On the asymptotic zero distribution of orthogonal polynomials on the unit circle with variable Verblunsky coefficients

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Hausdorff geometry of polynomials and polynomial sequences

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$$\lim_{j \to \infty} X_{n_j,N_j} = X,$$

for any $\{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ and $\{N_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ such that

$$n_j, N_j \to \infty \quad \text{and} \quad n_j / N_j \to t, \quad \text{as} \quad j \to \infty.$$
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(The meaning of the used topology will be always clear from the context.)
The variable coefficient OPRL

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- Fix $t > 0$. Let for all $N \in \mathbb{N}$, sequences

$$\{a_{n,N}\}_{n=0}^{\infty} \subset (0, \infty) \quad \text{and} \quad \{b_{n,N}\}_{n=0}^{\infty} \subset \mathbb{R}$$

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  are given. Suppose further that there exist functions $a, b \in C((0, t])$ such that

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  \lim_{n/N \to s} a_{n,N} = a(s) \quad \text{and} \quad \lim_{n/N \to s} b_{n,N} = b(s),
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  for all $s \in (0, t]$. 

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  for all $s \in (0, t]$.
- As a particular case, one can take

  $$ a_{n,N} := a \left( \frac{n + 1}{N} t \right) \quad \text{and} \quad b_{n,N} := b \left( \frac{n + 1}{N} t \right). $$
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  $$a_{n,N} := a\left(\frac{n + 1}{N} t\right) \quad \text{and} \quad b_{n,N} := b\left(\frac{n + 1}{N} t\right).$$
- Consider a family of OPRL $p_{n,N}$ determined by the recurrence

  $$p_{n+1,N}(x) = (x - b_{n,N}) p_{n,N}(x) - a_{n-1,N}^2 p_{n-1,N}(x)$$

  with initial conditions $p_{-1,N}(x) = 0$ and $p_{0,N}(x) = 1$. 
The variable coefficient OPRL

Denote

\[ \nu_{n,N} := \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}, \]

where \(x_k = x_{k,n,N}\) are roots of \(p_{n,N}\).
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- **Question:** What is the asymptotic distribution of roots of \( p_{n,N} \) if \( n/N \to t \)?

Remark: The measure \( \omega_{[\alpha,\beta]} \) is the equilibrium measure of \([\alpha,\beta]\) (log. pot. theory).
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**Theorem (Kuijlaars & Van Assche)**

One has

\[ \lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \omega[b(s) - 2a(s), b(s) + 2a(s)] ds, \]

where

\[ \frac{d\omega_{[\alpha,\beta]}(x)}{dx} = \frac{1}{\pi \sqrt{(x - \alpha)(\beta - x)}}, \quad \text{for } \alpha < x < \beta, \]

and \( \omega_{[\alpha,\beta]} = \delta_{\alpha} \), if \( \alpha = \beta \).
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**Locally Toeplitz matrices**

- Equivalently, the theorem gives the asymptotic eigenvalue distribution of Jacobi matrices

\[
J_{n,N} = \begin{pmatrix}
    b_{0,N} & a_{0,N} \\
    a_{0,N} & b_{1,N} & a_{1,N} \\
    a_{1,N} & b_{2,N} & a_{2,N} \\
    \vdots & \vdots & \vdots \\
    a_{n-3,N} & b_{n-2,N} & a_{n-2,N} \\
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as \( n/N \to t \).

- The concept of **locally Toeplitz matrices** (Tilli):

  \[
  \text{locally Toeplitz structure} \quad \leftrightarrow \quad a_{i,j} = a_{i+1,j+1} + o(1), \quad \text{as} \ n/N \to t,
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a_{n-1,N} & b_{n,N} & a_{n,N} & \cdots & \cdots & a_{2,N} & b_2,N \\
a_{n,N} & b_{n+1,N} & a_{n+1,N} & \cdots & \cdots & a_{3,N} & b_3,N \\
\end{pmatrix},
\]

as \( n/N \to t \).

