Classification of 6–dimensional real Manin triples

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Abstract

We present a complete list of 6–dimensional Manin triples or, equivalently, of 3–dimensional Lie bialgebras. We start from the well known classification of 3–dimensional real Lie algebras and assume the canonical bilinear form on the 6-dimensional Drinfeld double. Then we solve the Jacobi identities for the dual algebras. Finally we find mutually non–isomorphic Manin triples. The complete list consists of 78 classes of Manin triples, or 44 Lie bialgebras if one considers dual bialgebras equivalent.

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1 Introduction

In recent years, the study of T–duality in string theory has led to discovery of Poisson–Lie T–dual sigma models. Klimčík and Ševera have found a procedure allowing to construct such models from a given Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$, i.e. a decomposition of a Lie algebra $\mathcal{D}$ into two maximally isotropic subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$. The construction of the Poisson–Lie T–dual sigma models is described in [1] and [2]. The models have target spaces in the Lie groups G and $\tilde{G}$ and are defined by the Lagrangians

$$\mathcal{L} = E_{ij}(g)(g^{-1}\partial_- g)^i (g^{-1}\partial_+ g)^j \tag{1}$$

$$\tilde{\mathcal{L}} = \tilde{E}_{ij}(\tilde{g})(\tilde{g}^{-1}\partial_- \tilde{g})^i (\tilde{g}^{-1}\partial_+ \tilde{g})^j \tag{2}$$

where the matrices $E(g)$ and $\tilde{E}(\tilde{g})$ are constructed from a constant invertible matrix $E(e)$ by virtue of the adjoint representation of the group G resp $\tilde{G}$ on $\mathcal{D}$. It implies that any pair of Poisson–Lie T–dual sigma models is given (up to the constant matrix E) by the corresponding Manin triple and that's why it is interesting and useful to classify these structures.

One can easily see that the dimension of the Lie algebra $\mathcal{D}$ must be even. In the dimension two $\mathcal{D}$ must be abelian and there is just one Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}}) \equiv (\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ where $dim\mathcal{G} = dim\tilde{\mathcal{G}} = 1$. The classification of Manin triples for the four–dimensional Lie algebras together with the pairs of dual models was given in [3]. In this paper we are going to classify the Manin triples of the 6–dimensional real Lie algebras.

Important steps in this direction were made in [4] where a list of possible maximally isotropic subalgebras of the 6–dimensional Lie algebras can be found. It turns out that the subalgebras don't specify the Manin triple completely. For certain algebras there exist several rather different possible pairings, allowing to construct different Manin triples. In the present paper, we present a complete list of real 6–dimensional Manin triples, i.e. we give not only the possible subalgebras, but also the corresponding ad–invariant form (i.e we write dual bases of the algebras with respect to this form and their Lie brackets). The complex solvable Manin triples were classified in [5].

As we shall see Manin triples are equivalent to Lie bialgebras and the classification of the three–dimensional Lie bialgebras (i.e. six–dimensional Manin triples) was given in [6]. Our classification was done independently without knowledge of [6]. The consequent comparison proved that the results are identical even though we have started from a different description of the three–dimensional algebras and used a completely different method. It means that the present work can be considered as an independent check of [6] with the results expressed in a different form, namely as Manin triples.

In the following sections, we firstly recall the definitions of Manin triple, Drinfeld double and Lie bialgebra, then briefly explain the approach we have used to find all algebras of 6–dimensional Drinfeld doubles, and finally give a complete list of all 6–dimensional Manin triples.

2 Manin triples, Drinfeld doubles, Lie bialgebras

The Drinfeld double D is defined as a Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad–invariant nondegenerate bilinear form $\langle .,. \rangle$ can be decomposed into a pair of maximally isotropic subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ such that $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. This ordered triple of algebras $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ is called Manin triple.

One can see that the dimensions of the subalgebras are equal and that bases $\{X_i\}, \{\tilde{X}_i\}$ in the subalgebras can be chosen so that

$$\langle X_i, X_j \rangle = 0, \ \langle X_i, \tilde{X}^j \rangle = \langle \tilde{X}^j, X_i \rangle = \delta_i^j, \ \langle \tilde{X}^i, \tilde{X}^j \rangle = 0. \tag{3}$$

This canonical form of the bracket is invariant with respect to the transformations

$$X_i' = X_k A_i^k, \ \tilde{X}'^j = (A^{-1})_k^j \tilde{X}^k. \tag{4}$$

Due to the ad-invariance of $\langle .,. \rangle$ the algebraic structure of $\mathcal{D}$ is

$$[X_i, X_j] = f_{ij}{}^k X_k, \ [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}{}_k \tilde{X}^k,$$

$$[X_i, \tilde{X}^j] = f_{ki}{}^j \tilde{X}^k + \tilde{f}^{jk}{}_i X_k. \tag{5}$$

It is clear that to any Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ one can construct the dual one by interchanging $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$, i.e. interchanging the structure constants $f_{ij}{}^k \leftrightarrow \tilde{f}^{ij}{}_k$. All properties of Lie algebras (the nontrivial being the Jacobi identities) remain to be satisfied. On the other hand for given Drinfeld double more than two Manin triples can exist.

One can rewrite the structure of a Manin triple also in another, equivalent, but for certain considerations more suitable, form of Lie bialgebra.

