Continued Fractions Appearing Naturally in Spectral Analysis of Jacobi Operators

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Outline

1. Function $\psi$ and its properties

2. Stieltjes continued fractions and formal power series

3. The Rogers-Ramanujan continued fraction

4. A note on the role of continued fractions in spectral analysis of Jacobi operators
Function $\mathcal{E}$

**Definition**

Define $\mathcal{E} : \ell^1(\mathbb{N}) \to \mathbb{C}$ by

$$\mathcal{E}(y) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} y_{k_1} y_{k_2} \cdots y_{k_m}.$$

- For a finite number of complex variables we identify $\mathcal{E}(y_1, y_2, \ldots, y_n)$ with $\mathcal{E}(y)$ where $y = (y_1, y_2, \ldots, y_n, 0, 0, 0, \ldots)$. 

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- $\mathcal{E}$ is well defined on $\ell^1(\mathbb{N})$ due to estimation

$$|\mathcal{E}(y)| \leq \exp \|y\|_1.$$
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$\mathcal{E}$ is well defined on $\ell^1(\mathbb{N})$ due to estimation

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$\mathcal{E}$ is a continuous functional on $\ell^1(\mathbb{N})$ which is not linear. Especially, for any $y \in \ell^1(\mathbb{N})$, it satisfies limit relations

$$\lim_{n \to \infty} \mathcal{E}(y_1, y_2, \ldots, y_n) = \mathcal{E}(y) \quad \text{and} \quad \lim_{n \to \infty} \mathcal{E}(T^n y) = 1$$

where $T$ stands for unilateral right-shift operator on the space of complex sequence.
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  where $T$ stands for unilateral right-shift operator on the space of complex sequence.
- Function $\mathcal{E}$ fulfills many nice and simple algebraic and combinatorial identities.
- Function $\mathcal{E}$ have been developed for investigation of various spectral properties of Jacobi operators and it found many application here (not the scope of this talk).
For $y \in \ell^1(\mathbb{N})$, $\mathcal{E}$ satisfies recurrence rule

$$
\mathcal{E}(y) = \mathcal{E}(Ty) - y_1 \mathcal{E}(T^2y).
$$

Consequently, function $\mathcal{E}$ is related to the Stieltjes continued fraction (S-fraction). For a given $y \in \ell^1(\mathbb{N})$ such that $\mathcal{E}(y) \neq 0$, it holds

$$
\mathcal{E}(Ty) \mathcal{E}(y) = \frac{1}{1 - y_1 \frac{1}{1 - y_2 \frac{1}{1 - \ldots}}}
$$

The LHS of the last identity can be viewed as a formal power series in countably many indeterminates $y = \{y_k\}_{k=1}^{\infty}$. They form the ring $\mathbb{C}\llbracket y \rrbracket$.

People realized there is certain connection between an S-fraction and formal power series a long time ago:


Particularly, they study cases when $y_k = xe_k$ where $e_k$ is a fixed complex sequence and $x$ is a complex variable: L.J. Rogers: *On the representation of certain asymptotic series as convergent continued fractions*, 1907.
For \( y \in \ell^1(\mathbb{N}) \), \( \varepsilon \) satisfies recurrence rule

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\frac{\varepsilon(Ty)}{\varepsilon(y)} = \frac{1}{1 - \frac{y_1}{1 - \frac{y_2}{1 - \frac{y_3}{1 - \ldots}}}}.
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For $y \in \ell^1(\mathbb{N})$, $\mathcal{E}$ satisfies recurrence rule

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T.J. Stieltjes: *Recherches sur les fractions continues*, 1894-95. Particularly, they study cases when $y_k = xe^k$ where $e^k$ is a fixed complex sequence and $x$ is a complex variable:

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But how exactly one associates the S-fraction with a formal power series?

Let \( a = \{a_k\}_{k=1}^{\infty} \subset \mathbb{C} \). The formal Stieltjes continued fraction

\[
\frac{1}{1 - a_1 \frac{1}{1 - a_2 \frac{1}{1 - a_3 \ddots}}}
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is regarded here as a sequence of convergents \( \frac{A_n}{B_n}, n = 0, 1, 2, \ldots \), with \( A_n, B_n \in \mathbb{C}[a] \) defined by the usual recurrence rules

\[
A_0 = 0, \quad A_1 = 1, \quad B_0 = B_1 = 1, \quad A_n = A_{n-1} - a_{n-1} A_{n-2}, \quad B_n = B_{n-1} - a_{n-1} B_{n-2}, \quad n \geq 2.
\]

Since the constant term of \( B_n \) equals 1 for any \( n \), the convergents \( \frac{A_n}{B_n} \) can also be treated as formal power series \( A_n/B_n \).