The concept of **locally Toeplitz matrices** (Tilli):

- **locally Toeplitz structure** \( \leftrightarrow \) \( a_{i,j} = a_{i+1,j+1} + o(1) \), as \( n/N \to t \),
- **Toeplitz structure** \( \leftrightarrow \) \( a_{i,j} = a_{i+1,j+1} \).

Tilli deduced the asymptotic eigenvalue distribution for Hermitian locally Toeplitz matrices in LAA98 - a generalization of the result from JAT99.
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- The concept of **locally Toeplitz matrices** (Tilli):

  locally Toeplitz structure \( \iff \ a_{i,j} = a_{i+1,j+1} + o(1), \quad \text{as } n/N \to t, \)

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- Tilli deduced the asymptotic eigenvalue distribution for Hermitian locally Toeplitz matrices in LAA98 - a generalization of the result from JAT99.

- Tilli’s motivation stems from a discretization of a 1D Sturm–Liouville operator.
Kac–Murdock–Szegő matrices

The topic of JAT99 and LAA98 appeared much earlier in a paper by M. Kac, W. L. Murdock, and G. Szegő in 1953.
Kac–Murdoch–Szegő matrices

- They introduced the $n \times n$ matrices with entries:

  $$(T_n(a))_{i,j} = a_{j-i} \left(\frac{i+j}{2n}\right),$$

  (the “KMS matrix”)

  where $a_k \in C([0, 1])$ are given.
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(\dot{T}_n(a))_{i,j} = a_{j-i} \left( \frac{\min(i,j)}{n} \right) \quad \text{and} \quad (\ddot{T}_n(a))_{i,j} = a_{j-i} \left( \frac{\max(i,j)}{n} \right)
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Under the following assumptions:

i) $a_{-k} = \overline{a_k}$

ii) $\sum_{k \in \mathbb{Z}} \max\{|a_k(x)| \mid x \in [0,1]\} < \infty$,

all three matrices have the same asymptotic eigenvalue distribution.
Kac–Murdock–Szegő matrices

The symbol:

\[ a(x, t) := \sum_{k \in \mathbb{Z}} a_k(x) e^{ikt}. \]
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**Theorem (Kac, Murdock, Szegő)**

For all \( \phi \in C(\mathbb{R}) \), one has

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \phi(\lambda_k(a)) = \frac{1}{2\pi} \int_{0}^{1} \int_{0}^{2\pi} \phi(a(x, t)) dtdx,
\]

where \( \lambda_k(a) \) are eigenvalues of \( T_n(a) \) (or \( \dot{T}_n(a) \) or \( \ddot{T}_n(a) \)).
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- The tridiagonal case corresponds to the symbol: \( a(x, t) = a(x)e^{-it} + b(x) + a(x)e^{it} \).

By making use of the substitution \( t = \arccos(\xi - \frac{b(x)}{2a(x)}) \), \( t \in [0, \pi] \), in the integral on the RHS, one obtains the asymptotic zero distribution of variable OPRL.
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t = \arccos \left( \frac{\xi - b(x)}{2a(x)} \right), \quad t \in [0, \pi],
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\[ D := \{ z \in \mathbb{C} \mid |z| < 1 \} \quad \text{and} \quad T := \partial D. \]
**Goal of the talk:** Asymptotic zero distribution of OPUC with variable Verblunsky coefficients.

