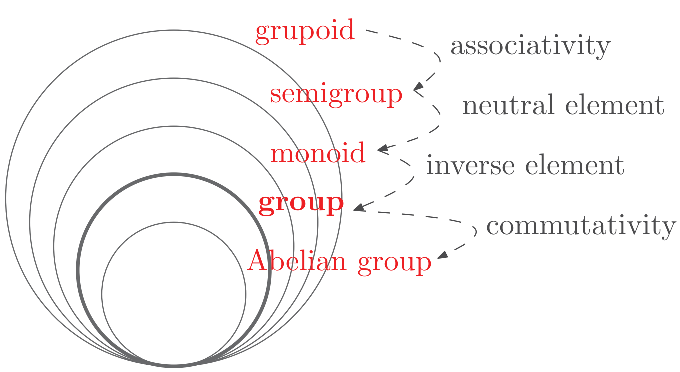
Chapter 1

MPI - lecture 5

1.1 Reminder and Motivation

Reminder of the last lecture

Hierarchy of structures of type "a set and a binary operation"



Example (1/4)

Example 1. *Consider the set* $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ *with the addition* $\mod 12$.

- *the set* $\mathbb{Z}_{12}$ *is closed under this operation, i.e., it is a* ***groupoid****groupoid;*
- *the operation is associative, so it is a* ***semigroup****semigroup;*

- *number* 0 *is the neutral element, so it is a* ***monoid****monoid;*
- *the inverse of a number* $k \neq 0$ *is*$12 - k$ *and the inverse of* 0 *is* 0*, so it is a* ***group****group;*
- *the operation is commutative, thus we have an* ***Abelian group****.*

Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the set of the residue classes modulo n.

The group $(\mathbb{Z}_n, +_{(\text{mod } n)})$ is the additive group modulo n; it is denoted by $\mathbb{Z}_n^+$.

Example (2/4)

Question: Which other set M forms a group with the addition $(\text{mod } 12)$?

In order for the operation to be well defined, we must have $M \subset \mathbb{Z}_{12}$:

Question (refined): Which subset of $\mathbb{Z}_{12}$ forms a group with the addition $(\text{mod } 12)$?

Answer: There are quite a lot of them. To find out how to discover them, let us ask this subquestion:

Sub-question: Which is the smallest subset of $\mathbb{Z}_{12}$ that forms a group with addition $(\text{mod } 12)$ and contains the number 2?

Example (3/4)

We are looking for a set $M \subset \mathbb{Z}_{12}$ such that $2 \in M$ and $(M, +(\text{mod } 12))$ is a group:

- M must be closed under addition $\text{mod } 12$:
  - it must contain $2 + 2 = \mathbf{4}$, $2 + 4 = \mathbf{6}$, $4 + 6 = \mathbf{10}$, ...
  - the set $\{0, 2, 4, 6, 8, 10\}$ is closed under this operation, so we have a groupoid;
- the operation remains associative, so it is a semigroup;
- 0 remains the neutral element, so it is a monoid;
- each element has its inverse in the set (the set is closed under inversion), so it is a group.

The wanted set is $M = \{0, 2, 4, 6, 8, 10\}$: we say that M is a subgroup generated by the set $\{2\}$.

Example (4/4)

Similarly, as we have generated the set for the element 2, we can proceed for others elements of $\mathbb{Z}_{12}$:

$$\begin{array}{lcl} \{0\} \rightarrow & \{0\} & \\ \{1\} \rightarrow & \{0,1,2,3,4,5,6,7,8,9,10,11\} & \leftarrow \{11\} \\ \{2\} \rightarrow & \{0,2,4,6,8,10\} & \leftarrow \{10\} \\ \{3\} \rightarrow & \{0,3,6,9\} & \leftarrow \{9\} \\ \{4\} \rightarrow & \{0,4,8\} & \leftarrow \{8\} \\ \{5\} \rightarrow & \{0,5,10,3,8,1,6,11,4,9,2,7\} & \leftarrow \{7\} \\ \{6\} \rightarrow & \{0,6\} & \end{array}$$

Back to the original question: there exist 6 different sets $M \subseteq \mathbb{Z}_{12}$ such that $(M, +(\text{mod } 12))$ is a group.

