Generalized Lyndon words

Francesco Dolce

INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

université PARIS DIDEROT PARIS 7

joint work with
Antonio Restivo and Christophe Reutenauer

Séminaire de l'Équipe Automates et Applications (IRIF)
Paris, 26 mars 2019

New Word Order(s)

$$(A^\infty, <)$$

$$u < v \;:\Leftrightarrow\; v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a < b \end{cases}$$

New Word Order(s)

$$(A^\infty, <)$$

$$u < v \;:\Leftrightarrow\; v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a < b \end{cases}$$

$$(A^\infty, <_g = (<_n)_{n \geq 1})$$

$$u <_g v \;:\Leftrightarrow\; v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a <_{|p|+1} b \end{cases}$$

New Word Order(s)

$$(A^\infty, <)$$

$$u < v \;:\Leftrightarrow\; v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a < b \end{cases}$$

$$(A^\infty, <_g = (<_n)_{n\geq 1})$$

$$u <_g v \;:\Leftrightarrow\; v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a <_{|p|+1} b \end{cases}$$

$$(A^\infty, \prec)$$

$$u \prec v \;:\Leftrightarrow\; \begin{cases} u^\omega < v^\omega \text{ or} \\ u^\omega = v^\omega \text{ and } |u| > |v| \end{cases}$$

New Word Order(s)

$$(A^\infty, <)$$

$$u < v \;:\Leftrightarrow\; v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a < b \end{cases}$$

$$(A^\infty, <_g = (<_n)_{n\geq 1})$$

$$u <_g v \;:\Leftrightarrow\; v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a <_{|p|+1} b \end{cases}$$

$$(A^\infty, \prec)$$

$$u \prec v \;:\Leftrightarrow \begin{cases} u^\omega < v^\omega \text{ or} \\ u^\omega = v^\omega \text{ and } |u| > |v| \end{cases}$$

$$(A^\infty, \prec_g)$$

$$u \prec_g v \Leftrightarrow \begin{cases} u^\omega <_g v^\omega \text{ or} \\ u^\omega = v^\omega \text{ and } |u| > |v| \end{cases}$$

Generalized lexicographical order

Let $(<_n)_{n\geq1}$ be a sequence of total orders on A.

Generalized lexicographical order

Let $(<_n)_{n\geq 1}$ be a sequence of total orders on A.

The *generalized lexicographical order* is defined as $u < v$ if either :

- u is a proper prefix of v, or
- $u = pas$, $v = pbt$ for some $p \in A^*$, $s, t \in A^\infty$, and $a, b \in A$ s.t. $a <_{|p|+1} b$.

Generalized lexicographical order

Let $(<_n)_{n\geq 1}$ be a sequence of total orders on A.

The *generalized lexicographical order* is defined as $u < v$ if either :

- u is a proper prefix of v, or
- $u = pas$, $v = pbt$ for some $p \in A^*$, $s, t \in A^\infty$, and $a, b \in A$ s.t. $a <_{|p|+1} b$.

Examples

- Classical order $(<)$: $a <_n b$ for all $n \geq 1$.

$$a < aa < ab < aba < baa < bab$$

Generalized lexicographical order

Let $(<_n)_{n\geq 1}$ be a sequence of total orders on A.

The *generalized lexicographical order* is defined as $u < v$ if either :

- u is a proper prefix of v, or
- $u = pas$, $v = pbt$ for some $p \in A^*$, $s, t \in A^\infty$, and $a, b \in A$ s.t. $a <_{|p|+1} b$.

Examples

- Classical order $(<)$: $a <_n b$ for all $n \geq 1$.
- Alternate order $(<_{alt})$: $\begin{cases} a <_n b & \text{if } n \equiv 1 \pmod 2 \\ b <_n a & \text{otherwise.} \end{cases}$

$$a <_{alt} ab <_{alt} aa <_{alt} b <_{alt} bba <_{alt} ba$$

Generalized lexicographical order

Let $(<_n)_{n\geq 1}$ be a sequence of total orders on A.

The *generalized lexicographical order* is defined as $u < v$ if either :

- u is a proper prefix of v, or
- $u = pas$, $v = pbt$ for some $p \in A^*$, $s, t \in A^\infty$, and $a, b \in A$ s.t. $a <_{|p|+1} b$.