A Lie bialgebra is a Lie algebra g equipped also by a Lie cobracket[1] $\delta : g \to g \otimes g : \delta(x) = \sum x_{[1]} \otimes x_{[2]}$ such that

$$\sum x_{[1]} \otimes x_{[2]} \quad = \quad -\sum x_{[2]} \otimes x_{[1]}, \tag{6}$$

[1] Summation index is suppressed

$$
\begin{aligned}
(id\otimes\delta)\circ\delta(x) &+ \text{cyclic permutations of tensor indices} = 0, &(7)\\
\delta([x,y]) &= \sum[x,y_{[1]}]\otimes y_{[2]} + y_{[1]}\otimes[x,y_{[2]}] - \\
&- [y,x_{[1]}]\otimes x_{[2]} - x_{[1]}\otimes[y,x_{[2]}] &(8)
\end{aligned}
$$

(for detailed account on Lie bialgebras see e.g. [7] or [8], Chapter 8).

The correspondence between a Manin triple and a Lie bialgebra can now be formulated in the following way. Because both subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ of the Manin triple are of the same dimension and are connected by nondegenerate pairing, it is natural to consider $\tilde{\mathcal{G}}$ as a dual $\mathcal{G}^*$ to $\mathcal{G}$ and to use the Lie bracket in $\tilde{\mathcal{G}}$ to define the Lie cobracket in $\mathcal{G}$; $\delta(x)$ is given by $\langle\delta(x), \tilde{y}\otimes\tilde{z}\rangle = \langle x, [\tilde{y},\tilde{z}]\rangle$, $\forall\tilde{y},\tilde{z}\in\mathcal{G}^*$, i.e. $\delta(X_i) = \tilde{f_i^{jk}} X_j\otimes X_k$. The Jacobi identities in $\tilde{\mathcal{G}}$

$$
\tilde{f_m^{kl}}\tilde{f_l^{ij}} + \tilde{f_m^{il}}\tilde{f_l^{jk}} + \tilde{f_m^{jl}}\tilde{f_l^{ki}} = 0 \qquad (9)
$$

are then equivalent to the property of cobracket (7) and the $\tilde{\mathcal{G}}$–component of the mixed Jacobi identities [2]

$$
\tilde{f^{jk}}_l f_{mi}{}^l + \tilde{f^{kl}}_m f_{li}{}^j + \tilde{f^{jl}}_i f_{lm}{}^k + \tilde{f^{jl}}_m f_{il}{}^k + \tilde{f^{kl}}_i f_{lm}{}^j = 0 \qquad (10)
$$

are equivalent to (8).

From now on, we will use the formulation in terms of Manin triples, Lie bialgebra formulation of all results can be easily derived from it. We also consider only algebraic structure, the Drinfeld doubles as the Lie groups can be obtained in principle by means of exponential map and usual theorems about relation between Lie groups and Lie algebras apply, e.g. there is a one to one correspondence between (finite–dimensional) Lie algebras and connected and simply connected Lie groups. The group structure of the Drinfeld double can be deduced e.g. by taking matrix exponential of adjoint representation of its algebra.

3 Method of classification

In this section we present the approach we have used to find all 6–dimensional Manin triples, i.e. 3–dimensional Lie bialgebras.

Starting point for our computations is the well known classification of 3–dimensional real Lie algebras (see e.g. [9] or [4]). Non–isomorphic Lie algebras are written in 11 classes, traditionally known as Bianchi algebras. Their commutation relations are:

$$
[X_1,X_2] = -aX_2 + n_3X_3,\ [X_2,X_3] = n_1X_1,\ [X_3,X_1] = n_2X_2 + aX_3, \qquad (11)
$$

[2]The Jacobi identities $[X_i,[\tilde{X}^j,\tilde{X}^k]] + \text{cyclic} = 0$ lead to both (10) (terms proportional to $\tilde{X}^l$) and (9) (terms proportional to X_l).

where the parameters a, n_1, n_2, n_3 have the following values

Class	a	n_1	n_2	n_3
I	0	0	0	0
II	0	1	0	0
VII_0	0	1	1	0
VI_0	0	1	-1	0
IX	0	1	1	1
$VIII$	0	1	1	-1
V	1	0	0	0
IV	1	0	0	1
$VII_a\,(a > 0)$	a	0	1	1
III	1	0	1	-1
$VI_a\,(a > 0, a \neq 1)$	a	0	1	-1

Therefore the 1st subalgebra $\mathcal{G}$ of the Manin triple $\mathcal{D}$ must be one of the Bianchi algebras given above and we can choose its basis so that the Lie brackets are of the form (11). In the 2nd subalgebra $\tilde{\mathcal{G}}$ we choose the dual basis $\tilde{X}^i$ so that (3) holds, and treat nine independent components of structure coefficients $\tilde{f}^{ij}_k$ of the 2nd subalgebra $\tilde{\mathcal{G}}$ in the basis $\tilde{X}^i$ as unknowns. We cannot assume that the $\tilde{f}^{ij}_k$ are of the form (11) as well because it can be incompatible with (3). Then we solve the mixed Jacobi identities (10) (these relations form a system of linear equations in $\tilde{f}^{ij}_k$) and the Jacobi identities for the dual algebra (9) (i.e. quadratic in $\tilde{f}^{ij}_k$).

As a result, we have found all structure coefficients of $\tilde{\mathcal{G}}$ consistent with the definition of Manin triple and the next step was to determine the Bianchi classes of obtained algebras $\tilde{\mathcal{G}}$. Finally we have found the the non–isomorphic Manin triples by considering Manin triples connected by the transformations (4) (i.e. change of basis in $\mathcal{G}$ accompanied by the dual change of basis in $\tilde{\mathcal{G}}$ with respect to $\langle , \rangle$) as equivalent and choosing one representant in each equivalence class.

In computations computer algebra systems Maple V and Mathematica 4 were independently used for manipulating expressions and solving sets of linear and quadratic equations, their results were checked one against the other.

