Non-self-adjoint Toeplitz matrices with purely real spectrum and related problems

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Toeplitz matrix

- **Toeplitz matrix:**

\[
T_n(a) = \left( a_{j-k} \right)_{j,k=0}^{n-1} = \begin{pmatrix}
a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\
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where \( a_n \in \mathbb{C} \).
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More precisely, let

\[ \Lambda(a) := \{ \lambda \in \mathbb{C} \mid \liminf_{n \to \infty} \text{dist} (\lambda, \text{spec}(T_n(a))) = 0 \} \]

i.e., \( \lambda \in \Lambda(a) \) if and only if \( \exists n_k \nearrow \infty \exists \lambda_k \in \text{spec}(T_{n_k}(a)) \) s.t. \( \lambda_k \to \lambda \).
Toeplitz matrices with real spectrum

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The question: determine the class of symbols $a$ for which

$$\Lambda(a) \subset \mathbb{R}.$$
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\[ T_n(a) = T_n^*(a), \; \forall n \in \mathbb{N} \iff a(\mathbb{T}) \subset \mathbb{R}. \]
Clearly, if $T_n(a)$ is Hermitian for all $n$, then $\Lambda(a) \subset \mathbb{R}$.

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In other words: \textit{In the Hermitian case, there exists a Jordan curve in $\mathbb{C}$ (namely, the unit circle) on which the symbol is a real-valued function.}
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**Theorem:**

1. Let the symbol $a$ be given by the Laurent series $\sum_n a_n z^n$ which is absolutely convergent in an annulus $r \leq |z| \leq R$, where $r \leq 1$ and $R \geq 1$. 

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Then $\Lambda(a) \subset \mathbb{R}$.
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$$\text{spec}(T_n(a)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N}.$$
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Remark:

If $a$ is analytic in $\mathbb{C} \setminus \{0\}$ (especially, if $a$ is a Laurent polynomial), then the assumption 1 can be omitted.
The case of banded Toeplitz matrices

**Question:** If $\Lambda(a) \subseteq \mathbb{R}$, can the set $\Lambda(a)$ be approached from the complex plane? That is, can $\text{spec}(T_n(a))$ contain non-real eigenvalues for some $n$?
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**Theorem:**

Let $b = b(z)$ be a Laurent polynomial which is neither a polynomial in $z$ nor in $1/z$. The following claims are equivalent:

1. $\Lambda(b) \subset \mathbb{R}$;
2. The set $b^{-1}(\mathbb{R})$ contains a Jordan curve (with $0$ in its interior);
3. For all $n \in \mathbb{N}$, $\text{spec}(T_n(b)) \subset \mathbb{R}$.

**Remark:** It is a very surprising feature of banded Toeplitz matrices that the asymptotic reality of the eigenvalues (claim 1) forces all eigenvalues of all $T_n(b)$ to be real (claim 3). Hence, if, for instance, the $2 \times 2$ matrix $T_2(b)$ has a non-real eigenvalue, there is no chance for the limiting set $\Lambda(b)$ to be real!
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Examples

1. **Tridiagonal Toeplitz matrix:**

   \[ b(z) = z^{-1} + az, \quad (a \in \mathbb{C} \setminus \{0\}). \]

   Then
   \[ \Lambda(b) \subset \mathbb{R} \iff a > 0. \]
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\[ b(z) = z^{-r} (1 + az)^{r+s}, \quad (r, s \in \mathbb{N}, a \in \mathbb{R} \setminus \{0\}). \]

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History on the topic

- We consider **banded** Toeplitz matrices only $\rightarrow$ the classical topic;

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- The set $\Lambda(b)$ can be described in terms of zeros of the polynomial $z \mapsto z^r (b(z) - \lambda)$ [Schmidt and Spitzer, 1960].