Examples

- Classical order $(<)$: $a <_n b$ for all $n \geq 1$.
- Alternate order $(<_{alt})$: $\begin{cases} a <_n b & \text{if } n \equiv 1 \pmod 2 \\ b <_n a & \text{otherwise.} \end{cases}$
- Prime order $(<_\pi)$: $\begin{cases} b <_n a & \text{if } n \text{ is prime} \\ a <_n b & \text{otherwise.} \end{cases}$

$$aba <_\pi abaa <_\pi aab <_\pi bab <_\pi baab$$

Generalized lexicographical order

inverse order

The *inverse (generalized) order* $\widetilde{<}$, obtained by reversing all the orders $<_n$, is also a generalized order.

Examples

- $aba \ <_\pi \ aab \ <_\pi \ bab \ <_\pi \ baa$
- $baa \ \widetilde{<}_\pi \ bab \ \widetilde{<}_\pi \ aab \ \widetilde{<}_\pi \ aba$.

Infinite order

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Infinite order

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

When $|u| = |v|$ one has $u < v \Leftrightarrow u^\omega < v^\omega$. In general, this is not true.

Example

$ab < aba$ but $(ab)^\omega > (aba)^\omega$.

Infinite order

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

When $|u| = |v|$ one has $u < v \Leftrightarrow u^\omega < v^\omega$. In general, this is not true.

Example

$ab < aba$ but $(ab)^\omega > (aba)^\omega$.

We can also consider the generalized lexicographical infinite order.

Example

$(ab)^\omega <_\pi a^\omega <_\pi b^\omega <_\pi (ba)^\omega$.

Infinite order

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

When $|u| = |v|$ one has $u < v \Leftrightarrow u^\omega < v^\omega$. In general, this is not true.

Example

$ab < aba$ but $(ab)^\omega > (aba)^\omega$.

We can also consider the generalized lexicographical infinite order.

Example

$(ab)^\omega <_\pi a^\omega <_\pi b^\omega <_\pi (ba)^\omega$.

$u^\omega = v^\omega \quad \Longleftrightarrow \quad u$ and v are power of a common word ($\Longleftrightarrow \quad uv = vu$).

Infinite order

Proposition [Reutenauer (2015)]

Let $<_g$ be a generalized order.
The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_g v^\omega$,

(2) $(uv)^\omega <_g v^\omega$,

(3) $u^\omega <_g (vu)^\omega$,

(4) $(uv)^\omega <_g (vu)^\omega$.

Infinite order

Proposition [Reutenauer (2015)]

Let $<_g$ be a generalized order.
The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_g v^\omega$,

(2) $(uv)^\omega <_g v^\omega$,

(3) $u^\omega <_g (vu)^\omega$,

(4) $(uv)^\omega <_g (vu)^\omega$.

Examples

Let $u = ab$ and $v = a$. Then

$$(ab.a)^\omega \;<_\pi\; (ab)^\omega \;<_\pi\; (a.ab)^\omega \;<_\pi\; a^\omega$$

Infinite order

Proposition [Reutenauer (2015)]

Let $<_g$ be a generalized order.
The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_g v^\omega$,

(2) $(uv)^\omega <_g v^\omega$,

(3) $u^\omega <_g (vu)^\omega$,

(4) $(uv)^\omega <_g (vu)^\omega$.

Examples

Let $u = ab$ and $v = a$. Then

$$(ab.a)^\omega <_\pi (ab)^\omega <_\pi (a.ab)^\omega <_\pi a^\omega$$

Infinite order

Proposition [Reutenauer (2015)]

Let $<_g$ be a generalized order.
The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_g v^\omega$,

(2) $(uv)^\omega <_g v^\omega$,

(3) $u^\omega <_g (vu)^\omega$,

(4) $(uv)^\omega <_g (vu)^\omega$.

Examples

Let $u = ab$ and $v = a$. Then

$$(ab.a)^\omega <_\pi (ab)^\omega <_\pi (a.ab)^\omega <_\pi a^\omega$$

Infinite order

Proposition [Reutenauer (2015)]

Let $<_g$ be a generalized order.
The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_g v^\omega$,

(2) $(uv)^\omega <_g v^\omega$,

(3) $u^\omega <_g (vu)^\omega$,

(4) $(uv)^\omega <_g (vu)^\omega$.

Examples

Let $u = ab$ and $v = a$. Then

$$(ab.a)^\omega \;<_\pi\; (ab)^\omega \;<_\pi\; (a.ab)^\omega \;<_\pi\; a^\omega$$

Infinite order

Proposition [Reutenauer (2015)]

Let $<_g$ be a generalized order.
The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_g v^\omega$,

(2) $(uv)^\omega <_g v^\omega$,

(3) $u^\omega <_g (vu)^\omega$,

(4) $(uv)^\omega <_g (vu)^\omega$.