Before listing our results, we shall give an example showing the progress of computation in some detail.

Example: Let us consider the algebra $VIII$, i.e. $\mathcal{G} = sl(2, \Re)$.

$$[X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

When one solves the mixed Jacobi identities (10), he finds that the 2nd subalgebra must have the form

$$[\tilde{X}^1, \tilde{X}^2] = -\alpha\tilde{X}^1 + \beta\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \gamma\tilde{X}^2 + \alpha\tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = -\gamma\tilde{X}^1 - \beta\tilde{X}^3.$$

The Jacobi identities in the 2nd subalgebra (9) in this case don't impose any further condition on the structure constants $\tilde{f^{ij}}_k$, i.e. we have already found the structure of all possible 2nd subalgebras $\tilde{\mathcal{G}}$ in the Manin triple.

Next we find the Bianchi forms of $\tilde{\mathcal{G}}$. It turns out that the 2nd algebra is of the Bianchi type I ($\tilde{f_k^{ij}} = 0$) if $\alpha = \beta = \gamma = 0$ and of type V otherwise.

Then we find values of $\tilde{f^{ij}}_k$ that allow transformation (4) leading to the rescaled Bianchi form V of the 2nd subalgebra $\tilde{\mathcal{G}}$ and leaving the Bianchi form of the 1st subalgebra $sl(2,\Re)$ invariant. This is possible only for

$$\alpha^2 + \beta^2 - \gamma^2 > 0$$

(for $\alpha^2 + \beta^2 - \gamma^2 < 0$ the transformation matrix would be complex, not real, for $\alpha^2 + \beta^2 - \gamma^2 = 0$ it would be singular). Therefore we have in the case $\alpha^2 + \beta^2 - \gamma^2 > 0$ a one–parametric set of non–equivalent Manin triples

$$[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0$$

and we must find representants of remaining classes of possible Manin triples. We choose the forms

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1,\ b > 0$$

for $\alpha^2 + \beta^2 - \gamma^2 < 0$ and

$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -(\tilde{X}^1 + \tilde{X}^3)$$

for $\alpha^2 + \beta^2 - \gamma^2 = 0$, $\alpha \neq 0 \vee \beta \neq 0 \vee \gamma \neq 0$ and easily verify that every possible 2nd subalgebra $\tilde{\mathcal{G}}$ can be taken to one of the given forms by transformation (4) which doesn't change the structure constants of the 1st subalgebra $\mathcal{G} = sl(2,\Re)$.

Details of computations for each Bianchi algebra are given in the Appendix.

4 Results: 6–dimensional Manin triples

The forms of non–equivalent Manin triples were choosen according to the following criteria: The 1st subalgebra is in the Bianchi form, the 2nd is in the form closest to Bianchi, i.e. Bianchi form if possible, or the structure constants are multiple of the Bianchi ones, or form a permution of the Bianchi ones, or, if neither is possible, are choosen to be as many zeros and small integers as possible.

In order to shorten the list, we have not explicitly written out the structure of algebras that can be found by the duality transform $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$ from the ones given in the list.

1. Dual algebras to Bianchi algebra IX:

$$[X_1, X_2] = X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

   Dual algebras:

   (a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

   (b) Bianchi algebra V

$$[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0.$$

2. Dual algebras to Bianchi algebra $VIII$:

$$[X_1, X_2] = -X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

   Dual algebras:

   (a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

   (b) Bianchi algebra V

      i. $[\tilde{X}^1, \tilde{X}^2] = -b\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b > 0.$

      ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1,\ b > 0.$

      iii. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -(\tilde{X}^1 + \tilde{X}^3).$

3. Dual algebras to Bianchi algebra VII_a:

$$[X_1, X_2] = -aX_2 + X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_2 + aX_3,\ a > 0.$$

   Dual algebras:

   (a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

   (b) Bianchi algebra II

      i. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

      ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

(c) Bianchi algebra $VII_{1/a}$

$[\tilde{X}^1, \tilde{X}^2] = b(-\frac{1}{a}\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2, \tilde{X}^3] = 0,$
$[\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \frac{1}{a}\tilde{X}^3),\ b \in \Re - \{0\}.$

4. Dual algebras to Bianchi algebra VII_0:

$$[X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0, [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3$, $[\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

ii. $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3$, $[\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

(c) Bianchi algebra IV

$$[\tilde{X}^1, \tilde{X}^2] = b(-\tilde{X}^2+\tilde{X}^3), [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3,\ b \in \Re-\{0\}.$$

(d) Bianchi algebra V

i. $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^2$, $[\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^3, .$

ii. $[\tilde{X}^1, \tilde{X}^2] = 0, [\tilde{X}^2, \tilde{X}^3] = b\tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = -b\tilde{X}^1, b > 0.$

5. Dual algebras to Bianchi algebra VI_a:

$$[X_1, X_2] = -aX_2 - X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_2 + aX_3,\ a > 0,\ a \neq 1.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0, [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

$$[\tilde{X}^1, \tilde{X}^2] = 0, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(c) Bianchi algebra $VI_{1/a}$

i. $[\tilde{X}^1, \tilde{X}^2] = -b(\frac{1}{a}\tilde{X}^2 + \tilde{X}^3), [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \frac{1}{a}\tilde{X}^3),\ b \in \Re - \{0\}.$

ii. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1$, $[\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a-1}(\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^1.$

iii. $[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^1,\ [\tilde{X}^2,\tilde{X}^3]=\frac{a-1}{a+1}(-\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^3,\tilde{X}^1]=-\tilde{X}^1.$

