Nevanlinna extremal measures for polynomials related to $q$-Fibonacci polynomials

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Symmetries of Discrete Systems and Processes

August 3, 2015
1 Preliminaries - Special functions

2 \(q\)-Fibonacci polynomials

3 Related orthogonal polynomials
Let $0 < q < 1$, $r, s \in \mathbb{Z}_+$. Recall the basic hypergeometric function

$$r\phi_s \left[ \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} ; q, z \right]$$

is defined by the power series

$$\sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \ldots (a_r; q)_n (-1)^{(s-r+1)n} q^{(s-r+1)n(n-1)/2}}{(b_1; q)_n (b_2; q)_n \ldots (b_s; q)_n (q; q)_n} z^n$$

where $z, a_1, a_2, \ldots, a_r \in \mathbb{C}$, $b_1, b_2, \ldots, b_s \in \mathbb{C} \setminus q\mathbb{Z}$ and

$$(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1})$$

is the $q$-Pochhammer symbol.
Two commonly known $q$-analogues to exponential function are due to Jackson:

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_n} z^n \quad \text{and} \quad e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n}.$$
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For $\alpha \geq 0$, Atakishiyev (1996) studied the one-parameter generalization

$$E_q^{(\alpha)}(z) = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2/2}}{(q; q)_n} z^n.$$ 

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$q$-exponential functions

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- Jackson’s $q$-exponential functions are particular cases corresponding to $\alpha = 0$ and $\alpha = 1$,

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Yet another particular case yields a very interesting function, this time $\alpha = 1/2$,

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One easily verifies that

$$\lim_{q \to 1^-} E_q((1 - q)z) = \exp(z).$$
Let us introduce the couple of \textit{q-sine} and \textit{q-cosine} such that the \textit{q}-version of Euler’s identity

\[
\mathcal{E}_q(iz) = C_q(z) + iq^{1/4} S_q(z)
\]

holds.
Let us introduce the couple of \( q \)-sine and \( q \)-cosine such that the \( q \)-version of Euler's identity

\[
\mathcal{E}_q(i z) = C_q(z) + iq^{1/4} S_q(z)
\]

holds.

The power series expansions for these functions then read

\[
S_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q; q)_{2n+1}} z^{2n+1}
\quad \text{and} \quad
C_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q; q)_{2n}} z^{2n}.
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\]

Alternatively, functions \( S_q \) and \( C_q \) can be written as the \( 1 \phi_1 \) function with the base \( q^2 \),
\[
S_q(z) = \frac{z}{1 - q} \, 1 \phi_1(0; q^3; q^2, q^2 z^2) \quad \text{and} \quad C_q(z) = 1 \phi_1(0; q; q^2, qz^2).
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Alternatively, functions $S_q$ and $C_q$ can be written as the $1\phi_1$ function with the base $q^2$,

$$S_q(z) = \frac{z}{1-q} 1\phi_1(0; q^3; q^2, q^2z^2) \quad \text{and} \quad C_q(z) = 1\phi_1(0; q; q^2, qz^2).$$

Functions $S_q$ and $C_q$ possess many nice properties. Let us only mention that they can be expressed with the aid of the third Jackson (or Hahn-Exton) $q$-Bessel function. In addition, they form a couple of linearly independent solution to a second-order $q$-difference equation.
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Functions $S_q$ and $C_q$ possess many nice properties. Let us only mention that they can be expressed with the aid of the third Jackson (or Hahn-Exton) $q$-Bessel function. In addition, they form a couple of linearly independent solution to a second-order $q$-difference equation.

At last, let us define the corresponding $q$-analogue to the hyperbolic sine and cosine:

$$Sh_q(z) = -iS_q(iz) \quad \text{and} \quad Ch_q(z) = C_q(iz).$$
A product formula

**Proposition**

For $u, v \in \mathbb{C}$, it holds

$$\mathcal{E}_q(u)\mathcal{E}_q(-v) = 3\phi_3 \left[ \begin{array}{c} 0, \ u^{-1}vq^{1/2}, \ uv^{-1}q^{1/2} \\ q^{1/2}, -q^{1/2}, \ -q \end{array} ; q, uvq^{1/2} \right]$$

$$+ q^{1/4} \frac{u - v}{1 - q} \ 3\phi_3 \left[ \begin{array}{c} 0, \ u^{-1}vq, \ uv^{-1}q \\ q^{3/2}, -q^{3/2}, \ -q \end{array} ; q, uvq \right].$$
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$$+q^{1/4}u - v \frac{1}{1 - q} 3\phi_3 \left[ \begin{array}{ccc} 0, & u^{-1}qv, & uv^{-1}q \\ q^{3/2}, & -q^{3/2}, & -q \end{array} ; q, uvq \right].$$

