}}^j = 0 \tag{8}$$

are equivalent to (6).

From now on, we will use the formulation in terms of Manin triples, Lie bialgebra formulation of all results can be easily derived from it. We also consider only algebraic structure, the Drinfeld doubles as the Lie groups can be obtained in principle by means of exponential map and usual theorems about relation between Lie groups and Lie algebras apply, e.g. there is a one to one correspondence between (finite-dimensional) Lie algebras and connected and simply connected Lie groups. The group structure of the Drinfeld double can be deduced e.g. by taking matrix exponential of adjoint representation of its algebra.

[a]Summation index is suppressed.
[b]The Jacobi identities $[X_i, [\tilde{X}^j, \tilde{X}^k]] + \text{cyclic} = 0$ lead to both (8) (terms proportional to $\tilde{X}^l$) and (7) (terms proportional to X_l).

We shall consider two Drinfeld doubles isomorphic if they have isomorphic algebraic structure and there is an isomorphism transforming one ad-invariant bilinear form to the other. As mentioned above we can always choose a basis so that the bilinear form have canonical form (1) and the Lie product is then given by (3). The Drinfeld doubles $\mathcal{D}$ and $\mathcal{D}'$ with these special bases $Y_a = (X_1, X_2, X_3, \tilde{X}^1, \tilde{X}^2, \tilde{X}^3)$, $Y'_a = (X'_1, X'_2, X'_3, \tilde{X}'^1, \tilde{X}'^2, \tilde{X}'^3)$ are isomorphic iff there is an invertible 6×6 matrix ${C_a}^b$ such that the linear map given by

$$Y'_a = {C_a}^b Y_b \tag{9}$$

transforms the Lie multiplication of $\mathcal{D}$ into that of $\mathcal{D}'$ and preserves the canonical form of the bilinear form $\langle\cdot,\cdot\rangle$. This is equivalent to

$${C_a}^p {C_b}^q B_{pq} = B_{ab}\,, \qquad {C_a}^p {C_b}^q {F_{pq}}^r = {F'_{ab}}^c {C_c}^r\,, \tag{10}$$

where ${F_{ab}}^c$, ${F'_{ab}}^c$, a, b, $c = 1, \ldots, 6$ are structure coefficients of the doubles $\mathcal{D}$ and $\mathcal{D}'$ and

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

3. Method and Result of Classification

As mentioned in the Introduction, there are 78 nonisomorphic classes of Manin triples.[8] If we take into account the duality transformation $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}}) \mapsto (\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ the number is reduced to 44. Their explicit form is given in App. B. It follows from (1) and (3) that the structure of the Manin triple can be given by the structure coefficients f_{ij}^k, $\widetilde{f^{ij}}_k$ of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ in the special basis where relations (1) hold. That is why we usually denote the Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ by $(\mathcal{G}|\tilde{\mathcal{G}})$ or $(\mathcal{G}|\tilde{\mathcal{G}}|b)$ when a scaling parameter b occurs in the definition of the Lie product. Let us note that $(\mathcal{G}|\tilde{\mathcal{G}}|b)$ and $(\mathcal{G}|\tilde{\mathcal{G}}|b')$ are isomorphic up to rescaling of $\langle\cdot,\cdot\rangle$.

It is clear that a direct check which of 44 Manin triples are decomposition of isomorphic Drinfeld doubles is a tremendous task. That is why we first evaluate as many invariants of the algebras as possible and then sort them into smaller subsets according to the values of the invariants. It is clear that only the Manin triples in these subsets can be decomposition of the same Drinfeld double. The invariants we have used are:

- signature (numbers of positive, negative and zero eigenvalues) of the Killing form,
- dimensions of the comutant $[\mathcal{D}, \mathcal{D}] \equiv \mathcal{D}^1 \equiv \mathcal{D}_1$ and subalgebras created by the repeated Lie multiplication $\mathcal{D}^{i+1} = [\mathcal{D}^i, \mathcal{D}]$, (up to $i = 3$, it turns out that for $i \geq 3$ $\mathcal{D}^{i+1} = \mathcal{D}^i$). (We have for completeness determined also dimensions of $\mathcal{D}_{i+1} = [\mathcal{D}^i, \mathcal{D}^i]$, but they does not lead to refinement of our partition.)

Table 1. Invariants of Manin triples.

Signature of K	Dim. of $[\mathcal{D},\mathcal{D}]$	Dim. of $\mathcal{D}^2$, $\mathcal{D}^3$	Dim. of $\mathcal{D}_2$, $\mathcal{D}_3$	Manin triples
$(3,3,0)$	6	$6,6$	$6,6$	$(9\|5\|b)$, $(8\|5.ii\|b)$, $(7_a\|7_{1/a}\|b)$, $(7_0\|5.ii\|b)$
$(4,2,0)$	6	$6,6$	$6,6$	$(8\|5.i\|b)$, $(6_a\|6_{1/a}.i\|b)$, $(6_0\|5.iii\|b)$
$(0,3,3)$	6	$6,6$	$6,6$	$(9\|1)$
$(2,1,3)$	6	$6,6$	$6,6$	$(8\|1)$, $(8\|5.iii)$, $(7_0\|4\|b)$, $(7_0\|5.i)$, $(6_0\|4.i\|b)$, $(6_0\|5.i)$, $(5\|2.ii)$, $(4\|2.iii\|b)$,
	3	$3,3$	$3,3$	$(3\|3.i)$
$(1,0,5)$	5	$5,5$	$1,0$	$(7_a\|1)$, $(7_a\|2.i)$, $(7_a\|2.ii)$, $a>1$ $(6_a\|1)$, $(6_a\|2)$, $(6_a\|6_{1/a}.ii)$, $(6_a\|6_{1/a}.iii)$, $(6_0\|1)$, $(6_0\|2)$, $(6_0\|4.ii)$, $(6_0\|5.ii)$, $(5\|1)$, $(5\|2.i)$, $(4\|1)$, $(4\|2.i)$, $(4\|2.ii)$
	3	$3,3$	$1,0$	$(3\|1)$, $(3\|2)$, $(3\|3.ii)$, $(3\|3.iii)$
$(0,1,5)$	5	$5,5$	$1,0$	$(7_a\|1)$, $(7_a\|2.i)$, $(7_a\|2.ii)$, $a<1$ $(7_0\|1)$, $(7_0\|2.i)$, $(7_0\|2.ii)$
$(0,0,6)$	5	$5,5$	$1,0$	$(7_a\|1)$, $(7_a\|2.i)$, $(7_a\|2.ii)$, $a=1$
	3	$0,0$	0,0	$(2\|2)$
		$2,0$	$0,0$	$(2\|2.i)$, $(2\|2.ii)$
	0	$0,0$	$0,0$	$(1\|1)$

The partition of the list of Manin triples according to the values of invariants is in Table 1. The final distinction between nonisomorphic Drinfeld doubles and their decomposition into Manin triples provides the following theorem.