6. Dual algebras to Bianchi algebra VI_0:

$$[X_1,X_2]=0,\ [X_2,X_3]=X_1,\ [X_3,X_1]=-X_2.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$$

(b) Bianchi algebra II

$$[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$$

(c) Bianchi algebra IV

i. $[\tilde{X}^1,\tilde{X}^2]=b(-\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=b\tilde{X}^3,\ b\in\Re-\{0\}.$

ii. $[\tilde{X}^1,\tilde{X}^2]=(-\tilde{X}^1+\tilde{X}^2+\tilde{X}^3),\ [\tilde{X}^2,\tilde{X}^3]=\tilde{X}^3,\ [\tilde{X}^3,\tilde{X}^1]=-\tilde{X}^3.$

(d) Bianchi algebra V

i. $[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^2,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=\tilde{X}^3.$

ii. $[\tilde{X}^1,\tilde{X}^2]=-\tilde{X}^1+\tilde{X}^2,\ [\tilde{X}^2,\tilde{X}^3]=\tilde{X}^3,\ [\tilde{X}^3,\tilde{X}^1]=-\tilde{X}^3.$

iii. $[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=-\tilde{X}^2,\ [\tilde{X}^3,\tilde{X}^1]=\tilde{X}^1.$

7. Dual algebras to Bianchi algebra V:

$$[X_1,X_2]=-X_2,\ [X_2,X_3]=0,\ [X_3,X_1]=X_3.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1,\tilde{X}^2]=0,\ [\tilde{X}^2,\tilde{X}^3]=\tilde{X}^1,\ [\tilde{X}^3,\tilde{X}^1]=0.$

ii. $[\tilde{X}^1,\tilde{X}^2]=\tilde{X}^3,\ [\tilde{X}^2,\tilde{X}^3]=0,\ [\tilde{X}^3,\tilde{X}^1]=0.$

and dual algebras ($\mathcal{G}\leftrightarrow\tilde{\mathcal{G}}$) to algebras given above for VI_0, VII_0, $VIII$, IX.

8. Dual algebras to Bianchi algebra IV:

$$[X_1,X_2]=-X_2+X_3,\ [X_2,X_3]=0,\ [X_3,X_1]=X_3.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

iii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^2,\ b \in \Re - \{0\}.$

and dual algebras ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to algebras given above for VI_0, VII_0.

9. Dual algebras to Bianchi algebra III:

$$[X_1, X_2] = -X_2 - X_3,\ [X_2, X_3] = 0,\ [X_3, X_1] = X_2 + X_3.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(c) Bianchi algebra III

i. $[\tilde{X}^1, \tilde{X}^2] = -b(\tilde{X}^2 + \tilde{X}^3),\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \tilde{X}^3),\ b \in \Re - \{0\}.$

ii. $[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2 + \tilde{X}^3,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

iii. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = -\tilde{X}^1.$

10. Dual algebras to Bianchi algebra II:

$$[X_1, X_2] = 0,\ [X_2, X_3] = X_1,\ [X_3, X_1] = 0.$$

Dual algebras:

(a) Bianchi algebra I

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$$

(b) Bianchi algebra II

i. $[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

ii. $[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3,\ [\tilde{X}^2, \tilde{X}^3] = 0,\ [\tilde{X}^3, \tilde{X}^1] = 0.$

and dual algebras ($\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$) to algebras given above for III, IV, VI_0, VI_a, VII_0, VII_a.

11. Dual algebras to Bianchi algebra I:

$$[X_1, X_2] = 0,\ [X_2, X_3] = 0,\ [X_3, X_1] = 0.$$

Dual algebras: all Bianchi algebras (in their Bianchi forms)

5 Conclusions

We have classified 6–dimensional Manin triples or, equivalently, 3–dimensional Lie bialgebras. In computations computer algebra systems Maple V and Mathematica 4 were used for solving the sets of linear and quadratic equations that follow from the Jacobi identities and similarity transformations. The results were calculated independently in both systems and afterwards were checked one against the other. The complete list consists of 78 classes of Manin triples (if one considers dual Lie bialgebras equivalent, then the count is 44). An open problem that remains is detecting the Manin triples that belong to the same Drinfeld double or, in other words, the classification of the 6-dimensional Drinfeld doubles.

One of interesting results is the number of possible Lie bialgebra structures for the algebra $VIII$, i.e. $sl(2,\Re)$. In this case there are up to rescaling 3 non–equivalent Manin triples. As mentioned in the Introduction, to every Manin triple correspond a pair of Poisson–Lie T–dual models. Therefore, there should exist 3 different pairs of non-abelian Poisson-Lie T-dual models for $sl(2,\Re)$. Only one of them appeared in the literature so far [10]. There is a natural question whether these models are equivalent (i.e. whether they correspond to the decomposition of one Drinfeld double) and if they lead after quantisation to the same quantum model.

Appendix: Most general form of $\tilde{\mathcal{G}}$ of Manin triple with given $\mathcal{G}$

In this Appendix we present our computations in some detail. For each Bianchi algebra we give solutions of the mixed Jacobi identities (10), i.e. linear equations in $\tilde{f}$, the remaining non–trivial Jacobi identities in $\tilde{\mathcal{G}}$ (9), i.e. in general quadratic equations in $\tilde{f}$ and their solutions, in general depending on several parameters $\alpha, \beta, \ldots$ Finally we specify values of parameters allowing transformation (4) of $\tilde{\mathcal{G}}$ into forms of $\tilde{\mathcal{G}}$ given in the list of non–isomorphic Manin triples.