**Notation:**

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Recall that OPUC is a family of monic polynomials \( \{ \Phi_n \}_{n=0}^{\infty} \) given by the Szegő recursion:

\[
\Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n \Phi_n^*(z), \quad n \in \mathbb{N}_0,
\]

and \( \Phi_0(z) = 1 \), where \( \Phi_n^*(z) = z^n \overline{\Phi_n(1/z)} \) and \( \{ \alpha_n \}_{n=0}^{\infty} \subset D^\infty \) are the Verblunsky coefficients.
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There is a 1–1 correspondence between probability measures on \( T \) with infinite support and the sequence \( \{ \alpha_n \}_{n=0}^{\infty} \subset D^\infty \).
OPUC - CMV matrix

The probability measure $\mu$ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^\infty$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(N_0)$ is given by the CMV matrix

$$C := \begin{pmatrix}
\overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \ldots \\
\rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\
0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \ldots \\
0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \ldots \\
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where $\rho_n = \sqrt{1 - |\alpha_n|^2}$. 
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The CMV matrix is a universal model for any unitary operator with simple spectrum.
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Denote $C_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $C$ from the upper-left corner.
The probability measure $\mu$ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^\infty$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(\mathbb{N}_0)$ is given by the CMV matrix

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Note that

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C_n = C_n(\alpha_0, \ldots, \alpha_{n-1}).
$$

It holds

$$
\Phi_n(z) = \det(z - C_n).
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0 & 0 & 0 & -\overline{\alpha}_4 \rho_3 & -\alpha_4 \rho_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

where $\rho_n = \sqrt{1 - |\alpha_n|^2}$.

The CMV matrix is a universal model for any unitary operator with simple spectrum.

Denote $C_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $C$ from the upper-left corner.

Note that

$$C_n = C_n(\alpha_0, \ldots, \alpha_{n-1}).$$

It holds

$$\Phi_n(z) = \det(z - C_n).$$

The zeros of $\Phi_n$ are located in $\mathbb{D}$. 
The probability measure $\mu$ associated with $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{D}^\infty$ is the spectral measure of a unitary operator whose matrix representation on $\ell^2(N_0)$ is given by the CMV matrix

$$C := \begin{pmatrix}
\bar{\alpha}_0 & \bar{\alpha}_1 \rho_0 & \rho_1 \rho_0 & 0 & 0 & \ldots \\
\rho_0 & -\bar{\alpha}_1 \alpha_0 & -\rho_1 \alpha_0 & 0 & 0 & \ldots \\
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\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
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The CMV matrix is a universal model for any unitary operator with simple spectrum.

Denote $C_n \in \mathbb{C}^{n,n}$ the cut-off CMV matrix, i.e., the $n \times n$ matrix obtained by truncation of $C$ from the upper-left corner.

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It holds

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The zeros of $\Phi_n$ are located in $\mathbb{D}$. $\Rightarrow C_n$ is not unitary.
If $\alpha_{n-1}$ is replaced by a parameter $\beta \in \mathbb{T}$, we arrive at the so-called para-orthogonal polynomials (= POPUC)

\[
\Phi_n^{(\beta)}(z) := \det \left( z - C_n^{(\beta)} \right),
\]

where

\[
C_n^{(\beta)} = C_n(\alpha_0, \ldots, \alpha_{n-2}, \beta).
\]
If \( \alpha_{n-1} \) is replaced by a parameter \( \beta \in \mathbb{T} \), we arrive at the so-called para-orthogonal polynomials (=POPUC)

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Matrix \( C_n^{(\beta)} \) is unitary.
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\[
\Phi^{(\beta)}_n(z) := \det \left( z - C^{(\beta)}_n \right),
\]

where

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C^{(\beta)}_n = C_n(\alpha_0, \ldots, \alpha_{n-2}, \beta).
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Matrix $C^{(\beta)}_n$ is unitary.

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Matrix $C_n^{(\beta)}$ is unitary.

The zeros of $\Phi_n^{(\beta)}$ are located in $\mathbb{T}$ and are simple.