Theorem 1. *Any six-dimensional real Drinfeld double belongs just to one of the following* 22 *classes and allows decomposition into all Manin triples listed in the class and their duals* ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$). *If the class contains parameter* a *or* b, *the Drinfeld doubles with different values of this parameter are nonisomorphic.*

(1) $(9|5|b) \cong (8|5.ii|b) \cong (7_0|5.ii|b)$, $b>0$,
(2) $(8|5.i|b) \cong (6_0|5.iii|b)$, $b>0$,
(3) $(7_a|7_{1/a}|b) \cong (7_{1/a}|7_a|b)$, $a \geq 1$, $b \in \mathbf{R} - \{0\}$,

(4) $(6_a|6_{1/a}.i|b) \cong (6_{1/a}.i|6_a|b)$, $a > 1$, $b \in \mathbf{R} - \{0\}$,
(5) $(9|1)$,
(6) $(8|1) \cong (8|5.iii) \cong (7_0|5.i) \cong (6_0|5.i) \cong (5|2.ii)$,
(7) $(7_0|4|b) \cong (4|2.iii|b) \cong (6_0|4.i|-b)$, $b \in \mathbf{R} - \{0\}$,
(8) $(3|3.i)$,
(9) $(7_a|1) \cong (7_a|2.i) \cong (7_a|2.ii)$, $a > 1$,
(10) $(6_a|1) \cong (6_a|2) \cong (6_a|6_{1/a}.ii) \cong (6_a|6_{1/a}.iii)$, $a > 1$,
(11) $(6_0|1) \cong (6_0|5.ii) \cong (5|1) \cong (5|2.i)$,
(12) $(6_0|2) \cong (6_0|4.ii) \cong (4|1) \cong (4|2.i) \cong (4|2.ii)$,
(13) $(3|1) \cong (3|2) \cong (3|3.ii) \cong (3|3.iii)$,
(14) $(7_a|1) \cong (7_a|2.i) \cong (7_a|2.ii)$, $0 < a < 1$,
(15) $(7_0|1)$,
(16) $(7_0|2.i)$,
(17) $(7_0|2.ii)$,
(18) $(7_1|1) \cong (7_1|2.i) \cong (7_1|2.ii)$,
(19) $(2|1)$,
(20) $(2|2.i)$,
(21) $(2|2.ii)$,
(22) $(1|1)$.

4. The Proof of Theorem 1

The essence of the proof is to find which of the 78 nonisomorphic Manin triples found in Ref. 8 and displayed in App. B yield isomorphic Drinfeld doubles. The isomorphisms are given by the explicit form of the transformation matrices C [see (9)] that were found by solution of Eq. (10). In this part we have used the computer programs Maple V and Mathematica 4. The solutions are not unique and here we present only a simple examples of them. The nonisomorphic Drinfeld doubles are distinguished by investigation of their various subalgebras and properties of $\langle \cdot, \cdot \rangle$ and the Killing form on them.

In the next subsection we analyze the subsets of nonisomorphic Manin triples characterized by invariants described in Sec. 3 and displayed in Table 1.

4.1. *Manin triples with the Killing form of signature* $(3, 3, 0)$

In this case the signature of the Killing form itself fixes the Lie algebra $\mathcal{D}$ of the Drinfeld double uniquely. It is the well-known so$(3, 1)$ which is simple as a real Lie algebra and its complexification is semisimple; it decomposes into two copies of sl$(2,\mathbf{C})$. The Drinfeld doubles corresponding to $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$, $(7_a|7_{1/a}|b)$ can consequently differ only by the bilinear form $\langle \cdot, \cdot \rangle$.

We can find a necessary condition for equivalence of semisimple Drinfeld doubles from the fact that any invariant symmetric bilinear form on a complex simple Lie algebra is a multiple of the Killing form and that any invariant symmetric bilinear form on a semisimple Lie algebra is a sum of invariant symmetric bilinear forms on its simple components. (*Proof*: Let $\mathcal{G} = \oplus_i \mathcal{G}_i$ be the decomposition into simple components, $X \in \mathcal{G}_i$, $Y \in \mathcal{G}_j$, $i \neq j$. Then $\exists A_k, B_k \in \mathcal{G}_j$ s.t. $Y = \sum_k [A_k, B_k]$ and from the ad-invariance of the form $\langle X, Y\rangle = \sum_k \langle X, [A_k, B_k]\rangle = -\sum_k \langle [A_k, X], B_k\rangle = -\sum_k \langle 0, B_k\rangle = 0$.)

We therefore consider the complexification $\mathcal{D}_\mathbf{C}$ of the Drinfeld double algebra and write both the Killing form on $\mathcal{D}_\mathbf{C}$ and the bilinear form $\langle \cdot, \cdot\rangle$ in terms of Killing forms K_1, K_2 of still unspecified simple components $\mathrm{sl}(2,\mathbf{C})_1$, $\mathrm{sl}(2,\mathbf{C})_2$ ($\mathcal{D}_\mathbf{C} = \mathrm{sl}(2,\mathbf{C})_1 \oplus \mathrm{sl}(2,\mathbf{C})_2$)

$$K = K_1 + K_2 \,, \quad \langle , \rangle = \alpha K_1 + \beta K_2 \,.$$

We trivially extend the Killing forms K_1, K_2 to the whole Drinfeld double algebra $\mathcal{D}_\mathbf{C}$ and express them as

$$K_1 = \frac{\langle , \rangle - \beta K}{\alpha - \beta} \,, \qquad K_2 = \frac{\alpha K - \langle , \rangle}{\alpha - \beta} \,.$$

Because K_1, K_2 are trivially extended Killing forms, they must have three-dimensional nullspace [$\mathrm{sl}(2,\mathbf{C})_2$ in the case of K_1 and $\mathrm{sl}(2,\mathbf{C})_1$ in the case of K_2]. These two conditions on dimensions of nullspaces fix the coefficients α, β uniquely up to a permutation. Therefore, *the necessary condition for equivalence of two semisimple six-dimensional Drinfeld doubles is the equality of their sets of coefficients* $\{\alpha, \beta\}$.

We compute the coefficients α, β for the Manin triples in this class and find that in three cases $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ is

$$\{\alpha, \beta\} = \left\{ \frac{i}{4b}, -\frac{i}{4b} \right\}$$

and for $(7_a|7_{1/a}|b)$ is

$$\{\alpha, \beta\} = \left\{ \frac{ia}{4b(a-i)^2}, -\frac{ia}{4b(i+a)^2} \right\}.$$

We see that the Manin triple $(7_a|7_{1/a}|b)$ defines for any a, b Drinfeld doubles different from any of the Drinfeld doubles associated to the Manin triples $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ and that Drinfeld doubles corresponding to $(7_a|7_{1/a}|b)$ with different values of a and b are different except the case $a' = 1/a$, $b' = b$. The Manin triples $(7_a|7_{1/a}|b)$ and $(7_{1/a}|7_a|b)$ are mutually dual, correspond to $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$ and therefore give the same Drinfeld double. The Manin triple $(7_1|7_1|b)$ is of course self-dual.