1.2 Subgroups

Definition of subgroup

Definition 2. *Let $G = (M, \circ)$ be a group. A subgroup of the group G is a pair $H = (N, \circ)$ such that:*

- $N \subseteq M$ *and* $N \neq \emptyset$,
- H *is a group.*
- Idea of substructures with the same properties as the original structure: compare for instance with a subspace of a linear (vector) space.
- Similarly, we can define subgroupoids, subsemigroups, submonoids,. . .
- A binary operation in the group $G = (M, \circ)$ is considered to be a mapping from $M \times M$ to M. The operation in a subgroup $H = (N, \circ)$ is, to be precise, the restriction of this operation to the set $N \times N$.

Trivial and proper subgroups

In each group $G = (M, \circ)$, there always exist at least two subgroups (if M contains only one element the two coincide):

- the group containing only the neutral element: $(\{e\}, \circ)$,
- and the group itself $G = (M, \circ)$.

These two groups are the trivial subgroups; other subgroups are non-trivial or proper subgroups.

Question 3. *If H is a subgroup of the group G, is the neutral element of H identical to the neutral element of G?*

Intersection of subgroups

Theorem 4. *Let $H_1, H_2, \ldots, H_n$, whith $n \geq 1$, be subgroups of a group $G = (M, \circ)$. Then*

$$H' = \bigcap_{i=1,2,\ldots,n} H_i$$

is also a subgroup of G.

Power of an element

Definition 5. *Let $G = (M, \circ)$ be a group with neutral element e. We define for each element $a \in M$ and each positive $n \in \mathbb{N}$ the n-th power of the element a as*

$$\begin{aligned} a^0 &= e \\ a^n &= \underbrace{a \circ a \circ \cdots \circ a}_{n \text{ times}} \\ a^{-n} &= (a^{-1})^n = \underbrace{a^{-1} \circ a^{-1} \circ \cdots \circ a^{-1}}_{n \text{ times}} \end{aligned}$$

Note that $a \circ a \circ \cdots \circ a$ can by written without brackets thanks to associativity (for a non-associative operation the result would depend on the order…).

For all $n, m \in \mathbb{N}$ the following “natural” equalities are true:

- $a^{n+m} = a^n \circ a^m$,

- $a^{nm} = (a^n)^m$.

For the additive notation of a group $G = (M, +)$, we define the n-th multiple of the element a and we denote it by $n \times a$ (resp. $-n \times a = n \times (-a)$).

Order of a (sub)group

Definition 6. *The order of a (sub)group $G = (M, \circ)$, denoted $|G|$, is its number of elements. If M is an infinite set, the order is infinite. According to the order we distinguish between finite and infinite groups.*

Example 7. *The group $\mathbb{Z}_{12}^{+}$ is of order 12. It has 6 subgroups:*

- *two trivial: $\{0\}$ and $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$;*
- *and four proper: $\{0, 6\}$, $\{0, 4, 8\}$, $\{0, 3, 6, 9\}$, and $\{0, 2, 4, 6, 8, 10\}$.*

of order respectively 1, 2, 3, 4, 6 and 12.

(Left) cosets of a subgroup

Let G be a group and H be one of its subgroups.
The (left) coset of H in G with respect to an element $g \in G$ is the set

$$gH = \{gh : h \in H\} \quad \text{(or } g + H \text{ in additive notation)}$$

Example 8. *Let us consider the subgroup $H = \{0, 3, 6, 9\}$ of $\mathbb{Z}_{12}$. Find $g + H$ for all $g \in \mathbb{Z}_{12}$.*

The index of H in G, denoted $[G : H]$, is the number of different cosets of H in G.

Lagrange's Theorem

Theorem 9. *Let H be a subgroup of a finite group G. The order of H divides the order of G.*
More precisely, $|G| = [G : H] \cdot |H|$.

To prove Lagrange's Theorem we need the following lemma.

Lemma 10. *For all $a, b \in G$ one has $|aH| = |bH|$.*

This statement connects the abstract structure of a group with divisibility and also with the notion of prime numbers!

Consequence: A group with prime order has only trivial subgroups!

Question 11. *Let G be a group of order n and $k \in \mathbb{N}$ be such that $k|n$.*
Is there any subgroup of G of order k?

Group generated by a set (1/2)

Question: How to find the smallest subgroup of a group $G = (M, \circ)$ containing a given nonempty set $N \subset M$?

Definition 12. *Let $G = (M, \circ)$ be a group and $N \subset M$ a nonempty set. The smallest subgroup of G containing N is the subgroup generated by N and is denoted by $\langle N \rangle$.*

In particular, for a singleton $N = \{a\}$ we use the notation $\langle a \rangle = \langle \{a\} \rangle$.