Two books on OPUC by B. Simon:
Contents

1 History - KMS matrices and variable coefficient OPRL

2 (P)OPUC

3 POPUC with variable Verblunsky coefficients

4 OPUC with variable Verblunsky coefficients
For $\{\alpha_{n,N} \in \bar{\mathbb{D}} \mid n, N \in \mathbb{N}_0\}$ and $\beta \in \mathbb{T}$, we introduce

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and

\[
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Variable (P)OPUC

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**Program:**

1. First we deduce the asymptotic zero distribution for \( \Phi^{(\beta)}_{n,N} \), as \( n/N \to t \), (2 proofs).
Variable (P)OPUC

For \( \{ \alpha_{n,N} \in \overline{D} \mid n, N \in \mathbb{N}_0 \} \) and \( \beta \in \mathbb{T} \), we introduce

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Program:

1. First we deduce the asymptotic zero distribution for \( \Phi^{(\beta)}_{n,N} \), as \( n/N \to t \), (2 proofs).

2. By making use of the result for \( \Phi^{(\beta)}_{n,N} \) we obtain the asymptotic zero distribution for \( \Phi_{n,N} \) (under an additional assumption).
Intermezzo: The equilibrium measure of a circular arc

Define the probability measure on $\mathbb{T}$ by

$$d\nu_a(e^{i\theta}) := \frac{1}{2\pi} \frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_a/2) - \cos^2(\theta/2)}} d\theta, \quad \theta \in (\theta_a, 2\pi - \theta_a),$$

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$$U_{\nu_a}(z) = \int_{\Gamma_a} \log |z - e^{i\theta}| \, d\nu_a \left(e^{i\theta}\right) = \log |G_a(z)|, \quad z \in \mathbb{C} \setminus \mathbb{T},$$
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where, for $0 < a \leq 1$,

$$G_a(z) = \frac{1}{2} \left(z + 1 + \sqrt{(z - e^{i\theta_a})(z - e^{-i\theta_a})}\right),$$

and, for $a = 0$,

$$G_0(z) = \begin{cases} 1, & \text{if } |z| < 1, \\ z, & \text{if } |z| > 1. \end{cases}$$
Asymptotic zero distribution of variable POPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$
\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].
$$

where the measure $\nu |_{\alpha(s)}$ is the equilibrium measure of a circular arc.
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Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \nu_{n,N}^{(\beta)} = \frac{1}{t} \int_0^t \nu|_{\alpha(s)}| \, ds,$$

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We indicate two ways of proving the theorem:
Asymptotic zero distribution of variable POPUC

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Let \( t > 0 \) and \( \alpha : [0, t] \to \mathbb{D} \) be continuous. Suppose further that \( \{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D} \) is such that

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\]

Then, for any \( \beta \in \mathbb{T} \),

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We indicate two ways of proving the theorem:

1. based on the ratio asymptotics and a potential-theoretic argument;
Asymptotic zero distribution of variable POPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \to \overline{D}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset D$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \nu_{\beta}^{(n,N)} = \frac{1}{t} \int_0^t \nu|_{\alpha(s)}| \, ds,$$

where the measure $\nu|_{\alpha(s)}|$ is the equilibrium measure of a circular arc.

We indicate two ways of proving the theorem:

1. based on the ratio asymptotics and a potential-theoretic argument;
2. a simple moment based proof.
Proposition:

Let \( t \geq 0 \), and \( \{\alpha_{n,N} | n, N \in \mathbb{N}_0\} \subset D \) be such that \( \lim_{n/N \to t} \alpha_{n,N} = \alpha \). Then, for any \( \beta \in \mathbb{T} \),

\[
\lim_{n/N \to t} \frac{\Phi^{(\beta)}_{n+1,N}(z)}{\Phi^{(\beta)}_{n,N}(z)} = G_{|\alpha|}(z),
\]

uniformly in \( z \) in compact subsets of \( \mathbb{C} \setminus \mathbb{T} \).

Computing the logarithmic potential...
Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:
Let \( t \geq 0 \), and \( \{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D} \) be such that \( \lim_{n/N \to t} \alpha_{n,N} = \alpha \). Then, for any \( \beta \in \mathbb{T} \),

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U_{\nu_{n, N}}(\beta)(z) =
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Computing the logarithmic potential...