Also one sees that the Manin triples $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ with different b cannot lead to the same Drinfeld double. For the Manin triples $(9|5|b)$, $(8|5.ii|b)$, $(7_0|5.ii|b)$ with equal b, the transformations (9) between Drinfeld doubles exist, but may contain complex numbers since up to now we have considered only complexifications of the original Manin triples.

However, one can check that the following real transformation matrices C guarantee the equivalence of the Drinfeld doubles in this class for fixed value of b.

$$(9|5|b) \to (8|5.ii|b):\ C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 1 & 0 & -\frac{1}{b} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$(9|5|b) \to (7_0|5.ii|b):\ C = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2b} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2b} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 & 0 \\ -b & 0 & -b & 0 & 1 & 0 \\ 0 & b & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As mentioned in the beginning of this section, the transformation matrices are not unique; they contain several free parameters. Here and further we give them in a simple form setting the parameters zero or one.

4.2. *Manin triples with the Killing form of signature* $(4, 2, 0)$

In this case the signature of the Killing form again fixes the Lie algebra $\mathcal{D}$ of the Drinfeld double uniquely, it is $\mathrm{sl}(2, \mathbf{R}) \oplus \mathrm{sl}(2, \mathbf{R})$, and the Drinfeld doubles may again differ only by the bilinear form $\langle \cdot, \cdot \rangle$. We use the criterion developed in the previous subsection for semisimple Drinfeld doubles and find

- $(8|5.i|b)$, $(6_0|5.iii|b) : \{\alpha, \beta\} = \left\{\frac{1}{4b}, -\frac{1}{4b}\right\}$,
- $(6_a|6_{1/a}.i|b) : \{\alpha, \beta\} = \left\{\frac{a}{4b(a-1)^2}, -\frac{a}{4b(1+a)^2}\right\}$.

This shows that the Manin triples might specify isomorphic Drinfeld doubles only in the following two cases:

(1) $(8|5.i|b)$ and $(6_0|5.iii|b)$ for the same value of b. In this case we have found the transformation matrix C

$$(8|5.i|b) \to (6_0|5.iii|b):\ C = \begin{pmatrix} 0 & 0 & -\frac{b}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{b}{2} & \frac{b}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -\frac{1}{b} \\ 0 & 0 & 1 & -\frac{1}{b} & 0 & 0 \\ 0 & 0 & b & 0 & -1 & 0 \end{pmatrix}.$$

This transformation is real and therefore the Drinfeld doubles are isomorphic, $(8|5.i|b) \cong (6_0|5.iii|b)$.

(2) $(6_a|6_{1/a}.i|b)$ and $(6_{1/a}|6_a.i|b)$. One can easily see that these Manin triples are dual (i.e. can be obtained one from the other by the interchange $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) and the Drinfeld doubles are therefore isomorphic.

4.3. *Manin triples with the Killing form of signature* $(0, 3, 3)$

This class contains only one Manin triple $(9|1)$ and its dual; the corresponding Drinfeld double is isomorphic to $\mathrm{so}(3) \triangleright \mathbf{R}^3$ since the Killing form has the signature $(0, 3, 3)$ and $\dim[\mathcal{D}, \mathcal{D}] = 3$.

4.4. *Manin triples with the Killing form of signature* $(2, 1, 3)$

We consider only the Manin triples with $\dim[\mathcal{D}, \mathcal{D}] = 6$, the other set in this class contains only one Manin triple $(3|3.i)$, which is isomorphic as a Lie algebra to $\mathrm{sl}(2, \mathbf{R}) \oplus \mathbf{R}^3$ since the Killing form has the signature $(2, 1, 3)$ and $\dim[\mathcal{D}, \mathcal{D}] = 3$.

The Manin triples in this set $(8|1)$, $(8|5.iii)$, $(7_0|4|b)$, $(7_0|5.i)$, $(6_0|4.i|b)$, $(6_0|5.i)$, $(5|2.ii)$, $(4|2.iii|b)$, are neither semisimple ($\mathrm{rank}K \neq 6$) nor solvable ($[\mathcal{D}, \mathcal{D}] = \mathcal{D}$). Therefore they have a nontrivial Levi–Maltsev decomposition into semidirect sum of a semisimple subalgebra S and radical N

$$\mathcal{D} = S \triangleright N\,,$$

both of them are three-dimensional. Knowledge of this decomposition turns out to be helpful in the investigation of equivalence of the Drinfeld doubles.

A rather simple computation shows that the radical is in all these Manin triples Abelian and maximally isotropic, e.g. for $(8|1)$ the radical is $N = \mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$, for $(4|2.iii|b)$ the radical is $N = \mathrm{span}\{X_3, \tilde{X}^1, \tilde{X}^2\}$.

Next we find the semisimple component. It turns out that the semisimple subalgebra S is in all cases $\mathrm{sl}(2, \mathbf{R})$, e.g. for $(8|1)$ it can be evidently chosen $S = \mathrm{span}\{X_1, X_2, X_3\}$, for $(4|2.iii|b)$ the most general form of the semisimple subalgebra is $S = \text{span}\{2X_1 - 2\alpha X_3 - \frac{2}{b}\tilde{X}^1 - 2\beta\tilde{X}^2, -\frac{2}{b}X_2 - \frac{2\gamma}{b}X_3 - \frac{2\beta}{b}\tilde{X}^1, \alpha\tilde{X}^1 + (2-\gamma)\tilde{X}^2 + \tilde{X}^3\}$ for any values of α, β, γ.

One can restrict the form $\langle\cdot,\cdot\rangle$ to the semisimple subalgebra S and finds that for $(8|1)$ $\langle\cdot,\cdot\rangle_S = 0$, i.e. S is maximally isotropic, whereas for $(4|2.iii|b)$ and any choice of α, β, γ is $\langle\cdot,\cdot\rangle_S = -1/bK_S$, K_S being the Killing form on S. This shows that as Drinfeld doubles $(8|1)$ and $(4|2.iii|b)$ and similarly $(4|2.iii|b)$ for different values of b are not isomorphic.

Performing the same computation for all Manin triples in this set, we find that they divide into two subsets.