Corollaries:

1. $C_q(u)C_q(v) + q^{1/2}S_q(u)S_q(v) = 3\phi_3 \left[ \begin{array}{ccc} 0, & u^{-1}qv^{1/2}, & uv^{-1}q^{1/2} \\ q^{1/2}, & -q^{1/2}, & -q \end{array} ; q, -uvq^{1/2} \right]$, \[ \]
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For \( u, v \in \mathbb{C} \), it holds

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\end{array} \right] \\
+ q^{1/4} \frac{u - v}{1 - q} 3\phi_3 \left[ \begin{array}{ccc}
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qu^{1/2}, & -q^{1/2}, & -q \\
\end{array} \right],
\]

2.

\[
S_q(u)C_q(v) - C_q(u)S_q(v) = \frac{u - v}{1 - q} 3\phi_3 \left[ \begin{array}{ccc}
0, & u^{-1}vq, & uv^{-1}q \\
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Proposition

For \( u, v \in \mathbb{C} \), it holds

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q^{1/2}, & -q^{1/2}, & -q \\
\end{array} \right] \]

\[
+ q^{1/4} \frac{u-v}{1-q} \left[ \begin{array}{ccc}
0, & u^{-1}vq, & uv^{-1}q \\
q^{3/2}, & -q^{3/2}, & -q \\
\end{array} \right] ; q, uvq .
\]

Corollaries:

1. \( \mathcal{C}_q(u)\mathcal{C}_q(v) + q^{1/2}S_q(u)S_q(v) = 3 \phi_3 \left[ \begin{array}{ccc}
0, & u^{-1}vq^{1/2}, & uv^{-1}q^{1/2} \\
q^{1/2}, & -q^{1/2}, & -q \\
\end{array} \right] , q, -uvq^{1/2} \]

2. \( S_q(u)\mathcal{C}_q(v) - \mathcal{C}_q(u)S_q(v) = \frac{u-v}{1-q} \left[ \begin{array}{ccc}
0, & u^{-1}vq, & uv^{-1}q \\
q^{3/2}, & -q^{3/2}, & -q \\
\end{array} \right] ; q, -uvq \].

By setting \( u = q^{1/2}v \) in 1. one gets

\[
\mathcal{C}_q(q^{1/2}v)\mathcal{C}_q(v) + q^{1/2}S_q(q^{1/2}v)S_q(v) = 1 .
\]
Carlitz (1975) introduced $q$-Fibonacci polynomials $\varphi_n(x; q)$ by

$$
\varphi_n(x; q) = \sum_{2k < n} \binom{n-k-1}{k}_q q^{k^2} x^{n-2k-1}
$$

where $n \in \mathbb{Z}_+$ and

$$
\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}
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is the $q$-binomial coefficient.
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Polynomials $\varphi_n(x; q)$ satisfy the second-order recurrence

$$\varphi_{n+1}(x; q) = x \varphi_n(x; q) + q^{n-1} \varphi_{n-1}(x; q), \quad n \in \mathbb{N},$$

with the initial conditions $\varphi_0(x; q) = 0$ and $\varphi_1(x; q) = 1$. 

Let us mention that $\varphi_n(1; q)$ are polynomials in $q$ first considered by I. Schur (1917) in conjunction with his proof of Rogers-Ramanujan identities. They are referred as $q$-Fibonacci numbers $F_n$ since clearly $\varphi_n(1; 1) = F_n$. 
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\]

where \( n \in \mathbb{Z}_+ \) and

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For convenience, we replace \( q \) with \( q^{-1} \) from now.

