Linear Algebra with Application (LAWA 2020)

Homework 3

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Exercise 1 [Example 3.21]

Let us consider the systems of linear equations

$$\begin{cases} x_1 - 2x_2 + 4x_3 - x_4 + 5x_6 &= 0 \\ -2x_1 + 4x_2 - 7x_3 + x_4 + 2x_5 - 8x_6 &= 0 \\ 3x_1 - 6x_2 + 12x_3 - 3x_4 + x_5 + 15x_6 &= 0 \\ 2x_1 - 4x_2 + 9x_3 - 3x_4 + 3x_5 + 12x_6 &= 0 \end{cases}.$$

The reduction of the augmented matrix to an equivalent matrix in reduced row-echelon form is the following:

$$\begin{pmatrix} 1 & -2 & 4 & -1 & 0 & 5 & 0 \\ -2 & 4 & -7 & 1 & 2 & -8 & 0 \\ 3 & -6 & 12 & -3 & 1 & 15 & 0 \\ 2 & -4 & 9 & -3 & 3 & 12 & 0 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1}]{iii)} \begin{pmatrix} 1 & -2 & 4 & -1 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - 4R_2 \\ R_4 \to R_4 - R_2}]{iii)} \begin{pmatrix} 1 & -2 & 0 & 3 & -8 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 + 8R_3 \\ R_2 \to R_2 - 2R_3 \\ R_4 \to R_4 - R_3}]{iii)} \begin{pmatrix} 1 & -2 & 0 & 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 2 [Example 3.25]

Let us consider the system of linear equations

$$\begin{cases} x_1 - 2x_2 + 2x_3 - x_4 & = & 1 \\ 2x_1 - 4x_2 + 3x_3 + x_4 & = & 2 \\ 3x_1 - 6x_2 + 5x_3 & = & 3 \end{cases}.$$

Let us first find a particular solution of the system. A possible one is

$$X_0 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

We can rewrite this system as the matrix equation

$$AX = B$$

where

$$A = \begin{pmatrix} 1 & -2 & 2 & -1 \\ 2 & -4 & 3 & 1 \\ 3 & -6 & 5 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

The associated homogeneous system is represented by the matrix equation

$$\begin{pmatrix} 1 & -2 & 2 & -1 \\ 2 & -4 & 3 & 1 \\ 3 & -6 & 5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By reduction of the augmented matrix of the homogeneous system in a reduced row-echelon form

$$\begin{pmatrix} 1 & -2 & 2 & -1 & 0 \\ 2 & -4 & 3 & 1 & 0 \\ 3 & -6 & 5 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}]{iii)} \begin{pmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 \to -R_2]{ii)} \begin{pmatrix} 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - R_2}]{iii)} \begin{pmatrix} 1 & -2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we find that the general solution of the homogeneous system is

$$X = \begin{pmatrix} 2s - 5t \\ s \\ 3t \\ t \end{pmatrix} = s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 0 \\ 3 \\ 1 \end{pmatrix}$$

where s and t are parameters representing arbitrary numbers.

If we thus define the basic solytions

$$X_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} -5 \\ 0 \\ 3 \\ 1 \end{pmatrix}$$

we can write the general solution of the original system as

$$X = X_0 + sX_1 + tX_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 0 \\ 3 \\ 1 \end{pmatrix}.$$