Examples

Let $u = ab$ and $v = a$. Then

$$(ab.a)^\omega \; <_\pi \; (ab)^\omega \; <_\pi \; (a.ab)^\omega \; <_\pi \; a^\omega$$

Generalized Lyndon words

A word $w \in A^+$ is a *(classical) Lyndon word* if for any nontrivial factorization $w = uv$ one has $w < vu$.

Generalized Lyndon words

A word $w \in A^+$ is a *generalized* *Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.

$$\begin{array}{l} abcab.abcab\cdots \\ bcaba.bcaba\cdots \\ cabab.cabab\cdots \\ ababc.ababc\cdots \\ babca.babca\cdots \end{array}$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.

$$\begin{array}{l} abcab.abcab\cdots \\ bcaba.bcaba\cdots \\ cabab.cabab\cdots \\ ababc.ababc\cdots \\ babca.babca\cdots \end{array}$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.

$$abcab.abcab\cdots$$
$$bcaba.bcaba\cdots$$
$$cabab.cabab\cdots$$
$$ababc.ababc\cdots$$
$$babca.babca\cdots$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.

$$abcab.abcab\cdots$$
$$bcaba.bcaba\cdots$$
$$cabab.cabab\cdots$$
$$ababc.ababc\cdots$$
$$babca.babca\cdots$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.

$$abcab.abcab\cdots$$
$$bcaba.bcaba\cdots$$
$$cabab.cabab\cdots$$
$$ababc.ababc\cdots$$
$$babca.babca\cdots$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.

$$\begin{array}{c} abcab.abcab\cdots \\ bcaba.bcaba\cdots \\ cabab.cabab\cdots \\ ababc.ababc\cdots \\ babca.babca\cdots \end{array}$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.
- cab is a $\widetilde{<}$–Lyndon word.

$$(cab)^\omega \,\widetilde{<}\, (bca)^\omega \,\widetilde{<}\, (abc)^\omega$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.
- cab is a $\widetilde{<}$–Lyndon word.
- $acab$ is a $<_{alt}$–Lyndon word (*Galois word*).

$$(acab)^\omega \;<_{alt}\; (abac)^\omega \;<_{alt}\; (baca)^\omega \;<_{alt}\; (caba)^\omega$$

Generalized Lyndon words

A word $w \in A^+$ is a *generalized Lyndon word* if for any nontrivial factorization $w = uv$ one has $w^\omega <_g (vu)^\omega$.

Examples

- $a, aab, abcd$ are classical Lyndon words.
- $abcab$ is a $<_\pi$–Lyndon word.
- cab is a $\widetilde{<}$–Lyndon word.
- $acab$ is a $<_{alt}$–Lyndon word (*Galois word*).

Classical Lyndon words are unbordered. This is not true for generalized Lyndon ones.

Example

$abcab$ is a $<_\pi$–Lyndon word.

Generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

A word w is a generalized Lyndon word if and only if for any non trivial factorization $w = uv$ one has :

- $u^\omega <_g v^\omega$
- $w^\omega <_g v^\omega$.

Generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

A word w is a generalized Lyndon word if and only if for any non trivial factorization $w = uv$ one has :

- $u^\omega <_g v^\omega$
- $w^\omega <_g v^\omega$.

Proof. $(uv)^\omega <_g (vu)^\omega \Leftrightarrow u^\omega <_g v^\omega \Leftrightarrow (uv)^\omega <_g v^\omega$.

Generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

A word w is a generalized Lyndon word if and only if for any non trivial factorization $w = uv$ one has :

- $u^\omega <_g v^\omega$
- $w^\omega <_g v^\omega$.

Proof. $(uv)^\omega <_g (vu)^\omega \quad \Leftrightarrow \quad u^\omega <_g v^\omega \quad \Leftrightarrow \quad (uv)^\omega <_g v^\omega$.

Example [$acab$ ($<_{alt}$),]

- $a^\omega <_{alt} (cab)^\omega$, $(ac)^\omega <_{alt} (ab)^\omega$, $(aca)^\omega <_{alt} b^\omega$

Generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

A word w is a generalized Lyndon word if and only if for any non trivial factorization $w = uv$ one has :

- $u^\omega <_g v^\omega$
- $w^\omega <_g v^\omega$.