Proposition

For all \( n \in \mathbb{Z}^+ \), one has

\[
\phi_n(x; q^{-1}) = \frac{1}{2} q^{-\frac{n}{2}} \left( \frac{n}{2} - 1 \right) \left[ E_q(x) E_q(\frac{n}{2} x) - (q^{-1})^n E_q(\frac{n}{2} x) E_q(-\frac{n}{2} x) \right].
\]

Corollary:

1. \( \phi_{2n+1}(x; q^{-1}) = q^{-n} \left( \frac{n}{2} - 1 \right) \left[ C_h_q(x) C_h_q(\frac{n}{2} x) - q^{-1} S_h_q(x) S_h_q(\frac{n}{2} x) \right] \),
2. \( \phi_{2n+2}(x; q^{-1}) = q^{-n} \left( \frac{n}{2} - 1 \right) \left[ S_h_q(x) C_h_q(\frac{n}{2} x) - C_h_q(x) S_h_q(\frac{n}{2} x) \right] \).

Yet another corollary:

1. \( \phi_{2n+1}(x; q^{-1}) = q^{-n} \left( \frac{n}{2} - 1 \right) \left[ \phi_3(x; q^{-1}) \right] \),
2. \( \phi_{2n+2}(x; q^{-1}) = q^{-n} \left( \frac{n}{2} - 1 \right) \left[ \phi_3(x; q^{-1}) \right] \).
Relation between $E_q$ and $q$-Fibonacci polynomials

For convenience, we replace $q$ with $q^{-1}$ from now.

**Proposition**

For all $n \in \mathbb{Z}_+$, one has

$$\varphi_n(x; q^{-1}) = \frac{1}{2} q^{-(n-1)^2/4} \left[ E_q(x) E_q(-q^{n/2} x) - (-1)^n E_q(-x) E_q(q^{n/2} x) \right].$$
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**Proposition**

For all $n \in \mathbb{Z}_+$, one has

$$\varphi_n(x; q^{-1}) = \frac{1}{2} q^{-(n-1)^2/4} \left[ \mathcal{E}_q(x)\mathcal{E}_q(-q^{n/2}x) - (-1)^n \mathcal{E}_q(-x)\mathcal{E}_q(q^{n/2}x) \right].$$

**Corollary:**

Yet another corollary:

$$\varphi_{2n+1}(x; q^{-1}) = q^{-n} \left( \frac{3}{4} \phi^3 \left[ 0, q^{-n+1}, q^n+1, q^{3/2}, -q^{3/2}, -q; q \right] \right).$$
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$$

**Corollary:**

$$
\varphi_{2n+1}(x; q^{-1}) = q^{-n^2} \left[ \mathcal{C}h_q(x)\mathcal{C}h_q(q^{n+1/2}x) - q^{1/2} \mathcal{S}h_q(x)\mathcal{S}h_q(q^{n+1/2}x) \right],
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$$

**Corollary:**

1. \(\varphi_{2n+1}(x; q^{-1}) = q^{-n^2} \left[ C_h(x) C_h(q^{n+1/2}x) - q^{1/2} S_h(x) S_h(q^{n+1/2}x) \right],\)

2. \(\varphi_{2n}(x; q^{-1}) = q^{-n(n-1)} \left[ S_h(x) C_h(q^n x) - C_h(x) S_h(q^n x) \right].\)
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**Corollary:**

1. $$\varphi_{2n+1}(x; q^{-1}) = q^{-n^2} \left[ \mathcal{C}_q(x) \mathcal{C}_q(q^{n+1/2} x) - q^{1/2} \mathcal{S}_q(x) \mathcal{S}_q(q^{n+1/2} x) \right],$$

2. $$\varphi_{2n}(x; q^{-1}) = q^{-n(n-1)} \left[ \mathcal{S}_q(x) \mathcal{C}_q(q^n x) - \mathcal{C}_q(x) \mathcal{S}_q(q^n x) \right].$$

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Relation between $E_q$ and $q$-Fibonacci polynomials

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**Proposition**

For all $n \in \mathbb{Z}_+$, one has

$$\varphi_n(x; q^{-1}) = \frac{1}{2} q^{-(n-1)^2/4} \left[ E_q(x)e_q(-q^{n/2}x) - (-1)^n E_q(-x)e_q(q^{n/2}x) \right].$$

**Corollary:**

1. $$\varphi_{2n+1}(x; q^{-1}) = q^{-n^2} \left[ Ch_q(x)Ch_q(q^{n+1/2}x) - q^{1/2} Sh_q(x)Sh_q(q^{n+1/2}x) \right],$$

2. $$\varphi_{2n}(x; q^{-1}) = q^{-n(n-1)} \left[ Sh_q(x)Ch_q(q^n x) - Ch_q(x)Sh_q(q^n x) \right].$$

Yet another corollary:

1. $$\varphi_{2n+1}(x; q^{-1}) = q^{-n^2} \phi_3 \left[ 0, q^{-n}, q^{n+1} ; q, q^{n+1} x^2 \right],$$
For convenience, we replace $q$ with $q^{-1}$ from now.