(1) $(8|1)$, $(8|5.iii)$, $(7_0|5.i)$, $(6_0|5.i)$, $(5|2.ii)$: $\langle\cdot,\cdot\rangle_S = 0$
(2) $(7_0|4|b)$, $(6_0|4.i|-b)$, $(4|2.iii|b)$: $\langle\cdot,\cdot\rangle_S = -1/bK_S$, $b \in \mathbf{R} - \{0\}$

We find the transformation matrices for Manin triples in these subsets and prove the equivalence of the corresponding Drinfeld doubles:

$$(8|1) \to (8|5.iii):\ C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(8|1) \to (7_0|5.i):\ C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$(8|1) \to (6_0|5.i):\ C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$(8|1) \to (5|2.ii):\ C = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix},$$

respectively

$$(4|2.iii|b) \to (7_0|4|b)\colon\ C = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & -\frac{1}{2b} & 0 & 1 & 0 \\ 0 & \frac{1}{2b} & 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$(4|2.iii|b) \to (6_0|4.i|-b)\colon\ C = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{2b} & 0 & 1 & 0 \\ 0 & -\frac{1}{2b} & 0 & 0 & 0 & 1 \\ -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Concerning the Lie structure of these Drinfeld doubles, it follows from the signature of the Killing form and dimension of $[\mathcal{D},\mathcal{D}]$ that the Lie algebra of D is isomorphic in both cases to $\mathrm{sl}(2,\mathbf{R}) \triangleright \mathbf{R}^3$ where commutation relations between subalgebras are given by the unique irreducible representation of $\mathrm{sl}(2,\mathbf{R})$ on $\mathbf{R}^3$.

4.5. *Manin triples with the Killing form of signature* $(1,0,5)$

4.5.1. *Case* $\dim[\mathcal{D},\mathcal{D}] = 5$

This set contains the greatest number of Manin triples: $(7_{a>1}|1)$, $(7_{a>1}|2.i)$, $(7_{a>1}|2.ii)$, $(6_a|1)$, $(6_a|2)$, $(6_a|6_{1/a}.ii)$, $(6_a|6_{1/a}.iii)$, $(6_0|1)$, $(6_0|5.ii)$, $(5|1)$, $(5|2.i)$, $(6_0|2)$, $(6_0|4.ii)$, $(4|1)$, $(4|2.i)$, $(4|2.ii)$. In order to shorten our considerations we firstly present the transformation matrices C showing the equivalence of following Drinfeld doubles and later we prove that the following classes of Drinfeld doubles are nonisomorphic:

(1) $(7_{a>1}|1) \cong (7_{a>1}|2.i) \cong (7_{a>1}|2.ii)$ for the same value of a

$$(7_a|1) \to (7_a|2.i)\colon\ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2a} & 0 & 1 & 0 \\ 0 & \frac{1}{2a} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(7_a|1) \to (7_a|2.ii):\ C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2a & 0 \\ 0 & 0 & 0 & 0 & 0 & 2a \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -\frac{1}{2a} & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2a} & 0 & 1 & 0 \end{pmatrix}.$$

(2) $(6_a|1) \cong (6_a|2) \cong (6_a|6_{1/a}.ii) \cong (6_a|6_{1/a}.iii)$ for the same value of a

$(6_a|1) \to (6_a|2)$:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2a} & 0 & 1 & 0 \\ 0 & \frac{1}{2a} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$(6_a|1) \to (6_a|6_{\frac{1}{a}}.ii)$:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-a & a-1 & 0 & 0 \\ 0 & 1-a & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \frac{1}{a-1} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{a-1} & 0 & 0 & 0 & -\frac{1}{a-1} & -\frac{1}{a-1} \end{pmatrix},$$

$(6_a|1) \to (6_a|6_{\frac{1}{a}}.iii)$:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1-a & a+1 & 0 & 0 \\ 0 & -1-a & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{a+1} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{a+1} & 0 & 0 & 0 & -\frac{1}{a+1} & \frac{1}{a+1} \end{pmatrix}.$$

(3) $(5|1) \cong (5|2.i) \cong (6_0|1) \cong (6_0|5.ii)$

$$(5|1) \to (5|2.i):\ C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$(5|1) \to (6_0|1):\ C = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(5|1) \to (6_0|5.ii):\ C = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(4) $(4|1) \cong (4|2.i) \cong (4|2.ii) \cong (6_0|2) \cong (6_0|4.ii)$

$$(4|1) \to (4|2.i):\ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(4|1) \to (4|2.ii)\colon\ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(4|1) \to (6_0|2)\colon\ C = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$(4|1) \to (6_0|4.ii)\colon\ C = \begin{pmatrix} 0 & 0 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & \frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the proof of inequivalence of the above given classes of Manin triples we exploit the fact that the Drinfeld doubles have at least one decomposition into Manin triple with the 2nd subalgebra $\tilde{\mathcal{G}}$ Abelian; we will use only these representantions $(7_a|1)$, $a > 1$, $(6_a|1)$, $(5|1)$, $(4|1)$ in our considerations.

Firstly we find all maximal isotropic Abelian subalgebras $\mathcal{A}$ of each of the given Drinfeld doubles. The dimension of any such $\mathcal{A}$ must be 3 from the maximal isotropy. The commutant is in all these cases $\mathcal{D}_1 = [\mathcal{D}, \mathcal{D}] = \mathrm{span}\{X_2, X_3, \tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$ and the centre is $Z(\mathcal{D}) = \mathrm{span}\{\tilde{X}^1\} = \mathcal{D}_2$. One can see that any element of the form $X_1 + Y$, $Y \in \mathcal{D}_1$ cannot occur in $\mathcal{A}$ because X_1 commutes only with $Z(\mathcal{D})$ and itself. Therefore, $\mathcal{A} \subset \mathcal{D}_1$. Further it follows from the maximality that $\mathcal{A}$ contains $Z(\mathcal{D})$ and we conclude that $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, Y_1, Y_2\}$ where Y_1, $Y_2 \in \mathrm{span}\{X_2, X_3, \tilde{X}^2, \tilde{X}^3\}$. Analyzing the maximal isotropy and replacing Y_1, Y_2 by their suitable linear combinations we find that $\mathcal{A}$ can be in general expressed in one of the following forms:

(1) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_2, \tilde{X}^3\}$,
(2) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_2 + \alpha\tilde{X}^3, X_3 - \alpha\tilde{X}^2\}$,
(3) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_2 + \alpha X_3, -\alpha\tilde{X}^2 + \tilde{X}^3\}$,
(4) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, X_3, \tilde{X}^2\}$,
(5) $\mathcal{A} = \mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$.

In the next step we check which of these subspaces really form a subalgebra of the given Manin triple.

- $(7_a|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2, X_3\}$ and $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$. One may easily construct for each of these maximal isotropic Abelian subalgebras the dual (w.r.t $\langle\cdot\rangle$) subalgebra by taking the remaining elements of the standard basis $X_1, \ldots, \tilde{X}^3$ and finds that it is isomorphic in both cases to Bianchi algebra 7_a. In other words, we have shown that this class of Drinfeld doubles is nonisomorphic to the other ones and are mutually nonisomorphic for different values of a.
- $(6_a|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2, X_3\}$, $\mathrm{span}\{\tilde{X}^1, X_2 + X_3, -\tilde{X}^2 + \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, X_2 - X_3, \tilde{X}^2 + \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$. By a slightly more complicated construction of the dual subalgebras we find that they are of the Bianchi type 6_a for the same a, i.e. this class of Drinfeld doubles is nonisomorphic to the other ones and are mutually nonisomorphic for different values of a.
- $(5|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2, \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, X_2, X_3\}$, $\mathrm{span}\{\tilde{X}^1, X_2 + \alpha X_3, -\alpha\tilde{X}^2 + \tilde{X}^3\}$, $\mathrm{span}\{\tilde{X}^1, X_3, \tilde{X}^2\}$ and $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$.
- $(4|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^1, X_2 + \alpha\tilde{X}^3, X_3 - \alpha\tilde{X}^2\}$, $\mathrm{span}\{\tilde{X}^1, X_3, \tilde{X}^2\}$ and $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$.