Proof. $(uv)^\omega <_g (vu)^\omega \quad \Leftrightarrow \quad u^\omega <_g v^\omega \quad \Leftrightarrow \quad (uv)^\omega <_g v^\omega$.

Example [$acab$ ($<_{alt}$), cab ($\tilde{<}$)]

- $a^\omega <_{alt} (cab)^\omega$, $(ac)^\omega <_{alt} (ab)^\omega$, $(aca)^\omega <_{alt} b^\omega$
- $(cab)^\omega \,\tilde{<}\, b^\omega$, $(ab)^\omega$

Factorization into generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

Each word $w \in A^+$ can be factorized in a unique way as $w = \ell_1 \ell_2 \cdots \ell_n$, with ℓ_i generalized Lyndon words s.t. $\ell_1^\omega \geq_g \ell_2^\omega \geq_g \cdots \geq_g \ell_n^\omega$.

Factorization into generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

Each word $w \in A^+$ can be factorized in a unique way as $w = \ell_1 \ell_2 \cdots \ell_n$, with ℓ_i generalized Lyndon words s.t. $\ell_1^\omega \geq_g \ell_2^\omega \geq_g \cdots \geq_g \ell_n^\omega$.

Example

- The factorization in classical Lyndon word of $acaabaa$ is $(ac)(aab)(a)(a)$, since

$$(ac)^\omega \geq (aab)^\omega \geq a^\omega \geq a^\omega$$

Factorization into generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

Each word $w \in A^+$ can be factorized in a unique way as $w = \ell_1 \ell_2 \cdots \ell_n$, with ℓ_i generalized Lyndon words s.t. $\ell_1^\omega \geq_g \ell_2^\omega \geq_g \cdots \geq_g \ell_n^\omega$.

Example

- The factorization in classical Lyndon word of $acaabaa$ is $(ac)(aab)(a)(a)$, since

$$(ac)^\omega \geq (aab)^\omega \geq a^\omega \geq a^\omega$$

- The factorization in $<_{alt}$–Lyndon word of $aabaabaabb$ is $(a)(aba)(aba)(abb)$, since

$$a^\omega \geq_{alt} (aba)^\omega \geq_{alt} (aba)^\omega \geq_{alt} (abb)^\omega$$

Factorization into generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

Each word $w \in A^+$ can be factorized in a unique way as $w = \ell_1 \ell_2 \cdots \ell_n$, with ℓ_i generalized Lyndon words s.t. $\ell_1^\omega \geq_g \ell_2^\omega \geq_g \cdots \geq_g \ell_n^\omega$.
Moreover, ℓ_n is

- the shortest suffix s of w s.t. s^ω is minimum,
- the longest suffix of w which is a generalized Lyndon word.

Example

- The factorization in classical Lyndon word of $acaabaa$ is $(ac)(aab)(a)(a)$, since

$$(ac)^\omega \geq (aab)^\omega \geq a^\omega \geq a^\omega$$

- The factorization in $<_{alt}$–Lyndon word of $aabaabaabb$ is $(a)(aba)(aba)(abb)$, since

$$a^\omega \geq_{alt} (aba)^\omega \geq_{alt} (aba)^\omega \geq_{alt} (abb)^\omega$$

Classical Lyndon words

Proposition

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega < v^\omega$,

(2) $(uv)^\omega < v^\omega$,

(3) $u^\omega < (vu)^\omega$,

(4) $(uv)^\omega < (vu)^\omega$.

Classical Lyndon words

Proposition

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega < v^\omega$,

(2) $(uv)^\omega < v^\omega$,

(3) $u^\omega < (vu)^\omega$,

(4) $(uv)^\omega < (vu)^\omega$.

(5) $u^\omega < (uv)^\omega$.

(6) $(vu)^\omega < v^\omega$.

Classical Lyndon words

Proposition

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega < v^\omega$,

(2) $(uv)^\omega < v^\omega$,

(3) $u^\omega < (vu)^\omega$,

(4) $(uv)^\omega < (vu)^\omega$.

(5) $u^\omega < (uv)^\omega$.

(6) $(vu)^\omega < v^\omega$.

Theorem [Bergman (1969)]

If $u^\omega < v^\omega$ then $u^\omega < (uv)^\omega < (vu)^\omega < v^\omega$.

