Spectral analysis of non-self-adjoint Jacobi operator associated with Jacobian elliptic functions

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Workshop on Operator Theory, Complex Analysis, and Applications 2016
June 21-24
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6 Intermezzo II - extremal properties of $|\text{sn}(uK(m) | m)|$
Jacobi operators associated with complex semi-infinite Jacobi matrix

To the semi-infinite Jacobi matrix

\[
\mathcal{J} = \begin{pmatrix}
  b_1 & a_1 & & \\
  a_1 & b_2 & a_2 & \\
  a_2 & b_3 & a_3 & \\
  & & & \ddots
\end{pmatrix}
\]

where \( b_n \in \mathbb{C} \) and \( a_n \in \mathbb{C} \setminus \{0\} \), we associate two operators \( J_{\text{min}} \) and \( J_{\text{max}} \) acting on \( \ell^2(\mathbb{N}) \).
Jacobi operators associated with complex semi-infinite Jacobi matrix

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\[ J = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 & a_2 \\ & a_2 & b_3 & a_3 \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \]

where \( b_n \in \mathbb{C} \) and \( a_n \in \mathbb{C} \setminus \{0\} \), we associate two operators \( J_{\text{min}} \) and \( J_{\text{max}} \) acting on \( \ell^2(\mathbb{N}) \).

\( J_{\text{min}} \) is the operator closure of \( J_0 \), an operator defined on \( \text{span}\{e_n \mid n \in \mathbb{N}\} \) by

\[ J_0 e_n := a_{n-1} e_{n-1} + b_n e_n + a_n e_{n+1}, \quad \forall n \in \mathbb{N}, \ (a_0 := 0). \]
Jacobi operators associated with complex semi-infinite Jacobi matrix

- To the semi-infinite Jacobi matrix

\[ J = \begin{pmatrix} \ldots & \ldots & \ldots \\ b_1 & a_1 & \ldots \\ a_1 & b_2 & a_2 \\ \vdots & \vdots & \vdots \\ \end{pmatrix} \]

where \( b_n \in \mathbb{C} \) and \( a_n \in \mathbb{C} \setminus \{0\} \), we associate two operators \( J_{\text{min}} \) and \( J_{\text{max}} \) acting on \( \ell^2(\mathbb{N}) \).

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- \( J_{\text{max}} \) acts as \( J_{\text{max}} x := J \cdot x \) (formal matrix product) on vectors from

\[ \text{Dom} \ J_{\text{max}} = \{ x \in \ell^2(\mathbb{N}) \mid J \cdot x \in \ell^2(\mathbb{N}) \} \].
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Both operators \( J_{\text{min}} \) and \( J_{\text{max}} \) are closed and densely defined.
Jacobi operators associated with complex semi-infinite Jacobi matrix

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where \(b_n \in \mathbb{C}\) and \(a_n \in \mathbb{C} \setminus \{0\}\), we associate two operators \(J_{\text{min}}\) and \(J_{\text{max}}\) acting on \(\ell^2(\mathbb{N})\).

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- Both operators \(J_{\text{min}}\) and \(J_{\text{max}}\) are closed and densely defined. They are related as

  \[
  J_{\text{max}}^* = C J_{\text{min}} C \quad \text{and} \quad J_{\text{min}}^* = C J_{\text{max}} C
  \]

  where \(C\) is the complex conjugation operator, \((Cx)_n = \overline{x_n}\).
Proper case and spectrum of Jacobi operator

- Any closed operator $A$ having $\text{span}\{e_n \mid n \in \mathbb{N}\} \subset \text{Dom}(A)$ and defined by the matrix product satisfies $J_{\text{min}} \subset A \subset J_{\text{max}}$. 

As a consequence, $\sigma_r(J) = \emptyset$.

We have the decomposition:

$$\sigma(J) = \sigma_p(J) \cup \sigma_c(J) = \sigma_p(J) \cup \sigma_{\text{ess}}(J)$$

where the essential spectrum has the simple characterization:

$$\sigma_{\text{ess}}(J) = \{z \in \mathbb{C} \mid \text{Ran}(J - z) \text{ is not closed}\}.$$
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- In general $J_{\text{min}} \neq J_{\text{max}}$. If $J_{\text{min}} = J_{\text{max}}$, the matrix $J$ is called proper and the operator $J := J_{\text{min}} \equiv J_{\text{max}}$ the Jacobi operator associated with $J$. 

Let $J_{\text{min}} = J_{\text{max}} =: J$. Then $J^* = CJC$.

