On the localization of spectra of complex sampling Jacobi matrices and open problems

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Quantum circle
Introduction

Experiments

Attempts to prove the Conjecture

The case of uniform grid

The story of Toeplitz matrices

The circle example

Equipotential measures
Definition:
- Let $a, b \in C([0, 1])$ be complex-valued functions.
Sampling matrices

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- We call the matrix

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J_{a,b}(\Delta_n) := \begin{pmatrix}
 b(t_1) & a(t_1) \\
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a sampling Jacobi matrix.
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Where they appear:

▶ Discrete approximations of 1-d BVP (grid, finite difference scheme),
▶ random matrices.

Problem:
Localization of $\text{spec}(J_{a,b}(\Delta_n))$ in terms of $a, b$. 

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\[ a(t) = \frac{i}{2} \left( -40320 + 198971 t^2 - 163647 t^4 + 53837 t^6 - 9488 t^8 \right) \]

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Fallen snowman: $a(t) = \ldots \text{complicated} \ldots$, $b(t) = \ldots \text{complicated} \ldots$
A random object:

\[ a(t) = (-4 - 2i) + (5 + 5i)t - (4 + 3i)t^2 + (4 + 5i)t^3 \]

\[ b(t) = (-4 + i) - 2t - (3 + i)t^2 - (3 + 2i)t^3 \]
It seems the eigenvalues are somewhat localized ...
Estimations for the localization domain

One has

\[ \|J(\Delta_n)\| \leq \|b\|_\infty + 2\|a\|_\infty, \quad \forall n, \forall \Delta_n, \forall a, b \in C([0,1]). \]
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Gerschrogin circle theorem:

Let \( A = (a_{i,j}) \in \mathbb{C}^{n,n} \) and
\[ R_i = \sum_{j \neq i} |a_{i,j}|, \]
then
\[ \text{spec}(A) \subset \bigcup_{i=1}^{n} D(a_{i,i}, R_i). \]
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Applying Gerschrogin’s theorem we obtain much better localization:
\[
\text{spec}(J_{a,b}(\Delta_n)) \subset \bigcup_{0 \leq t \leq 1} D(b(t), 2a(t)) \quad \forall n, \forall \Delta_n
\]
Let $\Delta = \{\Delta_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[0, 1]$. Put

$$\Lambda_{a,b}(\Delta) := \{ z \in \mathbb{C} | \liminf_{n \to \infty} \text{dist}(z, \text{spec}(J_{a,b}(\Delta_n))) = 0 \}.$$
Weaker formulation and the optimal localization

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So, $\lambda \in \Lambda_{a,b}(\Delta)$ iff

$$\exists \{n_k\} \subset \mathbb{N} \quad \exists \lambda_k \in \text{spec}(J_{a,b}(\Delta_{n_k})) \text{ such that } \lim_{k \to \infty} \lambda_{n_k} = \lambda.$$
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Conjecture:

For all $a, b \in C([0, 1])$ and $\Delta$ a sequence of partitions of $[0, 1]$, it holds

$$\Lambda_{a,b}(\Delta) \subset S_{a,b} := \bigcup_{0 \leq t \leq 1} [b(t) - 2a(t), b(t) + 2a(t)]$$

and this localization is optimal.
Let \( \Delta = \{\Delta_n\}_{n=1}^{\infty} \) be a sequence of partitions of \([0, 1]\). Put

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**Conjecture:**

For all \( a, b \in C([0, 1]) \) and \( \Delta \) a sequence of partitions of \([0, 1]\), it holds

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Equivalently the statement says: \( \forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \) one has

\[
\text{spec}(J_{a,b}(\Delta_n)) \subset U_\epsilon(S_{a,b}).
\]
Let’s take a look on pictures...
Square: \( a(t) = \frac{i}{2}, \ b(t) = 1 - 2t. \)
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An attempt to prove the Conjecture

Idea:

1. To replace $J_{a,b}(\Delta_n)$ by a matrix of “simpler structure” which is close (in norm) to $J_{a,b}(\Delta_n)$ and use some perturbation arguments, but in non-self-adjoint setting!
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2. Similar approach has been successfully used by Tilli in 1998 solving the similar problem for the so called locally Toeplitz matrices. However, all his results concerning eigenvalues are derived under the self-adjointness assumption!
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3. For instance, one can consider one can divide $[0, 1]$ to $m(\leq n)$ subintervals, decompose $n = n_1 + \cdots + n_m$, and introduce the following matrices (the frozen boxes idea):

$$A_n^{(m)} = \bigoplus_{i=1}^{m} J_{n_i}(a_i, b_i) + \sum_{i=1}^{m-1} x_i \left( e_{N_i} e_{N_i+1}^T + e_{N_i+1} e_{N_i}^T \right)$$

where $N_i = n_1 + \cdots + n_i$ and $a_i = a(t_{n_i})$, $b_i = b(t_{n_i})$ and $J_{n_i}(a_i, b_i)$ is a tridiagonal Toeplitz $n_i \times n_i$ matrix. Treat the problem for $A_n^{(m)}$. 
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4. However, it is to say that picture is very incomplete now and several pieces are missing!
Asymptotic eigenvalue distribution of $A_n^{(m)}$

Here we put $x_i = \sqrt{a_i a_{i+1}}$.