**Proposition**

For all $n \in \mathbb{Z}_+$, one has

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\varphi_n(x; q^{-1}) = \frac{1}{2} q^{-n(n-1)/4} \left[ \mathcal{E}_q(x)\mathcal{E}_q(-q^{n/2}x) - (-1)^n \mathcal{E}_q(-x)\mathcal{E}_q(q^{n/2}x) \right].
$$

**Corollary:**

1. $$
\varphi_{2n+1}(x; q^{-1}) = q^{-n^2} \left[ C_{\mathcal{H}}(x)C_{\mathcal{H}}(q^{n+1/2}x) - q^{1/2} S_{\mathcal{H}}(x)S_{\mathcal{H}}(q^{n+1/2}x) \right],
$$

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\varphi_{2n}(x; q^{-1}) = q^{-n(n-1)} \left[ S_{\mathcal{H}}(x)C_{\mathcal{H}}(q^n x) - C_{\mathcal{H}}(x)S_{\mathcal{H}}(q^n x) \right].
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Yet another corollary:

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\varphi_{2n+1}(x; q^{-1}) = q^{-n^2} \ {}_3\phi_3 \left[ \begin{array}{c} 0, \ q^{-n}, \ q^{n+1} \\ q^{1/2}, \ -q^{1/2}, \ -q \end{array} ; q, q^{n+1}x^2 \right],
$$

2. $$
\varphi_{2n}(x; q^{-1}) = xq^{-n(n-1)} \frac{1 - q^n}{1 - q} \ {}_3\phi_3 \left[ \begin{array}{c} 0, \ q^{-n+1}, \ q^{n+1} \\ q^{3/2}, \ -q^{3/2}, \ -q \end{array} ; q, q^{n+1}x^2 \right].
$$
The following limit relations are an immediate consequence of the previous formulas.
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**Proposition**

For all \( x \in \mathbb{C} \) and \( 0 < q < 1 \), the limit relations

\[
\lim_{n \to \infty} q^{n(n-1)} \varphi_{2n}(x; q^{-1}) = Sh_q(x) \quad \text{and} \quad \lim_{n \to \infty} q^{n^2} \varphi_{2n+1}(x; q^{-1}) = Ch_q(x)
\]

hold.
Asymptotic behavior of $q^{-1}$-Fibonacci polynomials

The following limit relations are an immediate consequence of the previous formulas.

**Proposition**

For all $x \in \mathbb{C}$ and $0 < q < 1$, the limit relations

$$\lim_{n \to \infty} q^{n(n-1)} \varphi_{2n}(x; q^{-1}) = S h_q(x)$$

and

$$\lim_{n \to \infty} q^{n^2} \varphi_{2n+1}(x; q^{-1}) = C h_q(x)$$

hold.

Let us note the asymptotic behavior in case of $q$-Fibonacci polynomials with $0 < q < 1$ is particularly different:

$$\lim_{n \to \infty} x^{-n} \varphi_{n+1}(x; q) = 0 \phi_1(0; q, qx^{-2}).$$
1 Preliminaries - Special functions

2 $q$-Fibonacci polynomials

3 Related orthogonal polynomials
Recall polynomials $\varphi_n(x; q)$ are a solution of the second-order recurrence

$$\varphi_{n+1}(x; q) = x\varphi_n(x; q) + q^{n-1}\varphi_{n-1}(x; q), \quad n \in \mathbb{N},$$

with the initial conditions $\varphi_0(x; q) = 0$ and $\varphi_1(x; q) = 1$. 