- The weak limit of the eigenvalue-counting measures of $T_n(b)$:

\[ \mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta \lambda_k^{(n)} \]

exists, as $n \rightarrow \infty$, and is absolutely continuous measure $\mu$ supported on $\Lambda(b)$ whose density can be expressed in terms of zeros of $z \mapsto z^r (b(z) - \lambda)$ [Hirschman Jr., 1967].
1. Let $T_n(b)$ be a banded Toeplitz matrix with real elements:

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Suppose the Jordan curve $\gamma$ is present in $b^{-1}(\mathbb{R})$ and assume, additionally, that $\gamma$ admits a polar parametrization:

$$\gamma(t) = \rho(t)e^{it}, \quad t \in [-\pi, \pi].$$
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Theorem:

Let $b'(\gamma(t)) \neq 0$ for all $t \in (0, \pi)$. Then $b \circ \gamma$ restricted to $(0, \pi)$ is either strictly increasing or decreasing; the limiting measure $\mu$ is supported on the interval $[\alpha, \beta] := b(\gamma([0, \pi]))$ and its density satisfies

$$\frac{d\mu}{dx}(x) = \pm \frac{1}{\pi} \frac{d}{dx}(b \circ \gamma)^{-1}(x),$$

for $x \in (\alpha, \beta)$, where the $+$ sign is used when $b \circ \gamma$ increases on $(0, \pi)$, and the $-$ sign is used otherwise.
Numerical illustration - the Jordan curve without critical points of $b$

$$b(z) = z^{-3} - z^{-2} + 7z^{-1} + 9z - 2z^2 + 2z^3 - z^4,$$
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The limiting measure and the Jordan curve with critical points

If $\gamma((0, \pi))$ contains some critical points of $b$, then the description of $\mu$ is slightly more complicated.
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Theorem:
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Theorem:

Suppose that \( b \) and \( \gamma \) are as before and let

\[
\ell \in \mathbb{N}_0 \text{ be the number of critical points of } b \text{ in } \gamma((0, \pi)) \quad \text{and} \quad 0 =: \phi_0 < \phi_1 < \cdots < \phi_{\ell} < \phi_{\ell+1} =: \pi
\]

such that \( b'(\gamma(\phi_j)) = 0 \) for all \( 0 \leq j \leq \ell+1 \).

Then \( b \circ \gamma \) restricted to \( (\phi_i-1, \phi_i) \) is strictly monotone for all \( 1 \leq i \leq \ell+1 \), and the limiting measure \( \mu = \mu_1 + \mu_2 + \cdots + \mu_{\ell+1} \), where \( \mu_i \) is an absolutely continuous measure supported on \( [\alpha_i, \beta_i] := b(\gamma([\phi_i-1, \phi_i])) \) whose density is given by

\[
d\mu_i d\nu(x) = \pm \frac{1}{\pi} d\nu(b \circ \gamma) - \frac{1}{\pi} (x)
\]

for all \( x \in (\alpha_i, \beta_i) \) and all \( i \in \{1, 2, \ldots, \ell+1\} \). The + sign is used when \( b \circ \gamma \) increases on \( (\alpha_i, \beta_i) \), and the − sign is used otherwise.
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Suppose that $b$ and $\gamma$ are as before and let $\ell \in \mathbb{N}_0$ be the number of critical points of $b$ in $\gamma((0, \pi))$ and $0 =: \phi_0 < \phi_1 < \cdots < \phi_\ell < \phi_{\ell+1} := \pi$ are such that $b'(\gamma(\phi_j)) = 0$ for all $0 \leq j \leq \ell + 1$. 

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Suppose that $b$ and $\gamma$ are as before and let $\ell \in \mathbb{N}_0$ be the number of critical points of $b$ in $\gamma((0, \pi))$ and $0 =: \phi_0 < \phi_1 < \cdots < \phi_\ell < \phi_{\ell+1} := \pi$ are such that $b'(\gamma(\phi_j)) = 0$ for all $0 \leq j \leq \ell + 1$. Then $b \circ \gamma$ restricted to $(\phi_{i-1}, \phi_i)$ is strictly monotone for all $1 \leq i \leq \ell + 1$, and the limiting measure $\mu = \mu_1 + \mu_2 + \cdots + \mu_{\ell+1}$, where $\mu_i$ is an absolutely continuous measure supported on $[\alpha_i, \beta_i] := b(\gamma([\phi_{i-1}, \phi_i]))$ whose density is given by

$$\frac{d\mu_i}{dx}(x) = \pm \frac{1}{\pi} \frac{d}{dx} (b \circ \gamma)^{-1}(x)$$

for all $x \in (\alpha_i, \beta_i)$ and all $i \in \{1, 2, \ldots, \ell + 1\}$. The $+$ sign is used when $b \circ \gamma$ increases on $(\alpha_i, \beta_i)$, and the $-$ sign is used otherwise.
Numerical illustration - the Jordan curve with critical points of $b$