Classical Lyndon words

Proposition

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega < v^\omega$,

(2) $(uv)^\omega < v^\omega$,

(3) $u^\omega < (vu)^\omega$,

(4) $(uv)^\omega < (vu)^\omega$.

(5) $u^\omega < (uv)^\omega$.

(6) $(vu)^\omega < v^\omega$.

Theorem [Bergman (1969)]

If $u^\omega < v^\omega$ then $u^\omega < (uv)^\omega < (vu)^\omega < v^\omega$.

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

Factorization into classical Lyndon words

Theorem [Ufnarovskij (1995)]

Let $w = \ell_1 \ell_2 \cdots \ell_n$ the unique non-increasing factorization of w in Lyndon word. Then

- ${\ell_1}^\omega > (\ell_2 \cdots \ell_n)^\omega$
- ℓ_1 is the shortest nontrivial prefix p s.t. $w = ps$ and $p^\omega \geq s^\omega$,
- ℓ_1 is the shortest nontrivial prefix p s.t. $w = ps$ and $p^\omega \geq w^\omega$.

Factorization into classical Lyndon words

Theorem [Ufnarovskij (1995)]

Let $w = \ell_1 \ell_2 \cdots \ell_n$ the unique non-increasing factorization of w in Lyndon word. Then

- ${\ell_1}^\omega > (\ell_2 \cdots \ell_n)^\omega$
- ℓ_1 is the shortest nontrivial prefix p s.t. $w = ps$ and $p^\omega \geq s^\omega$,
- ℓ_1 is the shortest nontrivial prefix p s.t. $w = ps$ and $p^\omega \geq w^\omega$.

Example ($w = ac.aab.a.a$)

- $(ac)^\omega > ((aab)(a)(a))^\omega$,
- $(ac)^\omega > (acaabaa)^\omega$.

Galois words

The *alternating lexicographical order* $<_{alt}$ (w.r.t. an order $<$) is the generalized lexicographical order defined by the sequence $(<_n)_{n\geq 1}$ with

$$<_n = \begin{cases} < & \text{if } n \equiv 1 \pmod 2 \\ \tilde{<} & \text{otherwise.} \end{cases}$$

Example

$(ab)^\omega <_{alt} a^\omega <_{alt} b^\omega <_{alt} (ba)^\omega$.

Galois words

The *alternating lexicographical order* $<_{alt}$ (w.r.t. an order $<$) is the generalized lexicographical order defined by the sequence $(<_n)_{n\geq 1}$ with

$$<_n = \begin{cases} < & \text{if } n \equiv 1 \pmod 2 \\ \tilde{<} & \text{otherwise.} \end{cases}$$

Example

Let $a_i, b_i \in \{0, 1, \ldots, 9\}$.

$$a_1 a_2 a_3 \cdots <_{alt} b_1 b_2 b_3 \cdots \iff a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} < b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}$$

Galois words

The *alternating lexicographical order* $<_{alt}$ (w.r.t. an order $<$) is the generalized lexicographical order defined by the sequence $(<_n)_{n \geq 1}$ with

$$<_n = \begin{cases} < & \text{if } n \equiv 1 \pmod 2 \\ \tilde{<} & \text{otherwise.} \end{cases}$$

Example

Let $a_i, b_i \in \{0, 1, \ldots, 9\}$.

$$a_1 a_2 a_3 \cdots <_{alt} b_1 b_2 b_3 \cdots \iff a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} < b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}$$

A *Galois word* is a generalized Lyndon word for an alternating lexicographical order.

Characterization of Galois words

Proposition

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_{alt} v^\omega$,

(2) $(uv)^\omega <_{alt} v^\omega$,

(3) $u^\omega <_{alt} (vu)^\omega$,

(4) $(uv)^\omega <_{alt} (vu)^\omega$,

Characterization of Galois words

Proposition

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_{alt} v^\omega$,

(2) $(uv)^\omega <_{alt} v^\omega$,

(3) $u^\omega <_{alt} (vu)^\omega$,

(4) $(uv)^\omega <_{alt} (vu)^\omega$,

(5') $\begin{cases} u^\omega <_{alt} (uv)^\omega & \text{if } |u| \text{ is even} \\ u^\omega >_{alt} (uv)^\omega & \text{if } |u| \text{ is odd} \end{cases}$,

(6') $\begin{cases} (vu)^\omega <_{alt} v^\omega & \text{if } |v| \text{ is even} \\ (vu)^\omega >_{alt} v^\omega & \text{if } |v| \text{ is odd} \end{cases}$.