For $q > 0$ The Favard's theorem is applicable to the family $\{T_n(x; q)\}$. It tells us that there exists a positive Borel measure such that polynomials $\{T_n(x; q)\}$ are OG w.r.t. this measure. In addition, it is not hard to show that

1. the measure of OG is unique iff $0 < q \leq 1$ (determinate case of Hmp) and
2. there infinitely many measures of OG iff $q > 1$ (indeterminate case of Hmp).
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If we put

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T_n(x; q) = (-i)^n q^{-n/2} \varphi_{n+1}(iq^{1/2}x; q), \quad n = -1, 0, 1, 2, \ldots,
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then $\{T_n(x; q)\}$ fulfills the second-order difference equation

$$T_{n+1}(x; q) = xT_n(x; q) - q^{n-1} T_{n-1}(x; q), \quad n \in \mathbb{Z}_+,$$

with the initial conditions $T_{-1}(x; q) = 0$ and $T_0(x; q) = 1$.

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1. the measure of OG is unique iff $0 < q \leq 1$ (determinate case of Hmp) and
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The measure of OG in the case $0 < q < 1$ has been found by Al-Salam and Ismail (1983):

$$\sum_{j=1}^{\infty} \frac{\Phi_q(qz_j(q))}{z_j(q)\Phi'_q(z_j(q))} T_n(\pm z_j^{-1/2}) T_m(\pm z_j^{-1/2}) = -2q^{n(n-1)/2} \delta_{mn},$$

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The case $q = 1$ corresponds to Chebyshev polynomials of the second kind. Their measure of orthogonality is very well known.
Towards OG in the indeterminate case - Nevanlinna Theorem

Assume the indeterminate case. Recall the **Nevanlinna Theorem**:

- All measures of orthogonality $\mu_\varphi$ are in one-to-one correspondence with functions $\varphi$ belonging to the one-point compactification of the space of Pick functions.
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- Recall that Pick functions are defined and holomorphic on the open complex halfplane $\Im z > 0$, with values in the closed halfplane $\Im z \geq 0$. 

Four entire functions $A$, $B$, $C$, $D$ are called Nevanlinna functions and they are determined by the leading term of the asymptotic expansion of corresponding OG polynomials of the first and second kind for large index.
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- Recall that Pick functions are defined and holomorphic on the open complex halfplane \( \Im z > 0 \), with values in the closed halfplane \( \Im z \geq 0 \).
- The correspondence is established by identifying the Stieltjes transform of the measure \( \mu_\varphi \),

\[
\int_{\mathbb{R}} \frac{d\mu_\varphi(x)}{z-x} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
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- Four entire functions $A, B, C, D$ are called Nevanlinna functions and they are determined by the leading term of the asymptotic expansion of corresponding OG polynomials of the first and second kind for large index.
The particular class of measures of orthogonality is composed by measures $\mu_t$ associated with the Pick function $\varphi(z) = t \in \mathbb{R} \cup \{\infty\}$. 

Measures $\mu_t$ are called N-extremal and are purely discrete. The support of $\mu_t$ is an unbounded set of isolated points which is known to be equal to the zero set $\text{supp} \mu_t = \{x \in \mathbb{R} | B(x) - D(x) = 0\}$. Hence $\mu_t = \sum_{x \in \text{supp} \mu_t} \rho(x) \delta_x$. For the weight function $\rho$ one has $\rho(x) = \frac{1}{B'(x)D(x) - B(x)D'(x)}$. 

František Štampach (CTU in Prague)
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For the weight function $\rho$ one has

$$\rho(x) = \frac{1}{B'(x)D(x) - B(x)D'(x)}.$$
Since we have the limit relation for polynomials $\varphi_n(x; q^{-1})$, for $n \to \infty$, we can express functions $A, B, C, D$ in terms of $S_q$ and $C_q$. 

Proposition

For $0 < q < 1$, Nevanlinna functions corresponding to OG polynomials $\{T_n(x; q^{-1})\}$ are as follows:

$$A(z) = q^{-1}/2 \ D(q^{1/2}z) = S_q(z)$$

and

$$C(z) = -B(q^{1/2}z) = C_q(z).$$
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These formulas are not new. They have already been obtained by Chen and Ismail (1998).
Recall the reproducing kernel for polynomials $T_n(x; q^{-1})$ is related with Nevanlinna functions $B$ and $D$ by formula

$$K(u, v) = \frac{B(u)D(v) - D(u)B(v)}{u - v}.$$
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**Proposition**

The formula for the reproducing kernel for orthogonal polynomials \( T_n(x; q^{-1}) \) reads

\[
K(u, v) = \frac{1}{1 - q} {}_3\phi_3 \left[ \begin{array}{c} 0, \ u^{-1}vq, \ uv^{-1}q \\ q^{3/2}, \ -q^{3/2}, \ -q \end{array} \right] ; q, -uv.
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**Proposition**