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For $\alpha \in \mathbb{C}$, the semi-infinite Jacobi matrix

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J = \begin{pmatrix}
0 & 1 & 2\alpha & \cdots \\
1 & 0 & 2\alpha & \cdots \\
2\alpha & 0 & 3 & \cdots \\
3 & 0 & 4\alpha & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

is proper, and hence it determines a unique densely defined closed operator $J(\alpha)$. 
The Jacobi matrix associated with Jacobian elliptic functions

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is proper, and hence it determines a unique densely defined closed operator $J(\alpha)$.

- The aim of this talk is the investigation of spectral properties of $J(\alpha)$ for $\alpha \in \mathbb{C}$. 
For $\alpha \in \mathbb{C}$, the semi-infinite Jacobi matrix

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is proper, and hence it determines a unique densely defined closed operator $J(\alpha)$.

The aim of this talk is the investigation of spectral properties of $J(\alpha)$ for $\alpha \in \mathbb{C}$.

We will restrict with $\alpha$ to the unit disk $|\alpha| \leq 1$. The spectral properties of $J(\alpha)$ for $|\alpha| > 1$ are very similar to those for $|\alpha| < 1$. 
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Intermezzo I - Jacobian elliptic functions

Jacobian elliptic functions

For $0 \leq \alpha \leq 1$, the integral (incomplete elliptic of 1st kind)

$$u = \int_{0}^{\varphi} \frac{d\theta}{\sqrt{1 - \alpha^2 \sin^2 \theta}}$$

measures the arc length of an ellipse.
Jacobian elliptic functions

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  u = \int_{0}^{\varphi} \frac{d\theta}{\sqrt{1 - \alpha^2 \sin^2 \theta}}
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- Its inverse \(\varphi(u) = \text{am}(u, \alpha)\) is known as the amplitude.
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The (copolar) triplet of JEF:

- $\text{sn}(u, \alpha) = \sin \text{am}(u, \alpha)$,
- $\text{cn}(u, \alpha) = \cos \text{am}(u, \alpha)$,
- $\text{dn}(u, \alpha) = \sqrt{1 - \alpha^2 \sin^2 \text{am}(u, \alpha)}$. 
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- The (copolar) triplet of JEF:
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  \begin{align*}
  \text{sn}(u, \alpha) &= \sin \text{am}(u, \alpha), \\
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  \end{align*}
  \]
- Complete elliptic integral of the first kind:
  \[ K(\alpha) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - \alpha^2 \sin^2 \theta}}. \]
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Complete elliptic integral of the first kind:

$$K(\alpha) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \alpha^2 \sin^2 \theta}}.$$

JEFs are meromorphic functions in $u$ (with $\alpha$ fixed) as well as meromorphic functions in $\alpha$ (with $u$ fixed). While $K$ is analytic in the cut-plane $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. 
Intermezzo I - Jacobian elliptic functions

Jacobian elliptic functions - plotting

![Graph of Jacobian elliptic functions](image)

Complex Jacobi Matrix associated with JEF

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Spectral analysis of $J(\alpha)$ in the self-adjoint case

We start with the identities

$$\langle e_1, J(\alpha)^{2n+1} e_1 \rangle = 0 \quad \text{and} \quad \langle e_1, J(\alpha)^{2n} e_1 \rangle = C_{2n}(\alpha)$$
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where \( C_{2n} \) are polynomials that can be defined via the generating function formula:

\[ c_{n}(z, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}(\alpha)}{(2n)!} z^{2n}. \]
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$$c_n(z, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}(\alpha)}{(2n)!} z^{2n}.$$

Hence we may write

$$c_n(z, \alpha) = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \langle e_1, J(\alpha)^n e_1 \rangle = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \int_{\mathbb{R}} x^n d\mu(x) = \int_{\mathbb{R}} e^{ixz} d\mu(x).$$

where we denote $\mu(\cdot) := \langle e_1, E_J(\cdot) e_1 \rangle$. 
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where $C_{2n}$ are polynomials that can be defined via the generating function formula:

$$cn(z, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}(\alpha)}{(2n)!} z^{2n}.$$ 

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where we denote $\mu(\cdot) := \langle e_1, E_J(\cdot)e_1 \rangle$.

- We get

$$\mathcal{F}[\mu](z) = cn(z, \alpha).$$
Spectral analysis of \( J(\alpha) \) in the self-adjoint case

- We start with the identities

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where \( C_{2n} \) are polynomials that can be defined via the generating function formula:

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c_n(z, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}(\alpha)}{(2n)!} z^{2n}.
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\]

where we denote \( \mu(\cdot) := \langle e_1, E J(\cdot) e_1 \rangle \).