- $\mathcal{G} = IX$

  The mixed Jacobi identities (10) imply

  $$\tilde{f}^{23}{}_3 = -\tilde{f}^{12}{}_1, \tilde{f}^{23}{}_2 = \tilde{f}^{13}{}_1, \tilde{f}^{13}{}_3 = \tilde{f}^{12}{}_2, \tilde{f}^{23}{}_1 = 0, \tilde{f}^{12}{}_3 = 0, \tilde{f}^{13}{}_2 = 0.$$

  The Jacobi identities in $\tilde{\mathcal{G}}$ (9) in this case don't impose any new condition. The general form of $\tilde{\mathcal{G}}$ is therefore

  $$[\tilde{X}^1, \tilde{X}^2] = \alpha \tilde{X}^1 + \beta \tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \gamma \tilde{X}^2 - \alpha \tilde{X}^3,$$

$$[\tilde{X}^3, \tilde{X}^1] = -\gamma\tilde{X}^1 - \beta\tilde{X}^3$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form IX (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra V in the rescaled standard form IX (b) with $b = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$ otherwise.

- $\mathcal{G} = VIII$
  The mixed Jacobi identities (10) imply

$$\tilde{f^{12}}_1 = -\tilde{f^{23}}_3, \tilde{f^{13}}_1 = \tilde{f^{23}}_2, \tilde{f^{12}}_2 = \tilde{f^{13}}_3, \tilde{f^{23}}_1 = 0, \tilde{f^{13}}_2 = 0, \tilde{f^{12}}_3 = 0.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) in this case don't impose any new condition. The general form of $\tilde{\mathcal{G}}$ is therefore

$$[\tilde{X}^1, \tilde{X}^2] = -\alpha\tilde{X}^1 + \beta\tilde{X}^2,\ [\tilde{X}^2, \tilde{X}^3] = \gamma\tilde{X}^2 + \alpha\tilde{X}^3,$$
$$[\tilde{X}^3, \tilde{X}^1] = -\gamma\tilde{X}^1 - \beta\tilde{X}^3.$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form $VIII$ (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra V
  * in the rescaled standard form $VIII$ (b) i. with $b = \sqrt{\alpha^2 + \beta^2 - \gamma^2}$ if $\alpha^2 + \beta^2 - \gamma^2 > 0$,
  * in the form $VIII$ (b) ii. with $b = \sqrt{-(\alpha^2 + \beta^2 - \gamma^2)}$ if $\alpha^2 + \beta^2 - \gamma^2 < 0$,
  * in the form $VIII$ (b) iii. if $\alpha^2 + \beta^2 - \gamma^2 = 0$, and $\alpha \neq 0 \vee \beta \neq 0 \vee \gamma \neq 0$[3].

- $\mathcal{G} = VII_a$
  The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_2 = a\tilde{f^{13}}_3, \tilde{f^{12}}_2 = \tilde{f^{13}}_3, \tilde{f^{23}}_3 = -\frac{a^2\tilde{f^{23}}_2 + a^2\tilde{f^{13}}_1 - \tilde{f^{23}}_2 + \tilde{f^{13}}_1}{2a},$$
$$\tilde{f^{12}}_1 = -\frac{a^2\tilde{f^{23}}_2 + a^2\tilde{f^{13}}_1 + \tilde{f^{23}}_2 - \tilde{f^{13}}_1}{2a}, \tilde{f^{12}}_3 = -a\tilde{f^{13}}_3.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$4a\tilde{f^{23}}_1\tilde{f^{13}}_3 + (a\tilde{f^{23}}_2)^2 + 2a^2\tilde{f^{23}}_2\tilde{f^{13}}_1 + (\tilde{f^{23}}_2)^2 - 2\tilde{f^{23}}_2\tilde{f^{13}}_1 +$$
$$+ (a\tilde{f^{13}}_1)^2 + (\tilde{f^{13}}_1)^2 = 0.$$

The solutions of this equation give the following general forms of $\tilde{\mathcal{G}}$:

---

[3] In order to avoid abundant parentheses, logical conjuctions written in terms of symbols are considered with higher priority than that written by words and, or.

1.

$$
\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\frac{1}{2a}(a^2\alpha + \beta a^2 + \alpha - \beta)\tilde{X}^1 + \gamma\tilde{X}^2 - \gamma a\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= -\frac{1}{4\gamma a}(a^2\alpha^2 + 2\alpha\beta a^2 + \alpha^2 - 2\alpha\beta + \beta^2 a^2 + \beta^2)\tilde{X}^1 \\
&\quad +\alpha\tilde{X}^2 - \frac{1}{2a}(a^2\alpha + \beta a^2 - \alpha + \beta)\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^1 - \gamma a\tilde{X}^2 - \gamma\tilde{X}^3.
\end{aligned}
$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra $VII_{1/a}$ in the rescaled standard form VII_a (c) with $b = -a\gamma$.

2. $[\tilde{X}^1, \tilde{X}^2] = 0$, $[\tilde{X}^2, \tilde{X}^3] = \alpha\tilde{X}^1$, $[\tilde{X}^3, \tilde{X}^1] = 0$.
$\tilde{\mathcal{G}}$ can be transformed into
   - Bianchi algebra I in the standard form VII_a (a) if $\alpha = 0$,
   - Bianchi algebra II
     * in the standard form VII_a (b) i. if $\alpha > 0$,
     * in the form VII_a (b) ii. if $\alpha < 0$.