The formula for the reproducing kernel for orthogonal polynomials \( T_n(x; q^{-1}) \) reads

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K(u, v) = \frac{1}{1 - q^3} \phi_3 \left[ 0, u^{-1}vq, \frac{uv^{-1}q}{q}; q, -uv \right].
\]

By applying the limit \( v \to u \) in the last expression one finds the formula for the weight function \( \rho \):
First application of the product formula - the reproducing kernel

- Recall the reproducing kernel for polynomials $T_n(x; q^{-1})$ is related with Nevanlinna functions $B$ and $D$ by formula
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**Proposition**

The formula for the reproducing kernel for orthogonal polynomials $T_n(x; q^{-1})$ reads

\[ K(u, v) = \frac{1}{1 - q} \, _3\phi_3 \left[ \begin{array}{c} 0, \quad u^{-1}vq, \quad uv^{-1}q \; q, -uv \\ q^{3/2}, \quad -q^{3/2}, \quad -q \end{array} \right]. \]

- By applying the limit $v \to u$ in the last expression one finds the formula for the weight function $\rho$:

\[ \frac{1}{\rho(u)} = B'(u)D(u) - D'(u)B(u) = \frac{1}{1 - q} \, _3\phi_3 \left[ \begin{array}{c} 0, \quad q, \quad q \; q, -u^2 \\ q^{3/2}, \quad -q^{3/2}, \quad -q \end{array} \right]. \]
Recall N-extremal measure $\mu_t$ is supported by zeros of the function

$$z \mapsto B(z)t - D(z).$$

where $B(z) = -C_q(q^{-1/2}z)$ and $D(z) = q^{1/2}S_q(q^{-1/2}z)$. 
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By applying the suitable reparametrization
\[ t = \frac{C_q(u)}{S_q(u)} \]
one arrives at another N-extremal measure $\nu_u$ supported by zeros of function
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Applying the product formula once more, we get the final complete description of all N-extremal measures of orthogonality of polynomials $T_n(x; q^{-1}) \ldots$
N-extremal measures of orthogonality of $T_n(x; q^{-1})$

**Theorem**

If $0 < q < 1$ and $u \in \mathbb{R}$, then the orthogonality relation for $T_n(x; q^{-1})$ reads

$$\sum_{k=1}^{\infty} \left( \genfrac{[}{]}{0pt}{}{3\phi_3}{q^{3/2}, -q^{3/2}, -q ; q, -\lambda_k^2(u)} \right) -1 T_n(\lambda_k(u); q^{-1}) T_m(\lambda_k(u); q^{-1}) = \frac{q^{-n(n-1)/2}}{1 - q} \delta_{mn}$$

where $\lambda_1(u), \lambda_2(u), \lambda_3(u), \ldots$ stand for zeros of the function

$$z \mapsto \genfrac{[}{]}{0pt}{}{3\phi_3}{0, u^{-1}z, uz^{-1}; q, -uz}.$$
Two particular orthogonality relations

Let the sequences

\[ 0 < s_1(q) < s_2(q) < s_3(q) < \ldots \quad \text{and} \quad 0 < c_1(q) < c_2(q) < c_3(q) < \ldots \]

denote all positive zeros of \( S_q \) and \( C_q \), respectively.
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One has the following orthogonality relations:

\[
(1 - q)T_n(0; q^{-1}) T_m(0; q^{-1}) - \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{S_q(qs_k(q))}{s_k(q)S'_q(s_k(q))} T_n(q^{\frac{1}{2}} s_k(q); q^{-1}) T_m(q^{\frac{1}{2}} s_k(q); q^{-1}) = q^{-n(n-1)/2} \delta_{mn}
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where $s_{-k}(q) = -s_k(q)$. 
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\]

where \( s_{-k}(q) = -s_k(q) \).

2. \[
- \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{C_q(qc_k(q))}{c_k(q)C_q'(c_k(q))} T_n(q^{1/2} c_k(q); q^{-1}) T_m(q^{1/2} c_k(q); q^{-1}) = q^{-n(n-1)/2} \delta_{mn}
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where \( c_{-k}(q) = -c_k(q) \).

Thank you!