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Contents

1. Toeplitz matrices with real spectrum
2. The asymptotic eigenvalue distribution
3. Connections to the Hamburger Moment Problem and Orthogonal Polynomials
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The limiting measure as a solution of the HMP

- We consider real Laurent polynomial symbols:

\[ b(z) = \sum_{k=-r}^{s} a_k z^k, \text{ where } a_{-r}a_s \neq 0 \text{ and } r, s \in \mathbb{N}. \]
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Proposition:

Let \( b^{-1}(\mathbb{R}) \) contains a Jordan curve. Then the limiting measure \( \mu \) coincides with the unique solution of the determinate HMP with moments

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(If a counter-example exists, \( \mathbb{C} \setminus \Lambda(b) \) has to be disconnected.)
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- If $b^{-1}(\mathbb{R})$ contains a Jordan curve, then there is a family of OGPs $\{p_n\}_{n=0}^{\infty}$ orthogonal w.r.t. the limiting measure $\mu$. 

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1. the Jordan curve intersects $\mathbb{R}$ at exactly two points whose $b$-images are the endpoints of the interval $\Lambda(b) = [\alpha, \beta]$;
2. the OGPs $\{p_n\}$ belong to the Blumenthal–Nevai class $M((\beta - \alpha)/2, (\alpha + \beta)/2)$, i.e.,

$$\lim_{n \to \infty} a_n = \frac{\beta - \alpha}{4} \quad \text{and} \quad \lim_{n \to \infty} b_n = \frac{\alpha + \beta}{2}.$$
Example 1/4

Let

\[ b(z) = \frac{1}{z^r} (1 + az)^{r+s}, \quad (a > 0, r, s \in \mathbb{N}). \]
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\[ \gamma(t) = \frac{\sin \frac{r}{r+s} t}{\sin \frac{s}{r+s} t} e^{it}, \quad t \in [-\pi, \pi]. \]
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h_m = \frac{1}{2\pi} \int_0^{2\pi} b^m (e^{i\theta}) \, d\theta = \binom{r + s}{rm}, \quad m \in \mathbb{N}_0.
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- To obtain $\mu$, one has to invert the function

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- But the main result yields that for the distribution function of $\mu$, $F_\mu := \mu ([0, \cdot))$, one has
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Explicit formulas for the Jacobi parameters $a_n$ and $b_n$ are not known in general but we have

$$2 \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \frac{(r + s)^{r+s}}{2r^r s^s}.$$
Example 3/4

- Special cases that admit more explicit results: \((r, s) = (1, 1), (1, 2), (2, 2)\).
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- Jacobi parameters:
  \[
  a_1^2 = 6a^2, \quad a_k^2 = \frac{9(6k - 5)(6k - 1)(3k - 1)(3k + 1)}{4(4k - 3)(4k - 1)^2(4k + 1)} a^2, \quad \text{for } k > 1.
  \]
  and
  \[
  b_1 = 3a, \quad b_k = \frac{3(36k^2 - 54k + 13)}{2(4k - 5)(4k - 1)} a, \quad \text{for } k > 1.
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Example 4/4

- Polynomials $p_n$ can be expressed as a combination of the associated Jacobi polynomials $P_n^{(\alpha,\beta)}(x; c)$ studied by J.Wimp (1987).
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$$
2^n p_n(x) = r_n^{(\alpha,\beta)}(x; c) - 4 \cdot \frac{r_{n-1}}{729} r_n^{(\alpha,\beta)}(x+1) - 256 \cdot \frac{729}{4} r_{n-2} r_n^{(\alpha,\beta)}(x+2), \quad n \in \mathbb{N},
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where $\alpha = \frac{1}{2}$, $\beta = -\frac{2}{3}$, and $c = -\frac{1}{6}$. This relation and the known properties of the associated Jacobi polynomials allow to derive other formulas for $p_n$ such as: an explicit representation, a generating function, ...
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it holds

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