Sequence \( A_n/B_n \) is always convergent in \( \mathbb{C}[a] \) equipped with canonical (product) topology.

Thus with every formal S-fraction there is naturally associated a unique formal power series \( f(a) \) in the indeterminates \( a \).

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Sequence \( A_nB_n^{-1} \) is always convergent in \( \mathbb{C}[[a]] \) equipped with canonical (product) topology.
Formal S-fraction and formal power series

But how exactly one associates the S-fraction with a formal power series?

Let $a = \{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$. The formal Stieltjes continued fraction

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Sequence $A_nB_n^{-1}$ is always convergent in $\mathbb{C}[[a]]$ equipped with canonical (product) topology.

Thus with every formal S-fraction there is naturally associated a unique formal power series $f(a)$ in the indeterminates $a$. 
Explicit expression for $f(a)$

- For a multiindex $m \in \mathbb{N}^\ell$ denote by $|m| = \sum_{j=1}^{\ell} m_j$ its order and by $d(m) = \ell$ its length.

\[ f(a) = 1 + \sum_{N=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \beta(m) d(m) \prod_{j=1}^{\ell} a_{m_j} \]

Surprisingly, the explicit formula for $f(a)$ has been derived much later:

- P. Flajolet: Combinatorial aspects of continued fractions, 1980.
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- For a multiindex $m \in \mathbb{N}^\ell$ put
  $$\beta(m) := \prod_{j=1}^{\ell-1} \left( m_j + m_{j+1} - 1 \right), \quad \alpha(m) := \frac{\beta(m)}{m_1}.$$
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**Theorem**

The formal power series $f(a) \in \mathbb{C}[[a]]$ associated with the formal Stieltjes continued fraction is given by the formula

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f(a) = 1 + \sum_{N=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \beta(m) \prod_{j=1}^{d(m)} a_j^{m_j}.\]
Explicit expression for \( f(a) \)

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Surprisingly, the explicit formula for \( f(a) \) has been derived much later:
Explicit expression for $f(a)$

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Surprisingly, the explicit formula for $f(a)$ has been derived much later:

Theorem

In the ring of formal power series in the indeterminates $y_1, \ldots, y_n$, one has

$$\log \varepsilon(y_1, \ldots, y_n) = - \sum_{N=1}^{\infty} \sum_{\substack{m \in \mathcal{M}(N) \colon \frac{d(m)}{d(m)} < n}} \alpha(m) \sum_{k=1}^{n-d(m)} \prod_{j=1}^{d(m)} y_{k+j-1}^m.$$  

For a complex sequence $y = \{y_k\}_{k=1}^{\infty}$ such that $\|y\|_1 < \log 2$ one has

$$\log \varepsilon(y) = - \sum_{N=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \alpha(m) \sum_{k=1}^{d(m)} \prod_{j=1}^{m_j} y_{k+j-1}^m.$$
Theorem

In the ring of formal power series in the indeterminates $y_1, \ldots, y_n$, one has

$$\log \mathcal{E}(y_1, \ldots, y_n) = -\sum_{N=1}^{\infty} \sum_{\substack{m \in \mathcal{M}(N) \\colon d(m) < n}} \alpha(m) \sum_{k=1}^{n-d(m)} \prod_{j=1}^{d(m)} y_{k+j-1}^{m_j}.$$ 

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Main ingredients for the proof:
**Theorem**

In the ring of formal power series in the indeterminates $y_1, \ldots, y_n$, one has

$$
\log \mathcal{C}(y_1, \ldots, y_n) = - \sum_{N=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \alpha(m) \sum_{k=1}^{n-d(m)} \prod_{j=1}^{d(m)} y_{k+j-1}^{m_j}.
$$

For a complex sequence $y = \{y_k\}_{k=1}^{\infty}$ such that $\|y\|_1 < \log 2$ one has

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Main ingredients for the proof:
- It holds identity $\mathcal{C}(y_1, \ldots, y_n) = \det(I + Y)$ where $Y$ is an $(n + 1) \times (n + 1)$ with elements in terms of $y_1, \ldots, y_n$. 
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- Since $\det \exp(A) = \exp(\text{Tr } A)$ and so $\log \det(I + Y) = \text{Tr } \log(I + Y)$, one gets

$$\log \mathcal{E}(y_1, \ldots, y_n) = \text{Tr } \log(I + Y) = - \sum_{N=1}^{\infty} \frac{1}{N} \text{Tr } Y^N.$$
Theorem