- We get

\[
\mathcal{F}[\mu](z) = cn(z, \alpha).
\]

Consequently, by applying the inverse Fourier transform to the function \( cn(z; \alpha) \), one may recover the spectral measure \( \mu \)!
Spectral analysis of \( J(\alpha) \) for \( \alpha \in (-1, 1) \)

For \( \alpha \in (-1, 1) \), the evaluation of the inverse Fourier transform yields

\[
\mu(t) = \frac{\pi}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \left[ \delta \left( t - \frac{(2n + 1)\pi}{2K} \right) + \delta \left( t + \frac{(2n + 1)\pi}{2K} \right) \right]
\]

where the nome \( q = q(\alpha) \) (\( |q| < 1 \)).
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where the nome $q = q(\alpha)$ ($|q| < 1$).

Hence the measure $\mu$ is discrete supported by the set

$$\text{supp } \mu = \frac{\pi}{2K} (2\mathbb{Z} + 1).$$
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Hence the measure $\mu$ is discrete supported by the set

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This implies that, for $\alpha \in (-1, 1)$, the spectrum of $J(\alpha)$ is discrete and

$$
\sigma(J(\alpha)) = \sigma_p(J(\alpha)) = \frac{\pi}{2K} (2\mathbb{Z} + 1).
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In addition, we can also compute the Weyl $m$-function $m(z; \alpha) := \langle e_1, (J(\alpha) - z)^{-1} e_1 \rangle$, since

$$m(z, \alpha) = i\mathcal{L}[\text{cn}(t, \alpha)](-iz), \quad \text{for } \Re z > 0.$$
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  where the nome $q = q(\alpha)$ ($|q| < 1$).

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  \[
  m(z, \alpha) = i\mathcal{L}[\text{cn}(t, \alpha)](-iz), \quad \text{for } \Re z > 0.
  \]

- It results in the formula
  \[
  m(z, \alpha) = \frac{2\pi z}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \frac{1}{\frac{(2n+1)^2\pi^2}{4K^2} - z^2}.
  \]
Spectral analysis of $J(\alpha)$ for $\alpha = \pm 1$

Recall that

$$\mathcal{F}[\mu](z) = \text{cn}(z, \pm 1) = \frac{1}{\cosh(z)}.$$
Recall that
\[ \mathcal{F}[\mu](z) = \text{cn}(z, \pm 1) = \frac{1}{\cosh(z)}. \]

By applying the inverse Fourier transform, one concludes that \( \mu \) is absolutely continuous measure supported on \( \mathbb{R} \) and its density equals
\[ \frac{d\mu}{dt} = \frac{1}{2 \cosh (\pi t/2)}, \quad \forall t \in \mathbb{R}. \]
Spectral analysis of $J(\alpha)$ for $\alpha = \pm 1$

- Recall that
  \[ \mathcal{F}[\mu](z) = \text{cn}(z, \pm 1) = \frac{1}{\cosh(z)}. \]

- By applying the inverse Fourier transform, one concludes that $\mu$ is absolutely continuous measure supported on $\mathbb{R}$ and its density equals
  \[ \frac{d\mu}{dt} = \frac{1}{2 \cosh(\pi t/2)}, \quad \forall t \in \mathbb{R}. \]

- This implies that the spectrum of $J(\pm 1)$ is purely absolutely continuous and
  \[ \sigma(J(\pm 1)) = \sigma_{ac}(J(\pm 1)) = \mathbb{R}. \]
Spectrum of $J(\alpha)$ in the self-adjoint case - animation
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1 Complex Jacobi matrices - generalities
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Spectral analysis of $J(\alpha)$ for $|\alpha| < 1$

- For $|\alpha| < 1$, the operator $J(\alpha)$ can be viewed as a perturbation of $J(0)$ with relative bound smaller than 1.
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- For $|\alpha| < 1$, the operator $J(\alpha)$ can be viewed as a perturbation of $J(0)$ with relative bound smaller than 1.
- Consequently, the spectrum of $J(\alpha)$ is discrete if $|\alpha| < 1$. 

\[
m(z, \alpha) = \frac{2\pi z}{\alpha K_\infty} \sum_{n=0}^{q_n+1} \frac{1}{1 + \frac{q_n+1}{2 \pi K^2 - z^2}}
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It implies (in the non-self-adjoint case, too!) that $\sigma(J(\alpha)) = \frac{\pi}{2} K(2Z + 1)$. 