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**Computing the logarithmic potential...**

\[
U_{\nu_n,N}^{(\beta)}(z) = \frac{1}{n} \log |\Phi_{n,N}^{(\beta)}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi_{k+1,N}^{(\beta)}(z)}{\Phi_{k,N}^{(\beta)}(z)} \right| =
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Proposition:

Let \( t \geq 0 \), and \( \{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D} \) be such that \( \lim_{n/N \to t} \alpha_{n,N} = \alpha \). Then, for any \( \beta \in \mathbb{T} \),

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\]

Since \( [ns]/N \to st \), as \( n/N \to t \), one has
Proof 1: the ratio asymptotics and a potential-theoretic argument

Proposition:
Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi_{n+1,N}(z)}{\Phi_{n,N}(z)} = G_{|\alpha|}(z),$$

uniformly in $z$ in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu_{n,N}}(\beta)(z) = \frac{1}{n} \log \left| \Phi_{n,N}(z) \right| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi_{k+1,N}(z)}{\Phi_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi_{[ns]+1,N}(z)}{\Phi_{[ns],N}(z)} \right| ds.$$

Since $[ns]/N \to st$, as $n/N \to t$, one has

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U_{\nu_{n,N}}^{(\beta)}(z) = \frac{1}{n} \log |\Phi^{(\beta)}_{n,N}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi^{(\beta)}_{k+1,N}(z)}{\Phi^{(\beta)}_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi^{(\beta)}_{[ns]+1,N}(z)}{\Phi^{(\beta)}_{[ns],N}(z)} \right| ds.
\]

Since $[ns]/N \to st$, as $n/N \to t$, one has

\[
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\lim_{n/N \to t} \frac{\Phi_{n+1,N}^{(\beta)}(z)}{\Phi_{n,N}^{(\beta)}(z)} = G_{|\alpha|}(z),
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uniformly in \( z \) in compact subsets of \( \mathbb{C} \setminus \mathbb{T} \).

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\]

Since \( [ns]/N \to st \), as \( n/N \to t \), one has

\[
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František Štampach (Stockholm University)
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Proof 1: the ratio asymptotics and a potential-theoretic argument

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Let $t \geq 0$, and $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ be such that $\lim_{n/N \to t} \alpha_{n,N} = \alpha$. Then, for any $\beta \in \mathbb{T}$,

$$\lim_{n/N \to t} \frac{\Phi_{n+1,N}(z)}{\Phi_{n,N}(z)} = G_{\alpha}(z),$$

uniformly in $z$ in compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

Computing the logarithmic potential...

$$U_{\nu_{\alpha,n,N}}(z) = \frac{1}{n} \log |\Phi_{n,N}^{(\beta)}(z)| = \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\Phi_{k+1,N}(z)}{\Phi_{k,N}(z)} \right| = \int_0^1 \log \left| \frac{\Phi_{[ns]+1,N}(z)}{\Phi_{[ns],N}(z)} \right| ds.$$ 

Since $[ns]/N \to st$, as $n/N \to t$, one has

$$\lim_{n/N \to t} U_{\nu_{\alpha,n,N}}^{(\beta)}(z) = \int_0^1 \log |G_{\alpha(st)}(z)| ds = \frac{1}{t} \int_0^t \log |G_{\alpha(s)}(z)| ds = \frac{1}{t} \int_0^t U_{\nu_{\alpha(s)}}(z) ds.$$
Proof 1: the ratio asymptotics and a potential-theoretic argument

In summary, we have proved that

$$\lim_{n/N \to t} U_{\nu_{n,N}}^{(\beta)}(z) = U_{\sigma_t}(z), \quad \forall z \in \mathbb{C} \setminus \mathbb{T},$$

where

$$\sigma_t := \frac{1}{t} \int_0^t |\nu_{\alpha(s)}| \, ds.$$
Proof 1: the ratio asymptotics and a potential-theoretic argument