- $\mathcal{G} = VII_0$
The mixed Jacobi identities (10) imply

$$
\tilde{f^{12}}_1 = -\tilde{f^{23}}_3, \tilde{f^{12}}_2 = \tilde{f^{13}}_3, \tilde{f^{23}}_2 = \tilde{f^{13}}_1, \tilde{f^{13}}_2 = 0, \tilde{f^{23}}_1 = 0.
$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$
\tilde{f^{12}}_3 \tilde{f^{13}}_1 = 0.
$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$
\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha\tilde{X}^1 + \beta\tilde{X}^2 + \gamma\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^3.
\end{aligned}
$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VII_0 (a) if $\gamma = \beta = \alpha = 0$,
- Bianchi algebra II
  * in the form VII_0 (b) i. if $\gamma > 0$ and $\beta = \alpha = 0$,
  * in the form VII_0 (b) ii. if $\gamma < 0$ and $\beta = \alpha = 0$,

- Bianchi algebra IV in the rescaled standard form VII_0 (c) with $b = -\frac{\beta^2+\alpha^2}{\gamma}$ if $\gamma \neq 0$ and $\beta \neq 0 \vee \alpha \neq 0$,
- Bianchi algebra V in the standard form VII_0 (d) i. with if $\gamma = 0$ and $\beta \neq 0 \vee \alpha \neq 0$.

2.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha\tilde{X}^1 + \beta\tilde{X}^2, \\
[\tilde{X}^2, \tilde{X}^3] &= \gamma\tilde{X}^2 + \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\gamma\tilde{X}^1 - \beta\tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VII_0 (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra V
  - in the standard form VII_0 (d) i. if $\gamma = 0$,
  - in the form VII_0 (d) ii. with $b = |\gamma|$ if $\gamma \neq 0$.

- $\mathcal{G} = VI_a$

The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_1 = -\frac{-a^2\tilde{f^{12}}_1 + a^2\tilde{f^{23}}_3 - \tilde{f^{23}}_3 - \tilde{f^{12}}_1}{2a}, \tilde{f^{12}}_3 = a\tilde{f^{12}}_2, \tilde{f^{13}}_3 = \tilde{f^{12}}_2,$$

$$\tilde{f^{13}}_2 = a\tilde{f^{12}}_2, \tilde{f^{23}}_2 = \frac{-a^2\tilde{f^{12}}_1 + a^2\tilde{f^{23}}_3 + \tilde{f^{23}}_3 + \tilde{f^{12}}_1}{2a}.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$4a\tilde{f^{23}}_1\tilde{f^{12}}_2 + (a\tilde{f^{12}}_1)^2 - 2a^2\tilde{f^{12}}_1\tilde{f^{23}}_3 - 2\tilde{f^{12}}_1\tilde{f^{23}}_3 - (\tilde{f^{12}}_1)^2 +$$

$$+(a\tilde{f^{23}}_3)^2 - (\tilde{f^{23}}_3)^2 = 0.$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha\tilde{X}^1 + \beta\tilde{X}^2 + a\beta\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= -\frac{a^2\alpha^2 - 2\alpha\gamma a^2 - 2\alpha\gamma - \alpha^2 + \gamma^2 a^2 - \gamma^2}{4a\beta}\tilde{X}^1 \\
&\quad + \frac{(-a^2\alpha + \gamma a^2 + \gamma + \alpha)}{2a}\tilde{X}^2 + \gamma\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= \frac{-a^2\alpha + \gamma a^2 - \gamma - \alpha}{2a}\tilde{X}^1 - a\beta\tilde{X}^2 - \beta\tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

– Bianchi algebra $VI_{1/a}$ in the rescaled standard form VI_a (c) with $b=-a\beta$.

2.

$$\begin{aligned} [\tilde{X}^1,\tilde{X}^2] &= \alpha\tilde{X}^1, \\ [\tilde{X}^2,\tilde{X}^3] &= \beta\tilde{X}^1+\alpha\frac{a+1}{a-1}\tilde{X}^2+\alpha\frac{a+1}{a-1}\tilde{X}^1, \\ [\tilde{X}^3,\tilde{X}^1] &= \alpha\tilde{X}^1. \end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

– Bianchi algebra I in the standard form VI_a (a) if $\alpha=\beta=0$,

– Bianchi algebra II in the standard form VI_a (b) if $\alpha=0$ and $\beta\neq 0$.

– Bianchi algebra $VI_{1/a}$ in the form VI_a (c) ii. if $\alpha\neq 0$.

3.

$$\begin{aligned} [\tilde{X}^1,\tilde{X}^2] &= \alpha\tilde{X}^1, \\ [\tilde{X}^2,\tilde{X}^3] &= \beta\tilde{X}^1-\alpha\frac{a-1}{a+1}\tilde{X}^2+\alpha\frac{a-1}{a+1}\tilde{X}^3, \\ [\tilde{X}^3,\tilde{X}^1] &= -\alpha\tilde{X}^1. \end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

– Bianchi algebra I in the standard form VI_a (a) if $\alpha=\beta=0$,

– Bianchi algebra II in the standard form VI_a (b) if $\alpha=0$ and $\beta\neq 0$.

– Bianchi algebra $VI_{1/a}$ in the form VI_a (c) ii. if $\alpha\neq 0$.