Example 13. *For the group $\mathbb{Z}_{12}^{+}$, we have proven that $\langle 2 \rangle = (\{0, 2, 4, 6, 8, 10\}, (+mod\ 12))$.*

Definition 14. *If for a set M it holds that $\langle M \rangle = G$, we say that M is a generating set of G.*

Group generated by a set (2/2)

Example 15. *The group $\mathbb{Z}_{12}^{+}$ is generated, for instance, by the sets $\{1\}$ and $\{5\}$, i.e.*

$$\langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_{12}^{+}.$$

Theorem 16. *Let $G = (M, \circ)$ be a group and $N \subset M$ a nonempty set. The following holds:*

- *the subgroup $\langle N \rangle$ equals the intersection of all subgroups containing N, i.e.*
$$\langle N \rangle = \bigcap \{H \colon H \text{ is a subgroup of} G \text{ containing} N\}$$
- *all elements belonging to $\langle N \rangle$ can be obtained by means of "group span", i.e.,*
$$\left\{a_1^{k_1} \circ a_2^{k_2} \circ \cdots a_n^{k_n} \ : \ n \in \mathbb{N}, a_i \in N, k_i \in \mathbb{Z}\right\}.$$

1.3 Cyclic groups

Examples of groups generated by one element (1/2)

We have seen that the additive group $\mathbb{Z}_{12}^{+}$ is equal to $\langle 1\rangle$, $\langle 5\rangle$, $\langle 7\rangle$, and $\langle 11\rangle$.

The following theorem says this is true in general:

Theorem 17. *An additive group modulo n is equal to $\langle k\rangle$ if and only if k and n are coprime numbers.*

Proof. This statement is a consequence of a general theorem which will be proven later and of the fact that $\langle 1\rangle = \mathbb{Z}_n^{+}$ for all $n \geq 2$. □

Examples of groups generated by one element (2/2)

The multiplicative group modulo p, denoted $\mathbb{Z}_p^{\times}$, where p is a prime number, is the set $\{1, 2, \ldots, p-1\}$ with the operation of multiplication modulo p.

Example 18. *Is there a one-element set generating the group $\mathbb{Z}_{11}^{\times}$?*

Yes, for example $\langle 2\rangle = \mathbb{Z}_{11}^{\times}$.

On the other hand, $\langle 3\rangle = (\{1, 3, 4, 5, 9\}, \cdot \ (mod\ 11))$.

Finding the generator(s) of a multiplicative group $\mathbb{Z}_p^{\times}$ is more complicated than for an additive group $\mathbb{Z}_n^{+}$.

Multiplicative groups have more complicated and interesting structure.

Definition of cyclic group

Definition 19. *A group $G = (M, \circ)$ is cyclic if there exists an element $a \in M$ such that $\langle a\rangle = G$.*

This element is a generator of the cyclic group.

- $\mathbb{Z}_n^{+}$ is a cyclic group for every n and its generators are all positive numbers $k \leq n$ coprime with n.
- the infinite group $(\mathbb{Z}, +)$ is cyclic and it has just two generators: 1 and -1.
- $\mathbb{Z}_{11}^{\times}$ is cyclic, and 2 is a generator.

$$(\text{mod } 13) \quad \begin{array}{cccccccccccc} 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 & 2^{10} & 2^{11} & 2^{12} \\ 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 & 1 \end{array}$$

$$(\text{mod } 13) \quad \begin{array}{cccccccccccc} 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 & 2^{10} & 2^{11} & 2^{12} \\ 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 & 1 \end{array}$$

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Why “cyclic”?

Consider the multiplicative group $\mathbb{Z}_{13}^{\times}$.

Again, 2 is a generator. If we repeatedly compose the element 2 with itself we successively get all elements of the group: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 3$, …, $2^{12} = 1$.

The 13-th power is again the number 2 and so the sequence of powers is indeed stuck in a cycle.

subgroups: $\{1, 3, 4, 9, 10, 12\}$,$\{1, 5, 8, 12\}$,$\{1, 3, 9\}$,$\{1, 12\}$.

generators: 2, 6, 7, 11.