**Theorem:**

Let $m \in \mathbb{N}$ and for all $j \in \{1, \ldots, m\}$, let $n_j : \mathbb{N} \to \mathbb{N}$ be such that $n_j(n) \to \infty$, as $n \to \infty$, and $N = n_1 + \cdots + n_m$.

Then
\[
\lim_{n \to \infty} \det(A_n^{(m)}(N(n)) - z) \prod_{j=1}^m a_{n_j(n)} U_{n_j(n)}(b_j - z^2 a_j) = m - 1 \prod_{j=1}^m \left[1 - \frac{1}{f(b_j - z^2 a_j)} f(b_j + 1 - z^2 a_j + 1)\right]
\]

where $f(z) = z - \sqrt{z - 1} \sqrt{z + 1}$ and $U_n(\cdot)$ stands for the Chebyshev polynomials of the 2nd kind, and the convergence is local uniform in $z \in \mathbb{C} \cup m_j = 1 [b_j - 2 a_j, b_j + 2 a_j]$.

**Corollary:**

"The set of limit points of $\text{spec}(A_n^{(m)}(N(n)))$, as $n \to \infty" = m \bigcup_{j=1}^m [b_j - 2 a_j, b_j + 2 a_j]."
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Asymptotic eigenvalue distribution of $A^{(m)}_n$

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**Corollary:**

"The set of limit points of $\text{spec} \left( A^{(m)}_{N(n)} \right)$, as $n \rightarrow \infty$" $= \bigcup_{j=1}^{m} [b_j - 2a_j, b_j + 2a_j]$. 

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\lim_{n \to \infty} \frac{\det \left( A_{N(n)}^{(m)} - z \right)}{\prod_{j=1}^{m} a_j^{n_j(n)} U_{n_j(n)} \left( \frac{b_j - z}{2a_j} \right)} = \prod_{j=1}^{m-1} \left[ 1 - f \left( \frac{b_j - z}{2a_j} \right) f \left( \frac{b_{j+1} - z}{2a_{j+1}} \right) \right]
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The limit of eigenvalue-counting measures of $A_n^{(m)}$

- In case of matrices $A_{N(n)}^{(m)}$, we can prove much more.
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- Denote $\mu_n^{(m)}$ the eigenvalue-counting measure of $A_{N(n)}^{(m)}$, i.e.,

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\mu_n^{(m)} = \sum_\lambda \frac{1}{\nu_a(\lambda)} \delta_\lambda
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where $\nu_a(\lambda)$ is the algebraic multiplicity of the eigenvalue $\lambda$. 


The limit of eigenvalue-counting measures of $A^{(m)}_n$

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Let $m \in \mathbb{N}$ and for all $j \in \{1, \ldots, m\}$, $n_j : \mathbb{N} \to \mathbb{N}$ be such that

$$\lim_{n \to \infty} n_j(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} n_j(n+1) - n_j(n) = \ell_j \in \mathbb{N}.$$
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- Denote $\mu_n^{(m)}$ the eigenvalue-counting measure of $A_{N(n)}^{(m)}$, i.e.,

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where $\nu_a(\lambda)$ is the algebraic multiplicity of the eigenvalue $\lambda$.

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Then

$$\omega - \lim_{n \to \infty} \mu_n^{(m)} = \sum_{j=1}^{m} \ell_j \omega_{a_j, b_j}$$

where $\omega_{a, b}$ is the absolutely continuous measure supported on $[b - 2a, b + 2a]$ with density

$$\frac{d\omega_{a,b}}{dz}(z) = \frac{1}{2a} \frac{d\omega}{dx}\left(\frac{b - z}{2a}\right) \quad \text{and} \quad \frac{d\omega}{dx}(x) = \frac{\chi(-1,1)(x)}{\pi \sqrt{1 - x^2}}.$$
The case of uniform grid

- Take the sequence $\Delta$ of uniform partitions of $[0, 1]$, i.e.,

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See the pictures . . .
The square
The square - uniform grid
The circle
The circle - uniform grid
The butterfly
The butterfly-uniform grid
The fish
The fish - uniform grid
Fallen snowman
Fallen snowman - uniform grid
The random object
The random object - uniform grid
Open problems

Previous numerical observations give rise to many questions:

▶ Is it possible to find a description of the curves in terms of $a$ and $b$?
▶ What are (topological, analytical,...) properties of these curves?
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▶ If so, what can be said about the limiting measure?