Already from comparison of number of possible maximal isotropic Abelian subalgebras for $(5|1)$ and $(4|1)$ one sees that the corresponding Drinfeld doubles are nonisomorphic.

It also follows that Drinfeld doubles corresponding to Manin triples $(7_a|1)$, $(6_a|1)$, $(5|1)$ and $(4|1)$ are different as Lie algebras, since any maximal isotropic Abelian subalgebra $\mathcal{A}$ of these Manin triples is in fact an Abelian ideal $\mathcal{I}$ such that $[\mathcal{D}, \mathcal{I}] = \mathcal{I}$ and any such three-dimensional ideal is maximal isotropic from ad-invariance of $\langle , \rangle$. Therefore we have in fact identified the nonisomorphic Drinfeld doubles from the knowledge of these ideals $\mathcal{I}$ (and in some cases $\mathcal{D}/\mathcal{I}$) which does not depend on the form $\langle , \rangle$ and the doubles differ already in their Lie algebra structure.

4.5.2. *Case* $\dim[\mathcal{D},\mathcal{D}] = 3$

All Manin triples of this subset are decomposition of one Drinfeld double, i.e. they can be transformed one into another by the transformation (9). Below are the corresponding matrices.

$$(3|1) \to (3|2):\; C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(3|1) \to (3|3.ii):\; C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 & 1 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(3|1) \to (3|3.iii):\; C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.6. *Manin triples with the Killing form of signature* $(0, 1, 5)$

This set contains Manin triples $(7_{a<1}|1)$, $(7_{a<1}|2.i)$, $(7_{a<1}|2.ii)$, $(7_0|1)$, $(7_0|2.i)$, $(7_0|2.ii)$. As in the Subsec. 4.5.1 we can show that Manin triples $(7_{a<1}|1)$, $(7_{a<1}|2.i)$, $(7_{a<1}|2.ii)$ are decomposition of isomorphic Drinfeld doubles for the same a; the transformation matrices given above for $a > 1$ are meaningful also in this case. It remains to be investigated whether the Drinfeld doubles induced by $(7_0|1)$, $(7_0|2.i)$, $(7_0|2.ii)$ are isomorphic as or not.

We again find all maximal isotropic Abelian subalgebras of these Manin triples. We find

- $(7_0|1)$: the maximal isotropic Abelian subalgebras are $\mathrm{span}\{\tilde{X}^3, X_1 + \alpha\tilde{X}^2, X_2 - \alpha\tilde{X}^1\}$, $\mathrm{span}\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$,
- $(7_0|2.i)$ the only maximal isotropic Abelian subalgebra is $\mathrm{span}\{\tilde{X}^3, X_1, X_2\}$, the dual subalgebra to it w.r.t $\langle\cdot,\cdot\rangle$ does not exist.
- $(7_0|2.ii)$ the only maximal isotropic Abelian subalgebra is $\mathrm{span}\{\tilde{X}^3, X_1, X_2\}$, the dual subalgebra to it w.r.t $\langle\cdot,\cdot\rangle$ does not exist.

This means that Drinfeld double induced by $(7_0|1)$ has only decompositions into Manin triple $(7_0|1)$ and that Drinfeld doubles corresponding to $(7_0|2.i)$, $(7_0|2.ii)$ are not isomorphic to the Drinfeld double corresponding to $(7_{a<1}|1)$ for any value of a. To prove that also $(7_0|2.i)$, $(7_0|2.ii)$ induce nonisomorphic Drinfeld doubles, we find all isotropic subalgebras of Bianchi type 7_0 in the Manin triple $(7_0|2.ii)$. They are

$$\mathrm{span}\{Y_1, Y_2, Y_3\}\,,$$

where

$$Y_1 = X_1 - \alpha\tilde{X}^3\,,\quad Y_2 = X_2 - \beta\tilde{X}^3\,,\quad Y_3 = X_3 + \alpha\tilde{X}^1 + \beta\tilde{X}^2\,,\quad \alpha,\ \beta \in \mathbf{R}\,,$$

and the dual subalgebra w.r.t. $\langle\cdot,\cdot\rangle$ is in general

$$\mathrm{span}\{\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3\}\,,$$

where

$$\tilde{Y}_1 = \gamma X_2 + \tilde{X}^1 - \gamma\beta\tilde{X}^3\,,\quad \tilde{Y}_2 = -\gamma X_1 + \tilde{X}^2 + \gamma\alpha\tilde{X}^3\,,\quad \tilde{Y}_3 = \tilde{X}^2\,,\quad \gamma \in \mathbf{R}\,.$$

Structure coefficients in this new basis $Y_1, \ldots, \tilde{Y}_3$ are identical with the original structure coefficients for any α, β, γ, therefore the Drinfeld double corresponding to $(7_0|2.ii)$ allows no decomposition into other Manin triples and similarly for $(7_0|2.i)$.

Concerning the Lie algebra structure, the Drinfeld doubles corresponding to $(7_0|2.i)$ and $(7_0|2.ii)$ are isomorphic as Lie algebras because they differ just by the sign of the bilinear form $\langle,\rangle$, and consequently the commutation relations implied by ad-invariance of $\langle,\rangle$ are the same. The other Drinfeld doubles specify different Lie algebras for the same reason as in Subsec. 4.5.1.

4.7. *Manin triples with the Killing form of signature* $(0, 0, 6)$

4.7.1. *Case* $\dim[\mathcal{D}, \mathcal{D}] = 5$

This set contains Manin triples $(7_1|1)$, $(7_1|2.i)$ and $(7_1|2.ii)$. They specify isomorphic Drinfeld doubles. For transformation matrices see Subsec. 4.5.1 and substitute $a = 1$.

4.7.2. *Case* $\dim[\mathcal{D},\mathcal{D}]=3$

In this set, the only Manin triples that can lead to the same Drinfeld double are $(2|2.i)$ and $(2|2.ii)$. To see that the Drinfeld doubles are different, it is sufficent to find the centres $Z(\mathcal{D})$ of these Manin triples and restrict the form $\langle\cdot,\cdot\rangle$ to them. These restricted forms $\langle\cdot,\cdot\rangle_{Z(\mathcal{D})}$ have different signatures, therefore the Drinfeld doubles are nonisomorphic:

(1) $(2|2.i) : Z(\mathcal{D}) = \mathrm{span}\{X_1, X_2 - \tilde{X}^2, \tilde{X}^3\}$, signature of $\langle\cdot,\cdot\rangle_{Z(\mathcal{D})} = (0, 1, 2)$.
(2) $(2|2.ii) : Z(\mathcal{D}) = \mathrm{span}\{X_1, X_2 + \tilde{X}^2, \tilde{X}^3\}$, signature of $\langle\cdot,\cdot\rangle_{Z(\mathcal{D})} = (1, 0, 2)$.

