Linear Algebra with Application (LAWA 2020)

Homework 8

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May 2020

Exercise 1 [Theorem 5.33] *Let A be a diagonalizable matrix. Let us suppose that P is a diagonalizing matrix and $D = P^{-1}AP$. Then, for any $k \in \mathbb{N}$ one has*

$$A^k = PD^kP^{-1}.$$

Proof. Let us prove the result by induction on k. If $k = 1$ then it is clear since

$$D = P^{-1}AP \quad \Longrightarrow \quad PDP^{-1} = A.$$

(Note that we could also have proved it for $k = 0$).

Let us now suppose that the result is true for $k - 1$, that is that we have $A^{k-1} = PD^{k-1}P^{-1}$. Then

$$\begin{array}{rcl} A^k & = & A^{k-1}A \\ & = & \left(PD^{k-1}P^{-1}\right) A \\ & = & \left(PD^{k-1}P^{-1}\right)\left(PDP^{-1}\right) \\ & = & PD^{k-1}\left(P^{-1}P\right) DP^{-1} \\ & = & PD^{k-1}DP^{-1} \\ & = & PD^kP^{-1}. \end{array}$$

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Exercise 2 [Example 5.45]

Let us consider the matrix

$$A = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{pmatrix}.$$

To compute A^{20} we first need to find the diagonal matrix D and the diagonalizing matrix P such that $A = PDP^{-1}$. To find the eigenvalues of A let us study the characteristic polynomial

$$\begin{array}{rcl} c_A(x) & = & \det(xI_3 - A) \\ & = & \det\left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{pmatrix}\right) \\ & = & \det\begin{pmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{pmatrix} \\ & = & (x-1)\det\begin{pmatrix} x+2 & -2 \\ 5 & x-5 \end{pmatrix} + \det\begin{pmatrix} 4 & -2 \\ 5 & x-5 \end{pmatrix} - \det\begin{pmatrix} 4 & -2 \\ x+2 & -2 \end{pmatrix} \\ & = & (x-3)\left((x+2)(x-5)+10\right) + (4x-20+10) - (-8+2x+4) \\ & = & (x-3)(x^2-3x) + 2(x-3) \\ & = & (x^2-3x+2)(x-3) \\ & = & (x-1)(x-2)(x-3) \end{array}.$$

Thus the eigenvalues of A are the roots of $c_A(x)$:

$$\lambda_1 = 1, \quad \lambda_2 = 2 \quad \text{and} \quad \lambda_3 = 3.$$

Since there are three distinct eigenvalues we are sure that the matix is diagonalizable.

To find the respective eigenvectors, let us study the homogenous systems of linear equations

$$(\lambda_1 I_3 - A)X = O, \quad (\lambda_2 I_3 - A)X = O \quad \text{and} \quad (\lambda_3 I_3 - A)X = O.$$

- For $\lambda_1 = 1$, the system

  $$(1 \cdot I - A)X = \begin{pmatrix} -2 & 4 & -2 \\ -1 & 3 & -2 \\ -1 & 5 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

  has general solution

  $$X = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

  with s an arbitrary number. Thus, an eigenvector to A relative to λ_1 is the column-vector

  $$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(one can actually check that $AX_1 = 1 \cdot X_1$).

- For $\lambda_2 = 2$, the system

$$(2 \cdot I - A)X = \begin{pmatrix} -1 & 4 & -2 \\ -1 & 4 & -2 \\ -1 & 5 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

has general solution

$$X = s \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

with s an arbitrary number. Thus, an eigenvector to A relative to λ_2 is the column-vector

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

(one can actually check that $AX_2 = 2 \cdot X_2$).

- For $\lambda_3 = 3$, the system

$$(3 \cdot I - A)X = \begin{pmatrix} 0 & 4 & -2 \\ -1 & 5 & -2 \\ -1 & 5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

has general solution

$$X = s \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

with s an arbitrary number. Thus, an eigenvector to A relative to λ_3 is the column-vector

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

(one can cactually check that $AX_3 = 3 \cdot X_3$).

Now that we have the three eigenvectors, we can construct a diagonalizing matrix

$$P = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Such a matrix is invertible, and we can find its inverse using the Inverse Matrix

Algorithm reducing $\begin{pmatrix} P & I \end{pmatrix} \to \begin{pmatrix} I & P^{-1} \end{pmatrix}$:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}]{iii)} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to -R_2]{ii)} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 - 2R_2 \\ R3 \to R_3 + R_2}]{iii)} \begin{pmatrix} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[R_1 \to R_1 - R_3]{iii)} \begin{pmatrix} 1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}.$$

So we have

$$P^{-1} = \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

We can check that

$$P^{-1}AP = \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} A = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Let us call

$$D = \operatorname{diag}(1, 2, 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then we have $A = PDP^{-1}$ and

$$\begin{aligned} A^{20} &= PD^{20}P^{-1} \\ &= \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20} \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 3^{20} \end{pmatrix} \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{20} - 1 & 3 - 2^{21} - 3^{20} & 3^{20} - 1 \\ 2^{20} - 1 & 3 - 2^{20} - 3^{20} & 3^{20} - 1 \\ 2^{20} - 1 & 3 - 2^{20} - 2 \cdot 3^{20} & 3^{20} - 1 \end{pmatrix} \end{aligned}$$

Exercise 3 [Example 5.48]

Let A, B be two square matrices such that $A \sim B$. If A is diagonalizable, then B is also diagonalizable.

Indeed, since A is diagonalizable, then there exists an invertible matrix P such that $A = P^{-1}DP$ with D a diagonal matrix. On the other hand, since $A \sim B$, then there exists an invertible matrix Q such that $B = Q^{-1}AQ$. Thus

$$B = Q^{-1}\left(P^{-1}DP\right)Q = \left(Q^{-1}P^{-1}\right)D\,(PQ) = (PQ)^{-1}\,D\,(PQ)$$

That is, B is diagonalizable with diagonalizing matrix the invertible matrix PQ.