Except few very special examples, all these questions remain open...
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Contents

Introduction

Experiments

Attempts to prove the Conjecture

The case of uniform grid

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- Let $T(b)$ stands for the banded Toeplitz operator determined by the symbol

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i.e.,

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    b_1 & b_0 & b_{-1} & & \ddots \\
    b_2 & b_1 & b_0 & \ddots & \ddots \\
    & \ddots & \ddots & \ddots & \ddots \\
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    \vdots & \ddots & \ddots & \ddots & \vdots \\
    b_s & \vdots & \ddots & b_0 & \ddots \\
\end{pmatrix}.$$

The $n \times n$ principle submatrix of $T(b)$ is denoted by $T_n(b)$. 
Towards the limiting set

The limiting set of spectra $\text{spec}(T_n(b))$:

$$\Lambda(b) = \{ z \in \mathbb{C} | \lim_{n \to \infty} \text{dist}(z, \text{spec}(T_n(b))) = 0 \}.$$
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- If

$$b_\rho(t) := b(\rho t), \quad \rho > 0,$$

then $T_n(b)$ and $T_n(b_\rho)$ are similar matrices since

$$T_n(b_\rho) = \text{diag}(\rho, \rho^2, \ldots, \rho^n) T_n(b) \text{ diag}(\rho^{-1}, \rho^{-2}, \ldots, \rho^{-n})$$
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- Therefore \( \text{spec}(T_n(b)) = \text{spec}(T_n(b_\rho)) \). Actually we have
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- However, there is a much more useful description of $\Lambda(b)$. Define

$$Q(z; \lambda) := z^r (b(z) - \lambda).$$

Theorem (Schmidt and Spitzer):

$$\Lambda(b) = \left\{ \lambda \in \mathbb{C} : |z^r(\lambda)| = |z^r+1(\lambda)| \right\}$$

Based on this description of $\Lambda(b)$, it was proved that...

Theorem (Schmidt, Spitzer, Ullman):

$\Lambda(b)$ is a connected set that equals the union of a finite number of pairwise disjoint open analytic arcs and a finite number of the so called exceptional points (roughly speaking: branching points and endpoints).
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An example (7-diagonal Toeplitz)
Towards the limiting measure

If $\lambda \notin \Lambda(b)$ then one can find $\rho > 0$ such that

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Define function $g : \mathbb{C} \setminus \Lambda(b) \to (0, \infty)$ by the formula

$$g(\lambda) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |b(\rho e^{i\theta}) - \lambda| \, d\theta \right).$$

It can be shown that $g(\lambda)$ does not depend on the specific choice of $\rho$. 

Theorem (Hirschman): The sequence of eigenvalue-counting measures of $T_n(b)$ converges weakly to a measure $\mu$ supported on $\Lambda(b)$. In addition, $d\mu(\lambda) = \frac{1}{2\pi} g(\lambda) |\partial g(\lambda)/\partial n_1 + \partial g(\lambda)/\partial n_2| ds(\lambda)$, for $\lambda \in \Lambda(b)$ a nonexceptional point (for such points, the outer normal vector derivatives $\partial g(\lambda)/\partial n_1$ and $\partial g(\lambda)/\partial n_2$ with respect to the two components separated by the respective arc of $\Lambda(b)$ exist) Here, $ds$ stands for the arc length measure.
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Contents

Introduction

Experiments

Attempts to prove the Conjecture

The case of uniform grid

The story of Toeplitz matrices

The circle example

Equipotential measures
Unit disk and the Szegö curve
Contents

Introduction

Experiments

Attempts to prove the Conjecture

The case of uniform grid

The story of Toeplitz matrices

The circle example

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The logarithmic potential

Let $\mu$ be a finite positive measure compactly supported in $\mathbb{C}$. The logarithmic potential is defined as

$$U^\mu(z) = \int_{\mathbb{C}} \log |z - \xi| d\mu(\xi).$$

($U^\mu$ is harmonic in $\mathbb{C} \setminus \text{supp } \mu$ and subharmonic in $\mathbb{C}$.)
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Two measures $\mu$ and $\nu$ are called equipotential iff

$$U^\mu(z) = U^\nu(z), \quad \forall z \in \mathbb{C} \setminus (\text{supp } \mu \cup \text{supp } \nu).$$
Theorem

Let $\mu_n$ be the eigenvalue-counting measures of $J_{a,b}(\Delta_n)$ with uniform partitions $\Delta_n$. Then there is a neighborhood $U$ of $\infty$ such that

$$\lim_{n \to \infty} U^{\mu_n}(z) = U^{\sigma}(z), \quad \forall z \in U$$
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$$\frac{d\omega_{a,b}}{dz}(z) = \frac{1}{2a} \frac{d\omega}{dx} \left( \frac{b - z}{2a} \right) \quad \text{and} \quad \frac{d\omega}{dx}(x) = \frac{\chi(-1,1)(x)}{\pi \sqrt{1 - x^2}}.$$
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**Corollary**

If the Conjecture stating $\Lambda_{a,b}(\Delta) \subset S_{a,b}$ holds true and the weak* limit $\mu$ of measures $\mu_n$ exists. Then the measures $\mu$ and $\sigma$ are equipotential.
Veselé Velikonoce