New explicitly diagonalizable Hankel matrices

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joint with P. Šťovíček

International Workshop on Operator Theory and its Applications
July 22, 2019

Acknowledgement: Supported by Europ. Reg. Development Fund-Project “Center for Advanced Applied Science” No. CZ.02.1.01/0.0/0.0/16_019/0000778.
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1 Introduction - the Hilbert matrix

2 New results - Hankel matrices and Jacobi matrices from the Askey scheme

3 New results - New diagonalizable Hankel matrices

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The semi-infinite matrix $H$ with entries

$$H_{m,n} = h_{m+n},$$

i.e.,

$$H = \begin{pmatrix}
h_0 & h_1 & h_2 & h_3 & \ldots \\
h_1 & h_2 & h_3 & h_4 & \ldots \\
h_2 & h_3 & h_4 & h_5 & \ldots \\
h_3 & h_4 & h_5 & h_6 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

is called the Hankel matrix.
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- Although the general spectral theory of Hankel operators is deeply developed, only very few concrete interesting (=not of finite rank) Hankel matrices with “explicitly” solvable spectral problem.
The Hilbert matrix

- The Hilbert matrix:

\[(H_0)_{m,n} = \frac{1}{m + n + 1},\]

i.e.,

\[
H_0 = \begin{pmatrix}
1 & 1 & 1 & 1 & \
\frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
The Hilbert matrix

- The *(generalized)* Hilbert matrix:

\[
(H_\lambda)_{m,n} = \frac{1}{m + n + 1 + \lambda},
\]

i.e.,

\[
H_\lambda = \begin{pmatrix}
\frac{1}{1+\lambda} & \frac{1}{2+\lambda} & \frac{1}{3+\lambda} & \frac{1}{4+\lambda} & \cdots \\
\frac{1}{1+\lambda} & \frac{1}{2+\lambda} & \frac{1}{3+\lambda} & \frac{1}{4+\lambda} & \cdots \\
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History of the Hilbert matrix

- Hilbert’s inequality (1908): There is $M > 0$ such that

$$0 \leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m a_n}{m + n + 1} \leq M \sum_{n=0}^{\infty} a_n^2,$$

for all real $a \in \ell^2(\mathbb{N}_0)$. 

Schur 1911: The optimal value of the constant $M = \pi$.

Perhaps first proof of $\|H_0\| = \pi$.

Magnus 1949 (also Schur): $\|H_\lambda\| = \pi$, $\lambda \geq -\frac{1}{2}$, and $\|H_\lambda\| = \pi |\sin(\lambda \pi)|$, $-1 < \lambda < -\frac{1}{2}$.

Magnus 1950: $\text{spec}(H_0) = [0, \pi]$.

Rosenblum 1958: A complete explicit spectral representation of $H_\lambda$ for $\lambda > -\frac{1}{2}$.

(Rosenblum applied ideas of the commutator method.)
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  (Rosenblum applied ideas of the commutator method.)
An alternative proof to Rosenblum’s approach

The Hilbert matrix $H_0$ commutes with the Jacobi matrix

$$J = \begin{pmatrix}
\beta_0 & \alpha_0 \\
\alpha_0 & \beta_1 & \alpha_1 \\
\alpha_1 & \beta_2 & \alpha_2 \\
& \ddots & \ddots & \ddots 
\end{pmatrix}$$

where

$$\alpha_n = -(n + 1)^2 \quad \text{and} \quad \beta_n = 2n(n + 1) + 3/4.$$
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- The associated sequence of ON polynomials $P = \{P_n\}_{n \in \mathbb{N}_0}$, is unambiguously defined as the formal eigenvector of $J$:

$$JP(x) = xP(x)$$

normalized such that $P_0(x) = 1$. 

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- In this case, $P_n$ is a particular case of the Continuous dual Hahn polynomials - a three-parameter family of hypergeometric OG polynomials listed in the Askey scheme.
An alternative proof to Rosenblum’s approach

As a result, we know that \( \{P_n\}_{n \in \mathbb{N}_0} \) is an ONB of \( L^2((0, \infty), \rho(x)dx) \), where

\[
\rho(x) = \frac{\pi \sinh(\pi \sqrt{x})}{\cosh^2(\pi \sqrt{x})}.
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Moreover, the unitary mapping

\[
U : \ell^2(\mathbb{N}_0) \to L^2((0, \infty), \rho(x)dx) : e_n \mapsto P_n
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diagonalizes the Jacobi operator \( J \), i.e, \( U J U^{-1} = T_x \).
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Since \( J \) is a self-adjoint operator with simple spectrum commuting with \( H_0 \),