These Drinfeld doubles are isomorphic as Lie algebras because they differ just by the sign of the bilinear form $\langle,\rangle$ and the commutation relations are due to the ad-invariance the same.

5. Conclusions

In this work we have constructed the complete list of six-dimensional real Drinfeld doubles up to their isomorphisms i.e. maps preserving both the Lie structure and an ad-invariant symmetric bilinear form $\langle,\rangle$ that define the double. The result is summarized in the theorem at the end of Sec. 3 and claims that there just 22 classes of the nonisomorphic Drinfeld doubles. Some of them contain one or two real parameters denoted a and b. The number 22 is in a way conditional because e.g. the classes 9,14,18 could be united into one. The reason why they are given as separate classes is that they have different values of their invariants, in this case the signature of the Killing form.

An important point that follows from the classification is that there are several different Drinfeld doubles corresponding to Lie algebras $so(1,3)$, $sl(2,\mathbf{R})\oplus sl(2,\mathbf{R})$, $sl(2,\mathbf{R})\triangleright\mathbf{R}^3$ whereas on solvable Lie algebras the Drinfeld double is unique (in some cases up to the sign of the bilinear form). On the other hand there are Manin triples with one isotropic subalgebra Abelian that are equivalent as Drinfeld doubles even though the other subalgebras are different [see $(6_0|1)$ and $(5|1)$]. That is why it is necessary to investigate the (non)equivalence of the Manin triples of this form. Moreover the above given examples indicate the diversity of Drinfeld double structures one may encounter in higher dimensions.

Beside that from the present classification procedure one can find whether a given six-dimensional Lie algebra can be equipped by a suitable ad-invariant bilinear form and turned into a Drinfeld double (and how many such forms exist). The decisive aspects are the signature of the Killing form and the dimensions of the ideals $\mathcal{D}_j, \mathcal{D}^j$. The necessary condition is that they have the values occuring in Table 1. The investigation then can be reduced to a direct check of equivalence with a particular six-dimensional Lie algebra (possibly after determination of Abelian ideals and the factor algebras as in the Subsec. 4.5.1).

One can see that for many Drinfeld doubles there are several decompositions into Manin triples. For each Manin triple there is a pair of dual sigma models. Their equation of motion [2]

$$\langle \partial_\pm l l^{-1}, \mathcal{E}^\pm \rangle = 0 \qquad (12)$$

are given by the Drinfeld double and a three-dimensional subspace $\mathcal{E}^+ \subset \mathcal{D}$ so that all these models (for fixed $\mathcal{E}^+$) are equivalent. Moreover the scaling of $\langle , \rangle$ does not change the equations of motion (12) and consequently all the models corresponding to (nonisomorphic) Drinfeld doubles with different b are equal. We can construct the explicit forms of the equations of motion for every Drinfeld double but without a physical motivation this does not make much sense.

Let us note that the complete sets of the equivalent sigma models for a fixed Drinfeld double are given by the so called modular space of the double. The construction of all nonisomorphic Manin triples for the double is the first step in the construction of the modular spaces.

Appendix A. Bianchi Algebras

It is known that any three-dimensional real Lie algebra can be brought to one of 11 forms by a change of basis. These forms represent nonisomorphic Lie algebras and are conventionally known as Bianchi algebras. They are denoted by $\mathbf{1}, \ldots, \mathbf{5}$, $\mathbf{6_a}$, $\mathbf{6_0}$, $\mathbf{7_a}$, $\mathbf{7_0}$, $\mathbf{8}$, $\mathbf{9}$ (see e.g. Ref. 11, in literature often uppercase roman numbers are used instead of arabic ones). The list of Bianchi algebras is given in decreasing order starting from simple algebras.

$$\mathbf{9}: \ [X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2, \ (\text{i.e. so}(3)),$$

$$\mathbf{8}: \ [X_1, X_2] = -X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2, \ (\text{i.e. sl}(2, \mathbf{R})),$$

$$\mathbf{7_a}: \ [X_1, X_2] = -aX_2 + X_3, \ [X_2, X_3] = 0, \ [X_3, X_1] = X_2 + aX_3, \ a > 0,$$

$$\mathbf{7_0}: \ [X_1, X_2] = 0, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2,$$

$$\mathbf{6_a}: \ [X_1, X_2] = -aX_2 - X_3, \ [X_2, X_3] = 0,$$

$$[X_3, X_1] = X_2 + aX_3, \ a > 0, \ a \neq 1,$$

$$\mathbf{6_0}: \ [X_1, X_2] = 0, \ [X_2, X_3] = X_1, \ [X_3, X_1] = -X_2,$$

$$\mathbf{5}: \ [X_1, X_2] = -X_2, \ [X_2, X_3] = 0, \ [X_3, X_1] = X_3,$$

$$\mathbf{4}: \ [X_1, X_2] = -X_2 + X_3, \ [X_2, X_3] = 0, \ [X_3, X_1] = X_3,$$

$$\mathbf{3}: \ [X_1, X_2] = -X_2 - X_3, \ [X_2, X_3] = 0, \ [X_3, X_1] = X_2 + X_3,$$

$$\mathbf{2}: \ [X_1, X_2] = 0, \ [X_2, X_3] = X_1, \ [X_3, X_1] = 0,$$

$$\mathbf{1}: \ [X_1, X_2] = 0, \ [X_2, X_3] = 0, \ [X_3, X_1] = 0.$$

One might use also another classification (used e.g. in Ref. 7). In this notation the basis of the Lie algebra is usually written as (e_0, e_1, e_2) and the classification is:

$$\mathbf{R}^3 = \mathbf{1}: \ [e_1, e_2] = 0, \ [e_0, e_i] = 0,$$

$$n_3 = \mathbf{2}: \ [e_1, e_2] = e_0, \ [e_0, e_i] = 0,$$

$$r_3(\rho): \ [e_1, e_2] = 0, \ [e_0, e_1] = e_1, \quad [e_0, e_2] = \rho e_2, \ -1 \leq \rho \leq 1.$$

This algebra is isomorphic to $\mathbf{6_0}$ for $\rho = -1$, $\mathbf{6}_{\frac{\rho+1}{\rho-1}}$ for $0 < |\rho| < 1$, $\mathbf{3}$ for $\rho = 0$ and $\mathbf{5}$ for $\rho = 1$.

$$r'_3(1) = \mathbf{4}: \ [e_1, e_2] = 0, \ [e_0, e_1] = e_1, \ [e_0, e_2] = e_1 + e_2,$$

$$s_3(\mu): \ [e_1, e_2] = 0, \ [e_0, e_1] = \mu e_1 - e_2, \ [e_0, e_2] = e_1 + \mu e_2, \ \mu \geq 0.$$

This algebra is isomorphic to $\mathbf{7_0}$ for $\mu = 0$ and $\mathbf{7}_\mu$ for $\mu > 0$.

$$\mathrm{sl}(2, \mathbf{R}) = \mathbf{8}, \qquad \mathrm{so}(3) = \mathbf{9}.$$

It is clear that this classification is more compact, on the other hand the classes in this classification contain algebras with different properties such as dimensions of commutant etc. and surprisingly the special cases of parameters we need to distinguish correspond in most cases to different Bianchi algebras. Therefore we use the Bianchi classification.