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\lim_{n/N \to t} U_{\nu(\beta)}^{n,N}(z) = U_{\sigma t}(z), \quad \forall z \in \mathbb{C} \setminus \mathbb{T},
\]

where

\[
\sigma_t := \frac{1}{t} \int_0^t \nu_{|\alpha(s)}| ds.
\]
Proof 1: the ratio asymptotics and a potential-theoretic argument

- In summary, we have proved that
  \[
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  \]
  where
  \[
  \sigma_t := \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} \, ds.
  \]
- Hence
  \[
  \lim_{n/N \to t} \nu_{n,N}^{(\beta)} = \sigma_t
  \]
  by the application of Widom's lemma.
Proof 2: a simple moment based proof

The idea is to show that

$$\lim_{n/N \to t} \frac{1}{n} \text{Tr} \left( C_{n,N}^{(\beta)} \right)^k = \int_0^{2\pi} e^{ik\theta} \sigma_t \left( e^{i\theta} \right), \quad \forall k \in \mathbb{Z}. $$
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Then all integer moments of \( \nu_{n,N}^{(\beta)} \) converge to the moments of \( \sigma_t \) and the Stone–Weierstrass theorem implies the statement.
Proof 2: a simple moment based proof

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Steps:
Proof 2: a simple moment based proof

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Steps:

1. It suffices to show the formula for \( k \in \mathbb{N}_0 \) because \( C_{n,N}^{(\beta)} \) is unitary.
Proof 2: a simple moment based proof

The idea is to show that

\[ \lim_{n/N \to t} \frac{1}{n} \text{Tr} \left( C_{n,N}^{(\beta)} \right)^k = \int_0^{2\pi} e^{i k \theta} \sigma_t \left( e^{i \theta} \right), \quad \forall k \in \mathbb{Z}. \]

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1. It suffices to show the formula for \( k \in \mathbb{N}_0 \) because \( C_{n,N}^{(\beta)} \) is unitary.
2. \[ \lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( C_n(\alpha) \right)^k = \int_0^{2\pi} e^{i k \theta} d\nu_{|\alpha|} \left( e^{i \theta} \right) \quad \text{(known from OPUC theory)} \]
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3. 

\[
\lim_{n \to \infty} \frac{1}{n} \left( \text{Tr} \left( C_{n,N} \right)^k - \int_0^1 \text{Tr} \left( C_n(\alpha(s)) \right)^k ds \right) = 0, \quad \text{(slightly combinatorial arguments)}
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Proof 2: a simple moment based proof

The idea is to show that

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Then all integer moments of $\nu_{n,N}^{(\beta)}$ converge to the moments of $\sigma_t$ and the Stone–Weierstrass theorem implies the statement.

Steps:

1. It suffices to show the formula for $k \in \mathbb{N}_0$ because $C_{n,N}^{(\beta)}$ is unitary.
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   $$\lim_{n \to \infty} \frac{1}{n} \text{Tr} \left( C_n (\alpha) \right)^k = \int_0^{2\pi} e^{ik\theta} d\nu_{|\alpha|} (e^{i\theta}) \quad \text{(known from OPUC theory)}$$

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4 OPUC with variable Verblunsky coefficients
The previous arguments cannot be used to deduce the asymptotic zero distribution for variable OPUC.
The previous arguments cannot be used to deduce the asymptotic zero distribution for variable OPUC.

The ratio asymptotic formula

$$\lim_{n/N \to t} \frac{\Phi_{n+1,N}(z)}{\Phi_{n,N}(z)} = G_{|\alpha|}(z)$$

is valid only for $z \notin \mathbb{C} \setminus \mathbb{D}$, in general! As a consequence, we do not know the logarithmic potential of the limiting measure (if exists) on a sufficiently “large” set to recover the limiting measure.
The previous arguments cannot be used to deduce the asymptotic zero distribution for variable OPUC.