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where \( f \) is a Borel function.
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Determination of \( f \) using a generating function formula for \( P_n \):

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  \]
A summary of the commutator method

In total, this approach shows that $H_0$ is unitarily equivalent to the multiplication operator by function

$$f(x) = \frac{\pi}{\cosh(\pi x)}$$

acting on $L^2((0, \infty), \rho(x)dx)$. This yields the spectral representation of $H_0$. Particularly,

$$\text{spec}(H_0) = \text{spec}_{ac}(H_0) = [0, \pi].$$

For details, see [Otte 2005 - slides; T. Kalvoda and P. Štovíček 2016].
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1. Finding a self-adjoint operator $J$ with simple spectrum and solvable spectral problem that commutes with $H$.
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2. Finding the spectral mapping $f$. 
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1 Introduction - the Hilbert matrix

2 New results - Hankel matrices and Jacobi matrices from the Askey scheme

3 New results - New diagonalizable Hankel matrices
The scope

Goal of the project: To extend the set of known Hankel matrices with explicitly solvable spectral problem.
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The Askey scheme

- Wilson
- Racah
- Continuous dual Hahn
- Continuous Hahn
- Hahn
- Dual Hahn
- Meixner - Pollaczek
- Jacobi
- Meixner
- Krawtchouk
- Laguerre
- Charlier
- Hermite

New results - Hankel matrices and Jacobi matrices from the Askey scheme

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The Askey scheme - semi-infinite Jacobi matrices
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**Theorem (A prominent role of the Hilbert matrix)**

Let $H = (h_{m+n})$ be a Hankel matrix with rank $H > 1$ and $h \in \ell^2(\mathbb{N}_0)$. 
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**Theorem (A prominent role of the Hilbert matrix)**

Let $H = (h_{m+n})$ be a Hankel matrix with rank $H > 1$ and $h \in \ell^2(\mathbb{N}_0)$. Let $J$ be a hermitian semi-infinite non-decomposable Jacobi matrix from the Askey scheme.
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A class of Jacobi matrices

We consider the class of Jacobi matrices $J = J(a, b, c; \sigma, k)$ with:

$$
\alpha_n = -\sqrt{(n + 1)(n + a + 1)(n + b + 1)(n + c + 1)},
$$

$$
\beta_n = (k^{-1} + k)n(n + \sigma),
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for $a, b, c > -1$, $\sigma \in \mathbb{R}$, and $k \in (0, 1)$. 

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A motivation:

- The corresponding OGPs are closely related to Heun functions.
New results - New diagonalizable Hankel matrices

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Theorem

The Jacobi matrix $J$ commutes with a non-trivial Hankel matrix if and only if $\alpha_n$ is a polynomial function of $n$. 

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Characterization of commuting Hankel matrices

**Theorem**

Let $J$ is the Jacobi matrix with

$$
\alpha_n = -(n + 1)(n + a + 1), \quad \beta_n = (k + k^{-1})n(n + \sigma).
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Characterization of commuting Hankel matrices

Theorem

Let $J$ is the Jacobi matrix with

$$\alpha_n = -(n + 1)(n + a + 1), \quad \beta_n = (k + k^{-1})n(n + \sigma).$$

Then, up to a constant multiplier, the Hankel matrix $H_{m,n} = h_{m+n}$ with entries

$$h_n = \frac{k^n \Gamma(n + a + 1)}{\Gamma(n + a + \omega(a, \sigma) + 1)} \, _2F_1(n + a + 1, \omega(a, \sigma) - 1; n + a + \omega(a, \sigma) + 1; k^2),$$

where

$$\omega(a, \sigma) = \frac{-2k^2 + (1 + k^2)(\sigma - a)}{1 - k^2},$$

is the only Hankel matrix with $h \in \ell^2(\mathbb{N}_0)$ commuting with $J$. 
New results - New diagonalizable Hankel matrices

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\[ h_n = \frac{k^n n! (n + a + 1)}{\Gamma(n + a + \omega(a, \sigma) + 1)} _2F_1(n + a + 1, \omega(a, \sigma) - 1; n + a + \omega(a, \sigma) + 1; k^2), \]

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is the only Hankel matrix with $h \in \ell^2(\mathbb{N}_0)$ commuting with $J$. Moreover, $H$ is a trace class operator on $\ell^2(\mathbb{N}_0)$.