Fermat's Theorem (1/2)

Theorem 20. *In a cyclic group $G = (M, \circ)$ of order n, for all elements*

$$(\text{mod } 13) \quad \begin{array}{cccccccccccc} 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 & 2^{10} & 2^{11} & 2^{12} \\ 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 & 1 \end{array}$$

$$(\text{mod } 13)\quad \begin{array}{cccccccccccc} 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 & 2^{10} & 2^{11} & 2^{12} \\ 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 & 1 \end{array}$$

$$(\text{mod } 13)\quad \begin{array}{cccccccccccc} 2^1 & \cancel{2^2} & \cancel{2^3} & \cancel{2^4} & 2^5 & \cancel{2^6} & 2^7 & \cancel{2^8} & \cancel{2^9} & \cancel{2^{10}} & 2^{11} & \cancel{2^{12}} \\ 2 & \cancel{4} & \cancel{8} & \cancel{3} & 6 & \cancel{12} & 11 & \cancel{9} & \cancel{5} & \cancel{10} & 7 & \cancel{1} \end{array}$$

$a \in M$*, it holds that*

$$a^n = e$$

Where e *is the neutral element of* G*.*

Proof. Consider a sequence of elements from M: $a, a^2, a^3, a^4, \ldots$

Denote q the smallest number such that $a^q = e$. Clearly $q \leq n$ (why?!)

The set $a, a^2, \cdots, a^q$ is the subgroup $\langle a \rangle$ and has order q By Lagrange's Theorem, we have that q divides n, i.e,. there exists $k \in \mathbb{N}$ such that $n = qk$.

We have $a^n = a^{qk} = (a^q)^k = e^k = e$. □

Fermat's Theorem (2/2)

$\mathbb{Z}_p^\times$ is always a cyclic group (it is not trivial to prove it) and the order of this group is $p-1$.

Applying the previous statement to $\mathbb{Z}_p^\times$ we obtain the well-known Fermat's Little Theorem.

Corollary 21 (Fermat's Little Theorem)**.** *For an arbitrary prime number* p *and an arbitrary* $1 \leq a < p$ *we have that*

$$a^{p-1} \equiv 1 \,(mod\ p).$$

How to find all generators (1/2)

Generally, to find all generators is not an easy task (e.g., in groups $\mathbb{Z}_p^\times$ we are not able to do it algorithmically); but if we have one, it is easy to find all the others.

Theorem 22. *If $(G, \circ)$ is a cyclic group of order n and a is one of its generator, then a^k is a generator if and only if k and n are coprime.*

To prove the Theorem we use the following

Lemma 23. *Let $D = \{mk + \ell n \mid m, \ell \in \mathbb{Z}\}$ and $d = \min\{|a| \mid a \in D \setminus \{0\}\}$.*
Then $d = \gcd(k, n)$.

How to find all generators (2/2)

Corollary 24. *In a cyclic group of order n, the number of all generators is equal to $\varphi(n)$.*

Where φ is the Euler's (totient) function), which assigns to each integer n the number of integers less than n that are coprime with n

$\mathbb{Z}_p^\times$ is a cyclic group of order $p - 1$ and thus it has $\varphi(p - 1)$ generators.

An effective algorithm for evaluating $\varphi(n)$ does not exist; if it existed, we would be able to find the integer factorization of arbitrarily large n and RSA would not be safe!

Subgroups of cyclic group are cyclic

Theorem 25. *Any subgroup of a cyclic group is again a cyclic group.*

Proof. Let $G = \langle a \rangle$ be a cyclic group and H one of its proper subgroup.
Let $p \in \mathbb{N}$ be the smallest non-zero number such that $a^p \in H$. We show that $H = \langle a^p \rangle$ and so finish the proof.

Surely it holds that $\langle a^p \rangle \subset H$. Let us thus prove that $H \subset \langle a^p \rangle$.

Let h be an element of H and q be such that $h = a^q$. Certainly there exist m, ℓ such that $d = gcd(q, p) = mq + \ell p$. Then $a^d \in H$ and thus $d \geq p$. But on the other hand $d \leq p$. Therefore $d = p$ and there exists k such that $q = kp$. It follows that $h = a^q = (a^p)^k \in \langle a^p \rangle$ and we are done. $\square$

$$(\text{mod } 13)\quad \begin{array}{cccccccccccc} 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & 2^9 & 2^{10} & 2^{11} & 2^{12} \\ 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 & 1 \end{array}$$

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Illustration of validity previous statements

Consider again the multiplicative group $\mathbb{Z}_{13}^{\times}$.

subgroups: $\{1,3,4,9,10,12\}$, $\{1,5,8,12\}$, $\{1,3,9\}$, $\{1,12\}$.
generators: 2, 6, 7, 11.

Order of an element

Let G be a group and $g \in G$. The order of g (in G) is the order of the group that is generated by g.

In the finite case, it is equivalent to $\text{order}(g) = \#\langle g \rangle$.

Example 26. *Find the order of all elements in $\mathbb{Z}_5^{\times}$ and in $\mathbb{Z}_7^{\times}$.*