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is valid only for \( z \in \mathbb{C} \setminus \overline{D} \), in general! As a consequence, we do not know the logarithmic potential of the limiting measure (if exists) on a sufficiently “large” set to recover the limiting measure.

Since \( C_{n,N} \) is not unitary, we know the key formula from the moment based proof:

\[ \lim_{n/N \to t} \frac{1}{n} \text{Tr} \left( C_{n,N} \right)^k = \int_0^{2\pi} e^{ik\theta} \sigma_t \left( e^{i\theta} \right), \]

for positive integers \( k \) only! This is also an insufficient information for recovering the limiting measure.
Recall that for any probability measure $\mu$ with support in $\overline{\mathbb{D}}$, there exists a unique probability measure $\mathcal{P}(\mu)$ supported on $\mathbb{T}$ such that

$$\int_{\overline{\mathbb{D}}} z^k d\mu(z) = \int_0^{2\pi} e^{ik\theta} d\mathcal{P}(\mu)(e^{i\theta}), \quad \forall k \in \mathbb{N}_0.$$
Balayage

Recall that for any probability measure $\mu$ with support in $\overline{D}$, there exists a unique probability measure $P(\mu)$ supported on $\mathbb{T}$ such that

$$\int_{\overline{D}} z^k d\mu(z) = \int_0^{2\pi} e^{ik\theta} dP(\mu)(e^{i\theta}), \quad \forall k \in \mathbb{N}_0.$$ 

Proposition:

Let $t > 0$ and $\alpha : [0, t] \to \overline{D}$ be continuous. Suppose further that $\{\alpha_{n,N} | n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

Then

$$\lim_{n/N \to t} P(\nu_{n,N}) = \frac{1}{t} \int_0^t \nu|\alpha(s)| ds.$$
Asymptotic eigenvalue distribution for variable OPUC

**Theorem:**

Let \( t > 0 \) and \( \alpha : [0, t] \to \mathbb{D} \) be continuous. Suppose further that \( \{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D} \) is such that

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\lim_{n/N \to s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].
\]

If \( \alpha(t) \neq 0 \), then

\[
\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu_{|\alpha(s)|} ds.
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If $\alpha(t) \neq 0$, then

$$\lim_{n/N \to t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu|\alpha(s)| \, ds.$$

**Proof:** 2 facts about OPUC:
Asymptotic eigenvalue distribution for variable OPUC

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**Proof:** 2 facts about OPUC:

1. If for any \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \# \{ k \in \{1, \ldots, n\} \mid |z_{k,n}| \geq 1 - \varepsilon \} = 1,
\]

where \( z_{1,n}, z_{2,n}, \ldots, z_{n,n} \) are zeros of \( \Phi_n \),
Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let \( t > 0 \) and \( \alpha : [0, t] \to \overline{\mathbb{D}} \) be continuous. Suppose further that \( \{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D} \) is such that
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\[
\lim_{n \to \infty} (\nu_n - \mathcal{P}(\nu_n)) = 0.
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Theorem:

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2. It holds

\[
-\overline{\alpha_{n-1}} = \Phi_n(0).
\]
Asymptotic eigenvalue distribution for variable OPUC

Theorem:

Let $t > 0$ and $\alpha : [0, t] \rightarrow \overline{\mathbb{D}}$ be continuous. Suppose further that $\{\alpha_{n,N} \mid n, N \in \mathbb{N}_0\} \subset \mathbb{D}$ is such that

$$\lim_{n/N \rightarrow s} \alpha_{n,N} = \alpha(s), \quad \forall s \in [0, t].$$

If $\alpha(t) \neq 0$, then

$$\lim_{n/N \rightarrow t} \nu_{n,N} = \frac{1}{t} \int_0^t \nu|\alpha(s)| \, ds.$$ 

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It suffices to show that the zeros of $\Phi_{n,N}$ cluster “mostly” on $\mathbb{T}$, as $n/N \rightarrow t$. 