Appendix B. List of Manin Triples

We present a list of Manin triples based on Ref. 8. The label of each Manin triple, e.g. (**8**|**5.ii**|**b**), indicates the structure of the first subalgebra $\mathcal{G}$, e.g. Bianchi algebra **8**, the structure of the second subalgebra $\tilde{\mathcal{G}}$, e.g. Bianchi algebra **5**; roman numbers *i*, *ii* etc. (if present) distinguish between several possible pairings $\langle \cdot, \cdot \rangle$ of the subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$ and the parameter b indicates the Manin triples differing by the rescaling of $\langle \cdot, \cdot \rangle$ (if such Manin triples are not isomorphic).

The Lie structures of the subalgebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are written out in mutually dual bases (X_1, X_2, X_3) and $(\tilde{X}^1, \tilde{X}^2, \tilde{X}^3)$ where the transformation (2) was used to bring $\mathcal{G}$ to the standard Bianchi form. Because of (3) this information specifies the Manin triple completely.

The dual Manin triples $(\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ are not written explicitly but can be easily obtained by $X_j \leftrightarrow \tilde{X}^j$.

(1) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{9}$:

$$(\mathbf{9}|\mathbf{1}): \ [X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0, \ [\tilde{X}^2, \tilde{X}^3] = 0, \ [\tilde{X}^3, \tilde{X}^1] = 0,$$

$$(\mathbf{9}|\mathbf{5}|\mathbf{b}): \ [X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2,$$
$$[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2, \ [\tilde{X}^2, \tilde{X}^3] = 0, \ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3, \ b > 0.$$

(2) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{8}$:

$$(\mathbf{8}|\mathbf{1}): \ [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,$$
$$\mathrm{v}[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{8}|\mathbf{5.i}|\mathbf{b}): \ [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0\,,$$
$$(\mathbf{8}|\mathbf{5.ii}|\mathbf{b}): \ [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1,\ b > 0\,,$$
$$(\mathbf{8}|\mathbf{5.iii}): \ [X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -(\tilde{X}^1 + \tilde{X}^3)\,.$$

(3) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{7_a}$:

$$(\mathbf{7_a}|\mathbf{1}): \ [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,$$
$$[X_3, X_1] = X_2 + aX_3,\ a > 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{7_a}|\mathbf{2.i}): \ [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,$$
$$[X_3, X_1] = X_2 + aX_3,\ a > 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{7_a}|\mathbf{2.ii}): \ [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,$$
$$[X_3, X_1] = X_2 + aX_3,\ a > 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{7_a}|\mathbf{7_{1/a}}|\mathbf{b}): \ [X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0\,,$$
$$[X_3, X_1] = X_2 + aX_3,\ a > 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = b\big(-\tfrac{1}{a}\tilde{X}^2 + \tilde{X}^3\big),\ [\tilde{X}^2, \tilde{X}^3] = 0\,,$$
$$[\tilde{X}^3, \tilde{X}^1] = b\big(\tilde{X}^2 + \tfrac{1}{a}\tilde{X}^3\big),\ b \in \mathbf{R} - \{0\}\,.$$

(4) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{7_0}$:

$$(\mathbf{7_0}|\mathbf{1}): \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{7_0}|\mathbf{2.i}): \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$

$$(\mathbf{7_0}|\mathbf{2.ii}): \ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0\,,$$
$$(\mathbf{7_0}|\mathbf{4}|\mathbf{b}): \ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=b(-\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3]=0\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=b\tilde{X}^3,\ b\in\mathbf{R}-\{0\}\,,$$
$$(\mathbf{7_0}|\mathbf{5.i}): \ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^2,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=\tilde{X}^3\,,$$
$$(\mathbf{7_0}|\mathbf{5.ii}|\mathbf{b}): \ [X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=X_2\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=b\tilde{X}^2,\ [\tilde{X}^3,\tilde{X}^1]=-b\tilde{X}^1\,,\ b>0\,.$$

(5) Manin triples with the first subalgebra $\mathcal{G}=\mathbf{6_a}$:

$$(\mathbf{6_a}|\mathbf{1}): \ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0\,,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0\,,$$
$$(\mathbf{6_a}|\mathbf{2}): \ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=\tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^1]=0\,,$$
$$(\mathbf{6_a}|\mathbf{6_{1/a}.i}|\mathbf{b}): \ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=-b\big(\tfrac{1}{a}\tilde{X}^2+\tilde{X}^3\big),\ [\tilde{X}^2,\tilde{X}^3]=0\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=b\big(\tilde{X}^2+\tfrac{1}{a}\tilde{X}^3\big),\ b\in\mathbf{R}-\{0\}\,,$$
$$(\mathbf{6_a}|\mathbf{6_{1/a}.ii}): \ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^3]=\tfrac{a+1}{a-1}(\tilde{X}^2+\tilde{X}^3)\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=\tilde{X}^1\,,$$
$$(\mathbf{6_a}|\mathbf{6_{1/a}.iii}): \ [X_1,X_2]=-aX_2-X_3,\ [X_2,X_3]=0\,,$$
$$[X_3,X_1]=X_2+aX_3,\ a>0,\ a\neq 1\,,$$
$$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^3]=\tfrac{a-1}{a+1}(-\tilde{X}^2+\tilde{X}^3)\,,$$
$$[\tilde{X}^3,\tilde{X}^1]=-\tilde{X}^1\,.$$

(6) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{6_0}$:

$$(\mathbf{6_0}|\mathbf{1}) : \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$

$$(\mathbf{6_0}|\mathbf{2}) : \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$

$$(\mathbf{6_0}|\mathbf{4.i}|\mathbf{b}) : \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = b(-\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2, \tilde{X}^3] = 0\,,$$
$$[\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b \in \mathbf{R} - \{0\}\,,$$

$$(\mathbf{6_0}|\mathbf{4.ii}) : \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = (-\tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3)\,,$$
$$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3\,,$$

$$(\mathbf{6_0}|\mathbf{5.i}) : \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^3\,,$$

$$(\mathbf{6_0}|\mathbf{5.ii}) : \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^1 + \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^3\,,$$

$$(\mathbf{6_0}|\mathbf{5.iii}|\mathbf{b}) : \ [X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = -X_2\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^1,\ b > 0\,.$$

(7) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{5}$:

$$(\mathbf{5}|\mathbf{1}) : \ [X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$

$$(\mathbf{5}|\mathbf{2.i}) : \ [X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0\,,$$

$$(\mathbf{5}|\mathbf{2.ii}) : \ [X_1, X_2] = -X_2,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{6_0}$, $\mathbf{7_0}$, $\mathbf{8}$, $\mathbf{9}$.