In the ring of formal power series in the indeterminates $y_1, \ldots, y_n$, one has

$$\log \mathcal{E}(y_1, \ldots, y_n) = -\sum_{N=1}^{\infty} \sum_{m \in M(N)} \sum_{d(m) < n}^{\infty} \alpha(m) \sum_{k=1}^{n-d(m)} \prod_{j=1}^{d(m)} y_{k+j-1}^{m_j}.$$ 

For a complex sequence $y = \{y_k\}_{k=1}^{\infty}$ such that $\|y\|_1 < \log 2$ one has

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Main ingredients for the proof:

- It holds identity $\mathcal{E}(y_1, \ldots, y_n) = \det(I + Y)$ where $Y$ is an $(n+1) \times (n+1)$ with elements in terms of $y_1, \ldots, y_n$.
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  $$\log \mathcal{E}(y_1, \ldots, y_n) = \text{Tr} \log(I + Y) = -\sum_{N=1}^{\infty} \frac{1}{N} \text{Tr} Y^N.$$ 
- To find an expression for $\text{Tr} Y^N$ is a hard combinatorial work of the proof.
As a consequence of the formula for logarithm of $\epsilon$ and its relation with an S-fraction one gets the following identity.

**Theorem**

Let $f(a) \in \mathbb{C}[a]$ be the formal power series expansion of the formal Stieltjes continued fraction. Then

$$\log f(a) = \sum_{N=1}^{\infty} \sum_{m \in M(N)} \alpha(m) d(m) \prod_{j=1}^{m} a_{m,j}.$$
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- This formula seems to be new. (Really?)
Logarithm of the power series of S-fraction

As a consequence of the formula for logarithm of $\mathcal{C}$ and its relation with an S-fraction one gets the following identity.

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This formula seems to be new. (Really?)

By using this result one can rediscover the power series expansion $f(a)$ which has been found in 1975.
Perhaps the simplest example is obtained if we set $a_j = z$, for all $j \in \mathbb{N}$, in the formal S-fraction. The formula for logarithm then yields

$$\log \left( \frac{1}{1 - \frac{z}{1 - \frac{z}{1 - \frac{z}{1 - \cdots}}} \right) = \sum_{N=1}^{\infty} \frac{1}{2N} \binom{2N}{N} z^N = \log \frac{2}{1 + \sqrt{1 - 4z}}.$$
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Since $c(z) = 2/(1 + \sqrt{1 - 4z})$ is known to be the generating function for the Catalan numbers, one derives this way an identity relating $\beta(m)$ with Catalan numbers,

$$\sum_{m \in \mathcal{M}(N)} \beta(m) = C_N := \frac{1}{N+1} \binom{2N}{N}.$$
The Rogers-Ramanujan continued fraction

The generalized Rogers-Ramanujan continued fraction

\[
\frac{1}{1 + \frac{z}{1 + \frac{qz}{1 + \frac{q^2z}{1 + \frac{q^3z}{1 + \cdots}}}}}
\]

represents a more involved example.
The Rogers-Ramanujan continued fraction

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\[
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\]

represents a more involved example.

**Theorem**

The power series expansion \( R(z; q) \) in the variable \( z \) of the generalized Rogers-Ramanujan continued fraction fulfills

\[
R(z; q) = 1 + \sum_{N=1}^{\infty} \left( \sum_{m \in \mathcal{M}(N)} \beta(m) q^{\epsilon_1(m)} \right) (-z)^N
\]

and

\[
\log R(z; q) = \sum_{N=1}^{\infty} \left( \sum_{m \in \mathcal{M}(N)} \alpha(m) q^{\epsilon_1(m)} \right) (-z)^N
\]

where

\[
\epsilon_1(m) = \sum_{j=1}^{d(m)} (j - 1)m_j.
\]
For $0 < q < 1$ and $z \in \mathbb{C}$, it can be shown

$$e \left( \{-zq^{k-1}\}_{k=1}^\infty \right) = _0\phi_1(0; q, z).$$
The Rogers-Ramanujan continued fraction

For $0 < q < 1$ and $z \in \mathbb{C}$, it can be shown

$$e \left( \{-zq^k\}_{k=1}^{\infty} \right) = \phi_1(0; q, z).$$

Consequently, for $R(z; q)$ it holds

$$R(z; q) = \phi_1(0; q, qz) / \phi_1(0; q, z).$$
For $0 < q < 1$ and $z \in \mathbb{C}$, it can be shown

$$
\mathcal{E} \left( \{-zq^k\}_{k=1}^{\infty} \right) = {}_0\phi_1(0; q, z).
$$

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$$
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$$

By putting $z = q$ one arrives at the Rogers-Ramanujan continued fraction

$$
\frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}.
$$
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By putting $z = q$ one arrives at the Rogers-Ramanujan continued fraction
\[ \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}}. \]