Characterization of Galois words

Proposition

The following conditions are equivalent for nonempty words $u, v \in A^*$.

(1) $u^\omega <_{alt} v^\omega$,

(2) $(uv)^\omega <_{alt} v^\omega$,

(3) $u^\omega <_{alt} (vu)^\omega$,

(4) $(uv)^\omega <_{alt} (vu)^\omega$,

(5') $\begin{cases} u^\omega <_{alt} (uv)^\omega & \text{if } |u| \text{ is even} \\ u^\omega >_{alt} (uv)^\omega & \text{if } |u| \text{ is odd} \end{cases}$,

(6') $\begin{cases} (vu)^\omega <_{alt} v^\omega & \text{if } |v| \text{ is even} \\ (vu)^\omega >_{alt} v^\omega & \text{if } |v| \text{ is odd} \end{cases}$.

Theorem [D., Restivo, Reutenauer (2018)]

w is a Galois word if and only if for any nontrivial factorization $w = ps$ one has

$$\begin{cases} p^\omega <_{alt} w^\omega & \text{if } |p| \text{ is even,} \\ p^\omega >_{alt} w^\omega & \text{if } |p| \text{ is odd.} \end{cases}$$

Factorization into Galois words

Theorem [D., Restivo, Reutenauer (2018)]

Let $w = g_1 g_2 \cdots g_n$ with g_i Galois words s.t. $g_1^\omega \geq_{alt} g_2^\omega \geq_{alt} \cdots \geq_{alt} g_n^\omega$.
Let m be the multiplicity of g_1.
Let p be the shortest nontrivial prefix of w s.t.

$$p^\omega \geq_{alt} w^\omega \text{ if } |p| \text{ is even} \quad \text{and} \quad p^\omega \leq_{alt} w^\omega \text{ if } |p| \text{ is odd.} \qquad (\star)$$

Then

(i) if $|g_1|$ is odd, m is even, and $m < n$, then $p = g_1^2$,

(ii) otherwise, $p = g_1$.

Example

Let $w = (abb)(abb)(abaa)$.
$((abb)^2)^\omega >_{alt} w^\omega$ and each proper prefix of $(abb)^2$ does not satisfy condition $(\star)$.

Complete trees

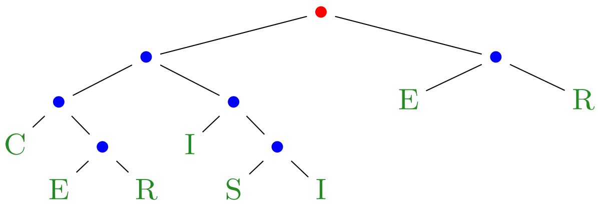
The set of *complete trees* over A is defined recursively as follows :

- each letter $a \in A$ is a tree ;
- if t_1, t_2 are trees, then (t_1, t_2) is a tree.

Complete trees

The set of *complete trees* over A is defined recursively as follows :

- each letter $a \in A$ is a tree ;
- if t_1, t_2 are trees, then (t_1, t_2) is a tree.

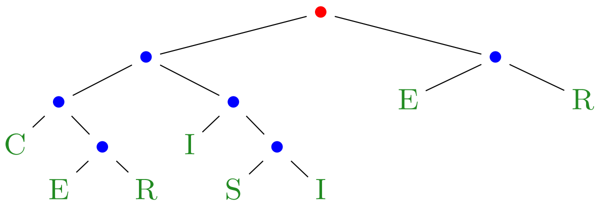


We will use the classical notions of *root*, *internal node* and *leaf* for a tree.

Complete trees

The set of *complete trees* over A is defined recursively as follows :

- each letter $a \in A$ is a tree ;
- if $\mathfrak{t}_1, \mathfrak{t}_2$ are trees, then $(\mathfrak{t}_1, \mathfrak{t}_2)$ is a tree.



We will use the classical notions of *root*, *internal node* and *leaf* for a tree.
The *foliage* $\varphi(\mathfrak{t})$ of a tree $\mathfrak{t}$ is defined as :

- $\varphi(a) = a$ for any $a \in A$,
- $\varphi((\mathfrak{t}_1, \mathfrak{t}_2)) = \varphi(\mathfrak{t}_1)\varphi(\mathfrak{t}_2)$ for any two trees $\mathfrak{t}_1, \mathfrak{t}_2$.