- $\mathcal{G}=VI_0$

The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_3=\tilde{f^{12}}_2,\tilde{f^{13}}_1=\tilde{f^{23}}_2,\tilde{f^{12}}_1=-\tilde{f^{23}}_3,\tilde{f^{13}}_2=0,\tilde{f^{23}}_1=0.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$\tilde{f^{12}}_3\tilde{f^{23}}_2=0.$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$\begin{aligned} [\tilde{X}^1,\tilde{X}^2] &= -\alpha\tilde{X}^1+\beta\tilde{X}^2+\gamma\tilde{X}^3, \\ [\tilde{X}^2,\tilde{X}^3] &= \alpha\tilde{X}^3, \\ [\tilde{X}^3,\tilde{X}^1] &= -\beta\tilde{X}^3. \end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VI_0 (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra II in the form VI_0 (b) if $\gamma \neq 0$ and $\alpha = \beta = 0$,
- Bianchi algebra IV
  * in the rescaled standard form VI_0 (c) i. with $b = \frac{\alpha^2-\beta^2}{\gamma}$ if $\gamma \neq 0$ and $\alpha^2 \neq \beta^2$,
  * in the form VI_0 (c) ii. with if $\gamma \neq 0$ and $\alpha^2 = \beta^2 \neq 0$,
- Bianchi algebra V
  * in the standard form VI_0 (d) i. if $\gamma = 0$ and $\alpha^2 \neq \beta^2$,
  * in the form VI_0 (d) ii. if $\gamma = 0$ and $\alpha^2 = \beta^2 \neq 0$,

2.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha\tilde{X}^1 + \beta\tilde{X}^2, \\
[\tilde{X}^2, \tilde{X}^3] &= \gamma\tilde{X}^2 + \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\gamma\tilde{X}^1 - \beta\tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form VI_0 (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra V
  * in the form VI_0 (d) i. if $\gamma = 0$ and $\alpha^2 \neq \beta^2$,
  * in the form VI_0 (d) ii. if $\gamma = 0$ and $\alpha^2 = \beta^2$,
  * in the form VI_0 (d) iii. with $b = |\gamma|$ if $\gamma \neq 0$.

- $\mathcal{G} = V$

The mixed Jacobi identities (10) imply

$$\tilde{f^{12}}_1 = \tilde{f^{23}}_3, \tilde{f^{13}}_3 = -\tilde{f^{12}}_2, \tilde{f^{23}}_2 = -\tilde{f^{13}}_1.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) in this case don't impose any new condition. The general form of $\tilde{\mathcal{G}}$ is therefore

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha\tilde{X}^1 + \beta\tilde{X}^2 + \gamma\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= \delta\tilde{X}^1 - \epsilon\tilde{X}^2 + \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\epsilon\tilde{X}^1 - \zeta\tilde{X}^2 + \beta\tilde{X}^3.
\end{aligned}$$

Finding the Bianchi forms of this algebra for all values of parameters seems to be rather complicated, because this case contains also all 2nd subalgebras of duals of Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ with $\tilde{\mathcal{G}} \equiv V$ given above. Therefore we compute only the values of parameters for which $\tilde{\mathcal{G}}$ is isomorphic to $I, \ldots, V$. We find that $\tilde{\mathcal{G}}$ can be transformed into[4]

- Bianchi algebra I in the standard form V (a) if $\alpha = \beta = \gamma = \delta = \epsilon = \zeta = 0$,
- Bianchi algebra II
  - in the form V (b) i. if
    - $\exists x, y$ s.t. $\alpha = x\gamma, \beta = y\gamma, \epsilon = -xy\gamma, \zeta = -y^2\gamma, \delta = x^2\gamma, \gamma \neq 0$
    - or $\alpha = \beta = \gamma = 0$ and $\exists x$ s.t. $\epsilon = -x\delta, \zeta = -x^2\delta, x \neq 0, \delta \neq 0$
    - or $\alpha = \beta = \gamma = \delta = \epsilon = 0, \zeta \neq 0$,
  - in the form V (b) ii. if $\alpha = \beta = \gamma = \epsilon = \zeta = 0$ and $\delta \neq 0$.

- $\mathcal{G} = IV$

The mixed Jacobi identities (10) imply

$$\tilde{f^{12}}_3 = 0, \tilde{f^{12}}_2 = 0, \tilde{f^{23}}_2 = -\tilde{f^{13}}_1 - 2\tilde{f^{12}}_1, \tilde{f^{23}}_3 = \tilde{f^{12}}_1, \tilde{f^{13}}_3 = 0.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$(\tilde{f^{12}}_1)^2 = 0.$$

The solution of this equation gives the most general form of $\tilde{\mathcal{G}}$:

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= 0, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^1 - \beta\tilde{X}^2, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^1 - \gamma\tilde{X}^2.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form IV (a) if $\alpha = \beta = \gamma = 0$,
- Bianchi algebra II
  - in the standard form IV (b) i. if $\gamma = \beta = 0$ and $\alpha > 0$,
  - in the form IV (b) ii. if $\gamma = \beta = 0$ and $\alpha < 0$,
  - in the form IV (b) iii. with $b = -\gamma$ if $\gamma \neq 0$ and $\beta^2 + \alpha\gamma = 0$,
- Bianchi algebra VI_0
  - in the rescaled standard form with $b = \gamma$ if $\gamma \neq 0$ and $\beta^2 + \alpha\gamma > 0$. The corresponding Manin triple is dual to the triple VI_0 (c) i.

[4]It is helpful to exploit the fact that the commutant of II is one–dimensional, i.e. suitably written matrix of structure coefficients has rank 1.

* in the form

$$[\tilde{X}^1, \tilde{X}^2] = 0,\ [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2,\ [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^1$$

  if $\gamma = 0$ and $\beta \neq 0$. The corresponding Manin triple is dual to the triple VI_0 (c) ii.

– Bianchi algebra VII_0 in the rescaled standard form with $b = \gamma$ if $\gamma \neq 0$ and $\beta^2 + \alpha\gamma < 0$. The corresponding Manin triple is dual to the triple VII_0 (c) i.