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Proof cont.

Let \( \{n_j\} \) and \( \{N_j\} \) such that \( n_j, N_j \to \infty \) and \( n_j/N_j \to t \), then

\[
-\overline{\alpha_{n_j-1,N_j}} = \Phi_{n_j,N_j}(0) = \prod_{k=1}^{n} (-z_{k,n_j,N_j}).
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- Then for any \( \varepsilon \in (0, 1) \),

  \[
  \frac{1}{n_j} \log |\alpha_{n_j-1,N_j}| = \frac{1}{n_j} \sum_{j=1}^{n_j} \log |z_{k,n_j,N_j}| \leq \log(1 - \varepsilon) \frac{\#\{k : |z_{k,n_j,N_j}| \leq 1 - \varepsilon\}}{n_j}
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\]

\[
\downarrow \quad \alpha(t) \neq 0
\]

\[
\downarrow \quad 0
\]

Hence

\[
\lim_{n/N \to t} \frac{\# \{k : |z_{k,n,N}| > 1 - \varepsilon \}}{n} = 1,
\]

and the previous theorem implies the result.
The case $\alpha(t) = 0$

- It is not clear what can happen when $\alpha(t) = 0$ at the moment.
The case $\alpha(t) = 0$

- It is not clear what can happen when $\alpha(t) = 0$ at the moment.
- Various situations can occur. Note that, as a special case, the have (standard) OPUC with

$$\alpha_{n,N} = \alpha_n \to 0, \quad \text{as} \ n \to \infty.$$  

In particular, it is known that the limiting measure need not be supported on $\mathbb{T}$ (or need no exists).
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In particular, it is known that the limiting measure need not be supported on $\mathbb{T}$ (or need no exists).
- Therefore let us consider the less general setting:

$$\alpha_{n,N} := \alpha \left( \frac{nt}{N} \right),$$

where $\alpha \in C([0,t])$. 

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- Therefore let us consider the less general setting:
  \[ \alpha_{n,N} := \alpha \left( \frac{nt}{N} \right), \]
  where $\alpha \in C([0, t])$.

- From now until the end of the talk, we investigate the asymptotic distribution of zeros of the polynomials
  \[ \Phi_N(z) := \Phi_{N,N}(z), \]
  for $N \to \infty$. (The notation is a bit confusing here!)
The case $\alpha(t) = 0$: polynomial vs. exponential decay

For a given $\alpha \in C([0, t])$, we would like to understand the situation when

$$\lim_{N \to \infty} \left| \alpha \left( \frac{N - 1}{N} t \right) \right|^{1/N} = A \in [0, 1].$$
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Example (Polynomial decay)
If $\alpha \in C([0, t])$ decays at $t$ as

$$\alpha(s) = a(s - t)^m + o((s - t)^m), \quad \text{as } s \to t-, $$

for some $m \in \mathbb{N}$ and $a \neq 0$. Then $A = 1$. 
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Example (Exponential decay)

If $\alpha \in C([0, t])$ decays at $t$ as

$$\alpha(s) = A^{t-s} (1 + o(1)), \quad \text{as } s \to t-,$$

for some $A \in (0, 1)$. Then $A < 1$. 
The polynomial decay and an open problem

**Theorem:**

Let $\alpha \in C([0, t])$ be such that $A = 1$. Then

$$\lim_{N \to \infty} \nu_N = \frac{1}{t} \int_0^t \nu |\alpha(s)| ds.$$
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Theorem:
Let $\alpha \in C([0, t])$ be such that $A = 1$. Then

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Proof: based on a modification of the previous arguments.
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**Proof:** based on a modification of the previous arguments.

**OPEN PROBLEM:**

What happens when $A < 1$?
Thank you!