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$$m(z, \alpha) = \frac{2\pi z}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \frac{1}{\left(\frac{(2n+1)^2 \pi^2}{4K^2} - z^2\right)}.$$ 

remains true for all $z \in \rho(J(\alpha))$ and $|\alpha| < 1$. 
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Let $0 < |\alpha| < 1$ and $N \in \mathbb{Z}$, then the vector $v^{(N)}$ given by formulas

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1. $\|v^{(N)}\| = ?$ or $\|v^{(N)}\| \sim ?$ for $N \to \pm \infty$.
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3. Then one can verify, indeed, that
   $$\lim_{a \to 1^-} \frac{\| (J(\alpha) - z) u(a) \|}{\| u(a) \|} = 0,$$
   and
   $$w-\lim_{a \to 1^-} u(a) = 0.$$
Spectral analysis of $J(\alpha)$ for $|\alpha| = 1$ - cont.

Essential for the verification of the “singular property” of the family $u(a) = a^n u_n$ are two main ingredients:
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   $$u \to |\sn(uK(\alpha), \alpha)|$$

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- It can be shown (not trivial!) that the function has unique global maximum at $u = 1$ for every $|\alpha| = 1, \alpha \neq \pm 1$. 

František Štampach (Stockholm University)
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On the extremal properties of $|\text{sn}(uK(m) | m)|$

All the necessary properties known when $m \in (0, 1)$
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

Transf. modulus $m$ from the unit circle to $(0, 1)$
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

\[
\text{sn}^2(uK(m) \mid m) = \frac{1}{\sqrt{m}} \frac{c_1^2 + s_1^2 s_1^2 \cos^2 \theta - cc_1 + iss_1 dd_1}{c_1^2 + s_1^2 s_1^2 \cos^2 \theta + cc_1 - iss_1 dd_1},
\]

where $\Im m > 0$, $m = e^{4i\theta}$ and $s = \text{sn}(uK(\cos^2 \theta) \mid \cos^2 \theta)$, $s_1 = \text{sn}(uK(\sin^2 \theta) \mid \sin^2 \theta)$, etc.
On the extremal properties of $|\text{sn}(uK(m) | m)|$

$|\text{sn}(uK(m) | m)| \leq 1$ for all $m \in \partial \mathbb{D}$ with the equality only for $m = 1$. 

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Another circle $\mathbb{D}_1 = \{z \mid |z - 1| = 1\}$
Another circle $D_1 = \{ z \mid |z - 1| = 1 \}$ and another transformation formula (not displayed) ...
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

$|\text{sn}(uK(m) \mid m)| < 1$ for all $m \in \partial \mathbb{D}_1$
On the extremal properties of \( |\text{sn}(uK(m) \mid m)| \)

\[
|\text{sn}(uK(m) \mid m)| \sim 0
\]

In addition,
\[
\lim_{\epsilon \to 0^+} |\text{sn}(uK(m \pm i\epsilon) \mid m \pm i\epsilon)| < 1
\]
for all \( m \geq 2 \) and

the function \( m \mapsto \text{sn}(uK(m) \mid m) \) is bounded.
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

\[ |\text{sn}(uK(m) \mid m)| \leq 1 \quad \text{for all} \quad m \notin \mathbb{D}_1 \setminus \mathbb{D} \quad \text{with the equality only for} \quad m = 1. \]
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If $0 < u \leq \frac{1}{2}$, the global maximum of $m \mapsto |\text{sn}(uK(m) \mid m)|$ is located at $m = 1$ with the value $= 1$. 
On the extremal properties of $\left| \text{sn}(uK(m) \mid m) \right|$

If $\frac{1}{2} < u < 1$, the global maximum of $m \mapsto \left| \text{sn}(uK(m) \mid m) \right|$ is located in $(1, 2)$ with the value $> 1$. 
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$ - main theorem

**Theorem:**

The following statements hold true.

1. For all $u \in (0, 1)$ and $m \not\in \{z \in \mathbb{C} \mid ||z - 1|| < 1 \land ||z|| > 1\}$, it holds $|\text{sn}(uK(m) \mid m)| < 1$.

2. For $u \in (0, 1/2]$ the function $m \mapsto |\text{sn}(uK(m) \mid m)|$ has unique global maximum located at $m = 1$ with the value equal to $1$, i.e., $|\text{sn}(K(1)u \mid 1)| = 1$ and $|\text{sn}(K(m)u \mid m)| < 1$ for all $m \neq 1$ (where the value at $m = 1$ is to be understood as the respective limit).

3. For $u \in (1/2, 1)$, the function $m \mapsto |\text{sn}(K(m)u \mid m)|$ has a global maximum located in the interval $(1, 2)$ with the value exceeding $1$, i.e., $\max_{m \in \mathbb{C}} |\text{sn}(K(m)u \mid m)| = |\text{sn}(K(m^*)u \mid m^*)| > 1$ for some $m^* \in (1, 2)$. 

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References:


Thank you!