Left standard factorization

Let w be a Lyndon word of length at least 2.
The *left standard factorization* of w is the factorization $w = uv$, where u is the longest nonempty proper prefix of w which is a Lyndon word.

Example

The left standard factorization of $aabaacab$ is $(aabaac)(ab)$.

Left standard factorization

Let w be a Lyndon word of length at least 2.
The *left standard factorization* of w is the factorization $w = uv$, where u is the longest nonempty proper prefix of w which is a Lyndon word.

Proposition

Both u and v are Lyndon words.
Moreover, either v is a letter or $v = v_1 v_2$, and $v_1 \leq u$.

Example

The left standard factorization of $aabaacab$ is $(aabaac)(ab)$.
The left standard factorization of ab is $(a)(b)$, and $a \leq aabaac$.

Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.

Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

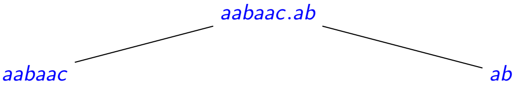
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.

$aabaacab$

Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

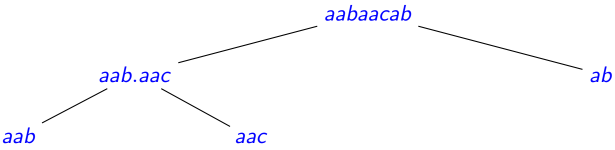
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

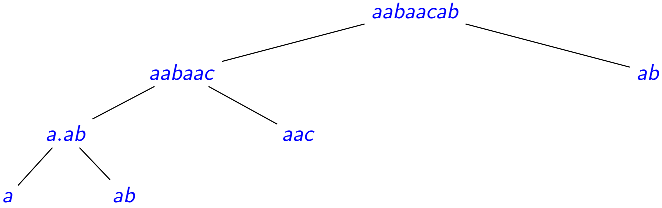
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

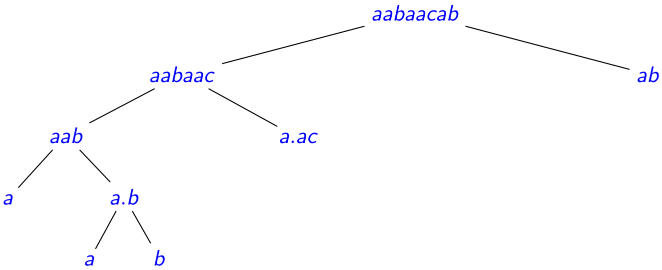
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

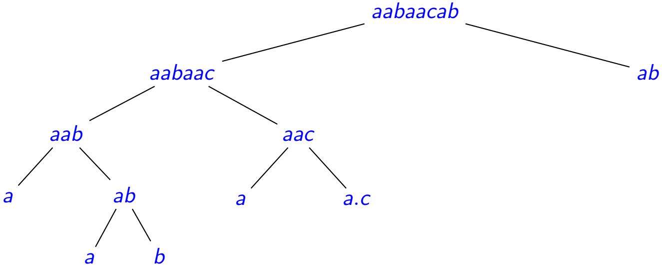
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

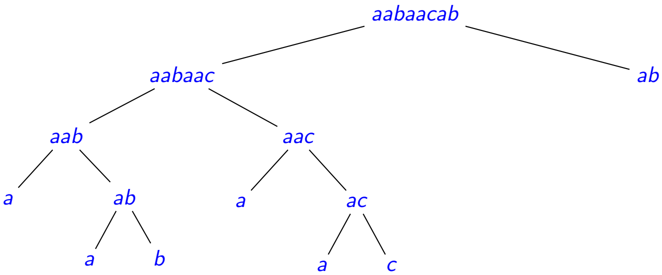
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

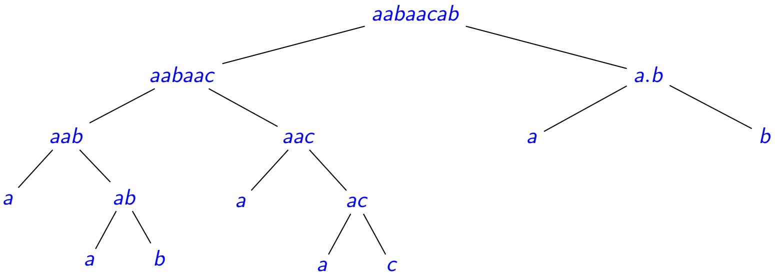
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