- For $\sigma = a+1$ and $k \to 1$, we arrive at the generalized Hankel matrix

  $$
h_n = \frac{1}{n+a+1}.
$$
Following the lines of the commutator method we seek for Jacobi matrices with
\[ \alpha_n = -(n+1)(n+a+1), \quad \beta_n = (k + k^{-1})n(n + \sigma) \]
whose spectral properties can be obtained explicitly (or in terms of special functions).
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- It turns out that there are at least 4 special Jacobi matrices whose spectral properties can be deduced from the known properties of the Stieltjes–Carlitz polynomials (Carlitz 1960).
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- These corresponds to the particular values of the parameters

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<td>(-1/2)</td>
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<tr>
<td>( \sigma )</td>
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<td>((1 + 2k^2)/(k^2 + 1))</td>
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Following the lines of the commutator method we seek for Jacobi matrices with
\[
\alpha_n = -(n + 1)(n + a + 1), \quad \beta_n = (k + k^{-1})n (n + \sigma)
\]
whose spectral properties can be obtained explicitly (or in terms of special functions).

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Basic elements of the theory of elliptic functions:

\[
K = K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad K' = K'(k) := K\left(\sqrt{1 - k^2}\right),
\]
and
\[
q = q(k) := \exp\left(-\pi K'(k)/K(k)\right).
\]
Four new diagonalizable Hankel matrices

We introduce four Hankel matrices $H^{(p)}$, $H^{(q)}$, $H^{(r)}$, $H^{(s)}$, depending on a parameter $k \in (0, 1)$,

$$H^{(j)}_{m,n} = h^{(j)}_{m+n}, \quad j = p, q, r, s,$$

for $m, n \in \mathbb{N}_0$, where

- $h^{(p)}_n := \frac{k^n \Gamma(n + 1/2)}{(n + 1)!} 2F_1 \left( \begin{array}{c} n + 1/2, 1/2 \\ n + 2 \end{array} \bigg| k^2 \right) = \frac{4k^n}{\sqrt{\pi}} \int_0^1 t^{2n} \sqrt{\frac{1 - t^2}{1 - k^2 t^2}} \, dt,$$

- $h^{(q)}_n := \frac{k^n \Gamma(n + 3/2)}{(n + 1)!} 2F_1 \left( \begin{array}{c} n + 3/2, -1/2 \\ n + 2 \end{array} \bigg| k^2 \right) = \frac{2k^n}{\sqrt{\pi}} \int_0^1 t^{2n+2} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt,$$

- $h^{(r)}_n := \frac{k^n \Gamma(n + 1/2)}{n!} 2F_1 \left( \begin{array}{c} n + 1/2, -1/2 \\ n + 1 \end{array} \bigg| k^2 \right) = \frac{2k^n}{\sqrt{\pi}} \int_0^1 t^{2n} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt,$$

- $h^{(s)}_n := \frac{k^n \Gamma(n + 3/2)}{(n + 2)!} 2F_1 \left( \begin{array}{c} n + 3/2, 1/2 \\ n + 3 \end{array} \bigg| k^2 \right) = \frac{4k^n}{\sqrt{\pi}} \int_0^1 t^{2n+2} \sqrt{\frac{1 - t^2}{1 - k^2 t^2}} \, dt.$
Diagonalization of $H^{(p)}$, $H^{(q)}$, $H^{(r)}$, $H^{(s)}$.

Theorem

Each of the Hankel matrices $H^{(j)}$, $j = p, q, r, s$, represents a positive trace class operator on $\ell^2(\mathbb{N}_0)$ with simple eigenvalues which are as follows:

\[
\nu^{(p)}_m = \frac{4\sqrt{\pi}}{k} \frac{q^{m+1/2}}{1 + q^{2m+1}}, \quad m \geq 0,
\]

\[
\nu^{(q)}_m = \frac{2\sqrt{\pi}}{k} \frac{q^{m+1/2}}{1 + q^{2m+1}}, \quad m \geq 0,
\]

\[
\nu^{(r)}_m = 2\sqrt{\pi} \frac{q^m}{1 + q^{2m}}, \quad m \geq 0,
\]

\[
\nu^{(s)}_m = \frac{4\sqrt{\pi}}{k^2} \frac{q^m}{1 + q^{2m}}, \quad m \geq 1.
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$$

Moreover, the corresponding eigenvectors and their $\ell^2$-norms are expressible in terms of the Stieltjes–Carlitz polynomials and elliptic integrals (not displayed).
Thank you!