- $\mathcal{G} = III$

The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_3 = \tilde{f^{12}}_2, \tilde{f^{12}}_1 = \tilde{f^{13}}_1, \tilde{f^{12}}_3 = \tilde{f^{12}}_2, \tilde{f^{13}}_2 = \tilde{f^{12}}_2, \tilde{f^{23}}_3 = \tilde{f^{23}}_2.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$\tilde{f^{23}}_1 \tilde{f^{12}}_2 - \tilde{f^{13}}_1 \tilde{f^{23}}_3 = 0.$$

The solutions of this equation give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha\tilde{X}^1 + \beta\tilde{X}^2 + \beta\tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= \frac{\alpha\gamma}{\beta}\tilde{X}^1 + \gamma\tilde{X}^2 + \gamma\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\alpha\tilde{X}^1 - \beta\tilde{X}^2 - \beta\tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

– Bianchi algebra III in the rescaled standard form III (c) i. with $b = 1/\beta$.

2.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= 0, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^1 + \beta\tilde{X}^2 + \beta\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= 0.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

– Bianchi algebra I in the standard form III (a) if $\alpha = \beta = 0$,

– Bianchi algebra II in the form III (b) i. if $\beta = 0$ and $\alpha \neq 0$,

– Bianchi algebra III in the form III (c) ii. if $\beta \neq 0$.

3.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha \tilde{X}^1, \\
[\tilde{X}^2, \tilde{X}^3] &= \beta \tilde{X}^1, \\
[\tilde{X}^3, \tilde{X}^1] &= -\alpha \tilde{X}^1.
\end{aligned}$$

$\tilde{\mathcal{G}}$ can be transformed into

- Bianchi algebra I in the standard form III (a) if $\alpha = \beta = 0$,
- Bianchi algebra II in the form III (b) i. if $\alpha = 0$ and $\beta \neq 0$,
- Bianchi algebra III in the form III (c) iii. if $\alpha \neq 0$.

- $\mathcal{G} = II$

Finding the Bianchi forms of the 2nd algebra for all values of parameters again seems to be rather complicated, because it contains also all 2nd subalgebras of duals of Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ with $\tilde{\mathcal{G}} \equiv II$ given above. Therefore we compute only the values of parameters for which possible $\tilde{\mathcal{G}}$s are isomorphic to I, II.

The mixed Jacobi identities (10) imply

$$\tilde{f^{13}}_1 = \tilde{f^{23}}_2, \tilde{f^{23}}_1 = 0, \tilde{f^{12}}_1 = -\tilde{f^{23}}_3.$$

The Jacobi identities in $\tilde{\mathcal{G}}$ (9) reduce to

$$-\tilde{f^{13}}_3 \tilde{f^{23}}_3 + \tilde{f^{23}}_3 \tilde{f^{12}}_2 - 2\tilde{f^{12}}_3 \tilde{f^{23}}_2 = 0,$$

$$-2\tilde{f^{13}}_2 \tilde{f^{23}}_3 - \tilde{f^{12}}_2 \tilde{f^{23}}_2 + \tilde{f^{23}}_2 \tilde{f^{13}}_3 = 0.$$

The solutions of these equations give the following most general forms of $\tilde{\mathcal{G}}$:

1.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha \tilde{X}^1 - \frac{2\beta\alpha - \gamma\delta}{\gamma} \tilde{X}^2 - \frac{\alpha^2\beta}{\gamma^2} \tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= \gamma \tilde{X}^2 + \alpha \tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\gamma \tilde{X}^1 - \beta \tilde{X}^2 - \delta \tilde{X}^3.
\end{aligned}$$

$\tilde{\mathcal{G}}$ of this form represents Bianchi algebras IV, V only.

2.

$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= \alpha \tilde{X}^2 + \beta \tilde{X}^3, \\
[\tilde{X}^2, \tilde{X}^3] &= 0, \\
[\tilde{X}^3, \tilde{X}^1] &= -\gamma \tilde{X}^2 - \delta \tilde{X}^3.
\end{aligned}$$

– Bianchi algebra I in the standard form V (a) if $\alpha = \beta = \gamma = \delta = 0$,
– Bianchi algebra II
  * in the form II (b) i. if $\exists x : \gamma = -x^2\beta, \delta = -x\beta, \alpha = x\beta, \beta > 0$ or $\delta = \alpha = \beta = 0, \gamma < 0$,
  * in the form II (b) ii. if $\exists x : \gamma = -x^2\beta, \delta = -x\beta, \alpha = x\beta, \beta < 0$ or $\delta = \alpha = \beta = 0, \gamma > 0$,

Bianchi algebras $III, IV, V, VI_a, VI_0, VII_a, VII_0$ otherwise.

3.
$$\begin{aligned}
[\tilde{X}^1, \tilde{X}^2] &= -\alpha\tilde{X}^1 + \beta\tilde{X}^2 + \gamma\tilde{X}^1, \\
[\tilde{X}^2, \tilde{X}^3] &= \alpha\tilde{X}^3, \\
[\tilde{X}^3, \tilde{X}^1] &= -\beta\tilde{X}^3.
\end{aligned}$$

– Bianchi algebra I in the standard form V (a) if $\alpha = \beta = \gamma = 0$,
– Bianchi algebra II
  * in the form II (b) i. if $\alpha = \beta = 0, \gamma > 0$,
  * in the form II (b) ii. if $\alpha = \beta = 0, \gamma < 0$,

Bianchi algebras IV, V otherwise.

• $\mathcal{G} = I$
$\tilde{\mathcal{G}}$ might be any 3–dimensional Lie algebra, it can be brought to its Bianchi form by the transformation (4).

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