(8) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{4}$:

$$(\mathbf{4}|\mathbf{1}): \; [X_1, X_2] = -X_2 + X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = 0,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{4}|\mathbf{2.i}): \; [X_1, X_2] = -X_2 + X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{4}|\mathbf{2.ii}): \; [X_1, X_2] = -X_2 + X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{4}|\mathbf{2.iii}|\mathbf{b}): \; [X_1, X_2] = -X_2 + X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = 0,\; [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^2,\; b \in \mathbf{R} - \{0\}$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{6_0}$, $\mathbf{7_0}$.

(9) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{3}$:

$$(\mathbf{3}|\mathbf{1}): \; [X_1, X_2] = -X_2 - X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_2 + X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = 0,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{3}|\mathbf{2}): \; [X_1, X_2] = -X_2 - X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_2 + X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{3}|\mathbf{3.i}): \; [X_1, X_2] = -X_2 - X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_2 + X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = -b(\tilde{X}^2 + \tilde{X}^3),\; [\tilde{X}^2, \tilde{X}^3] = 0\,,$$
$$[\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \tilde{X}^3),\; b \in \mathbf{R} - \{0\}\,,$$
$$(\mathbf{3}|\mathbf{3.ii}): \; [X_1, X_2] = -X_2 - X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_2 + X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2 + \tilde{X}^3,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{3}|\mathbf{3.iii}): \; [X_1, X_2] = -X_2 - X_3,\; [X_2, X_3] = 0,\; [X_3, X_1] = X_2 + X_3\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1,\; [\tilde{X}^2, \tilde{X}^3] = 0,\; [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^1\,.$$

(10) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{2}$:

$$(\mathbf{2}|\mathbf{1}): \; [X_1, X_2] = 0,\; [X_2, X_3] = X_1,\; [X_3, X_1] = 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0,\; [\tilde{X}^2, \tilde{X}^3] = 0,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{2}|\mathbf{2.i}): \; [X_1, X_2] = 0,\; [X_2, X_3] = X_1,\; [X_3, X_1] = 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\; [\tilde{X}^2, \tilde{X}^3] = 0,\; [\tilde{X}^3, \tilde{X}^1] = 0\,,$$
$$(\mathbf{2}|\mathbf{2.ii}): \; [X_1, X_2] = 0,\; [X_2, X_3] = X_1,\; [X_3, X_1] = 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3,\; [\tilde{X}^2, \tilde{X}^3] = 0,\; [\tilde{X}^3, \tilde{X}^1] = 0$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{3}$, $\mathbf{4}$, $\mathbf{6_0}$, $\mathbf{6_a}$, $\mathbf{7_0}$, $\mathbf{7_a}$.

(11) Manin triples with the first subalgebra $\mathcal{G} = \mathbf{1}$:

$$(\mathbf{1}|\mathbf{1}): \; [X_1, X_2] = 0, \; [X_2, X_3] = 0, \; [X_3, X_1] = 0\,,$$
$$[\tilde{X}^1, \tilde{X}^2] = 0, \; [\tilde{X}^2, \tilde{X}^3] = 0, \; [\tilde{X}^3, \tilde{X}^1] = 0$$

and dual Manin triples ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to Manin triples given above for $\mathcal{G} = \mathbf{2}$–$\mathbf{9}$.

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6.1 Errata

This paper is presented in the form of the proof, the final version is not yet available. Because it was impossible to edit the PDF file directly, I present here a list of corrections that will appear in the published text:

- On page 1, last line: “this this” should read “this”
- On page 4, last line: lower the indices in “$\mathcal{D}_{i+1} = [\mathcal{D}^i, \mathcal{D}^i]$”
- On page 5, Table 1, the third line from below “(2|2)” should read “(2|1)”
- On page 5, Theorem 1, “of this parameter” should read “of these parameters”
- On page 6, line (10) in Theorem 1: before “$a > 1$” should stand “and the Manin triples with $a \to \frac{1}{a}$”
- On page 6, section 4, 2nd paragraph, 1st line: “In the next subsection” should read “In the following subsections”
- On page 12, the line (2) “for the same value of a” should read “for the same value of a and for $a' = \frac{1}{a}$”
- On page 14, 1st paragraph, line 3: “representations” should read “representatives”
- On page 15, the part $(6_a|1)$ line 4: “they are of the Bianchi type 6_a for the same a” should read “they are of the Bianchi type 6_a for the same a (the duals of span$\{\tilde{X}^1, X_2, X_3\}$ and span$\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$) and $6_{\frac{1}{a}}$ (the duals of the rest)”.
- On page 15, the part $(6_a|1)$ last line: “for different values of a” should read “for different values of $a > 1$”
- On page 21, part (2) (8|1) 2nd line: “v” should be omitted

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2. Pre-Diploma work: Solution of the Yang–Baxter System for Quantum doubles, Prague 1998 (in czech).
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11. L. Šnobl, L. Hlavatý. Classification of 6–dimensional real Drinfeld doubles. accepted for publication in *International Journal of Modern Physics A*, e–preprint math.QA/0202210

12. L. Šnobl. On modular spaces of semisimple Drinfeld doubles, e–preprint hep-th/0204244.

List of conference talks

1. 8th Colloquium "Quantum Groups and Integrable Systems", Prague, June 1999; a contributed talk "Construction of quantum doubles from solutions of Yang-Baxter system".
2. XVIII Workshop on Geometric Methods in Physics, Bialowieža, July 1999; a contributed talk "Construction of quantum doubles from solutions of Yang-Baxter system".
3. 39. Internationale Universitätswochen für Kern– und Teilchenphysik "Methods of Quantization", Schladming, February-March 2000; a contributed talk "Construction of quantum doubles from solutions of Yang-Baxter system".
4. XIII International Congress on Mathematical Physics, Imperial College, London, July 2000; a contributed talk "Solution of the Yang-Baxter System for Quantum Doubles" and a poster "Construction of Quantum Doubles from Solutions of Yang-Baxter System".
5. Winter School "Geometry and Physics", Srní, January 2001; a contributed talk "Principal chiral models on non–semisimple groups".
6. 10th Colloquium "Quantum Groups and Integrable Systems", Prague, June 2001; a contributed talk "Principal chiral models with non–constant metric".
7. XX Workshop on Geometric Methods in Physics, Bialowieža, July 2001; a contributed talk "Principal chiral models with non–constant metric".
8. Winter School "Geometry and Physics", Srní, January 2002; a contributed talk "Poisson–Lie T–dual models with two dimensional targets".

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