Its convergents are expressible in terms of the q-Fibonacci numbers of the first and second kind:

\[ F_0(q) = 0, \quad F_1(q) = 1, \quad F_n(q) = F_{n-1}(q) + q^{n-2}F_{n-2}(q) \quad \text{for } n \geq 2, \]

and
\[ \hat{F}_0(q) = 0, \quad \hat{F}_1(q) = 1, \quad \hat{F}_n(q) = \hat{F}_{n-1}(q) + q^{n-1}\hat{F}_{n-2}(q) \quad \text{for } n \geq 2. \]

The Rogers-Ramanujan continued fraction

For $0 < q < 1$, there exists the limits

$$F_\infty(q) = \lim_{n \to \infty} F_n(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + \ldots,$$

$$\hat{F}_\infty(q) = \lim_{n \to \infty} \hat{F}_n(q) = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 + 4q^{10} + \ldots.$$
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By using the relation between $\zeta$ and $\zeta_0$ again, one finds well known identities

$$F_{\infty}(q) = \zeta_0( ; 0; q, q), \quad \hat{F}_{\infty}(q) = \zeta_0( ; 0; q, q^2).$$
The Rogers-Ramanujan continued fraction

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$$F_{\infty}(q) = 0 \phi_1( ; 0; q, q), \quad \hat{F}_{\infty}(q) = 0 \phi_1( ; 0; q, q^2).$$

Consequently, one arrives at the known relation

$$R(q) := R(q, q) = \hat{F}_{\infty}(q)/F_{\infty}(q).$$
The Rogers-Ramanujan continued fraction

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  \[
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  \]
  \[
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  \]

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  \[
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  \]

- Consequently, one arrives at the known relation
  \[
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  \]

- The celebrated Rogers-Ramanujan identities extend this identity to a much stronger result by showing
  \[
  0\phi_1( ; 0; q, q) = \prod_{\mathbb{N} \ni n \equiv 1, 4 \text{ mod } 5} (1 - q^n)^{-1}
  \]
  and
  \[
  0\phi_1( ; 0; q, q^2) = \prod_{\mathbb{N} \ni n \equiv 2, 3 \text{ mod } 5} (1 - q^n)^{-1}.
  \]
The Rogers-Ramanujan continued fraction

- Power series formulas for \( R(q) \) and \( \log R(q) \) yields

\[
R(q) = 1 + \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^\ell} (-1)^{|m|} \beta(m) q^{m_1+2m_2+\ldots+\ell m_\ell}
\]

and

\[
\log R(q) = \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^\ell} (-1)^{|m|} \alpha(m) q^{m_1+2m_2+\ldots+\ell m_\ell}.
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\]

- The summands can be expressed in terms of \( q \)-Fibonacci numbers:

\[
\sum_{m \in \mathbb{N}^{\ell}} (-1)^{|m|} \beta(m) q^{m_1 + 2m_2 + \ldots + \ell m_\ell} = (-1)^{\ell} q^{(\ell+1)\ell/2} \frac{F_{\ell+1}(q)F_{\ell+2}(q)}{F_{\ell+1}(q)F_{\ell+2}(q)}
\]

and

\[
\sum_{m \in \mathbb{N}^{\ell}} (-1)^{|m|} \alpha(m) q^{m_1 + 2m_2 + \ldots + \ell m_\ell} = \log \left( \frac{\hat{F}_{\ell+1}(q)F_{\ell+1}(q)}{\hat{F}_{\ell}(q)F_{\ell+2}(q)} \right),
\]

for \( \ell \in \mathbb{N} \).
Under some assumptions on sequences $\{a_n\}$ and $\{b_n\}$, the generalized continued fraction

$$
\frac{1}{z - b_1} - \frac{a_1^2}{z - b_2} - \frac{a_2^2}{z - b_3} - \frac{a_3^2}{z - b_4} - \ldots
$$

converges locally uniformly on $\mathbb{C} \setminus \mathbb{R}$. 
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This function, known as the Weyl \( m \)-function \( m(z) \) in the theory of Jacobi operators, plays a fundamental role in the spectral theory of those operators.
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Since \( m(z) \) is the Stieltjes transform of a Borel measure \( \mu_J \), which is closely related with the spectral measure of Jacobi operator \( J \), it encodes many information about the spectrum of \( J \).
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Function \( m(z) \) is of significant importance also in the theory of Orthogonal Polynomials or the Moment Problem.
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Consequently, results concerning continued fractions are of much interest even in the Mathematical Physics community!
Thank you, and enjoy Beskydy!