MIE-MPI: Tutorial 5

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Comments: The goal of this tutorial is to understand the notions of subgroup, cyclic group and generating set.

5.1 Subgroups

Exercise 5.1. Which of the following sets form a subgroup of the group $(\mathbb{Q} \setminus \{0\}, \cdot)$?

(a) set of all even numbers without zero;

(b) set of all odd numbers;

(c) $\{2^n : n \in \mathbb{Z}\}$;

(d) $\{2^n \cdot 3^m : n, m \in \mathbb{Z}\}$;

(e) $\left\{\frac{1+2n}{1+2m} : n, m \in \mathbb{Z}\right\}$.

Exercise 5.2. Find some other subgroup(s) of the group $(\mathbb{Q} \setminus \{0\}, \cdot)$ distinct from the ones in the previous exercise.

Exercise 5.3. Find all subgroups of the group given by following Cayley table.

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Hint: when looking for a subgroup, be more specific on what subgroup you look for.

Exercise 5.4. Is the set $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a \neq 0 \vee b \neq 0\}$ a subgroup of the group $(\mathbb{R} \setminus \{0\}, \cdot)$?

Exercise 5.5. Specify the following subgroups of $(\mathbb{Z}, +)$.

(a) $\langle 2 \rangle$;

(b) $\langle 5 \rangle$;

(c) $\langle\{2, 3\}\rangle$;

(d) $\langle\{2, 4\}\rangle$;

(e) $\langle\{6, 12\}\rangle$;

(f) $\langle\{n, m, \ell\}\rangle$ for $n, m, \ell \in \mathbb{N}^+$.

5.2 Cyclic groups and generators

Exercise 5.6. Find all generators and all subgroups of $\mathbb{Z}_{11}^{\times} = (\mathbb{Z}_{11}, \cdot)$.

Exercise 5.7. Find all generators and all subgroups of $\mathbb{Z}_{13}^{\times} = (\mathbb{Z}_{13}, \times_{13})$. Find the inverse elements of 12, 5, and 11.

Exercise 5.8. What is the probability that an element of $\mathbb{Z}_{23}^{\times}$ chosen randomly is generator?

Exercise 5.9. Is the number 5 a generator of the group $\mathbb{Z}_{23}^{\times}$?
What is the least number of "computation steps" needed to decide this question?