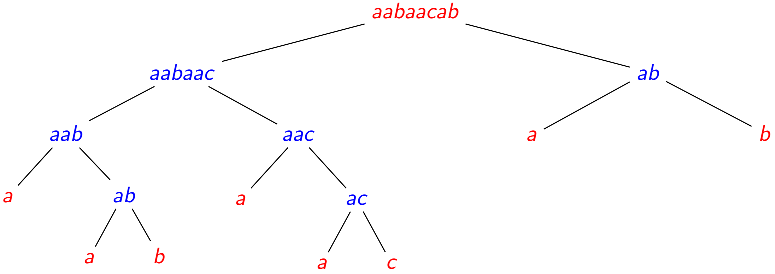
- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

- $\mathcal{L}(a) = a$ for each letter $a \in A$;
- $\mathcal{L}(w) = (\mathcal{L}(u), \mathcal{L}(v))$ for each Lyndon word w of length at least 2 with left standard factorization $w = uv$.



Clearly $\varphi(\mathcal{L}(w)) = w$.

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w = \quad aabaacab$

a
aa
aab
$aaba$
$aabaa$
$aabaac$
$aabaaca$
$aabaacab$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w = \quad aabaacab$

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w = aabaacab$

1

aa

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w = aabaacab$

21

$aa \prec a$

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w = aabaacab$

21 3

$aa \prec a \prec aabaa$

$a.aaaaaaaa\cdots$
$aa.aaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w =$ aabaacab

21 43

$aa \prec a \prec aabaa \prec aaba$

a.aaaaaaaa $\cdots$
aa.aaaaaaaa $\cdots$
aab.aabaaba $\cdots$
aaba.aabaaa $\cdots$
aabaa.aabaa $\cdots$
aabaac.aaba $\cdots$
aabaaca.aab $\cdots$
aabaacab.aa $\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w = aabaacab$

21543

$aa \prec a \prec aabaa \prec aaba \prec aab$

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w =$ $aabaacab$
21543 6

$aa \prec a \prec aabaa \prec aaba \prec aab \prec aabaaca$

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$w = aabaacab$

2154376

$aa \prec a \prec aabaa \prec aaba \prec aab \prec aabaaca \prec aabac$

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$$w = \begin{matrix} aabaacab \\ 21543768 \end{matrix}$$

$aa \prec a \prec aabaa \prec aaba \prec aab \prec aabaaca \prec aabac \prec w$

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

The *prefix standard permutation* associated to a word w is obtained by ordering the nonempty prefixes of w according to $\prec$.

Example

$$w = \begin{array}{l} aabaacab \\ 21543768 \end{array}$$

$aa \prec a \prec aabaa \prec aaba \prec aab \prec aabaaca \prec aabac \prec w$

$a.aaaaaaaaa\cdots$
$aa.aaaaaaaa\cdots$
$aab.aabaaba\cdots$
$aaba.aabaaa\cdots$
$aabaa.aabaa\cdots$
$aabaac.aaba\cdots$
$aabaaca.aab\cdots$
$aabaacab.aa\cdots$

Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$$aabaacab \longleftrightarrow 2154376\not{8}$$

Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$$aabaacab \quad \longleftrightarrow \quad 2154376$$

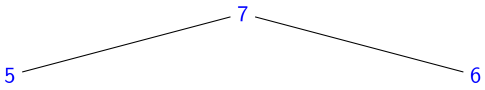
7

Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$aabaacab \quad \longleftrightarrow \quad 2154376$

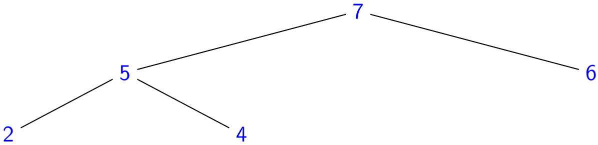


Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$aabaacab \quad \longleftrightarrow \quad 2154376$

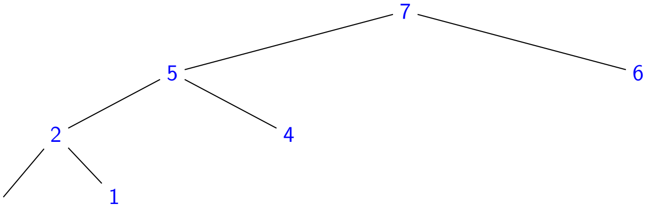


Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$aabaacab \quad \longleftrightarrow \quad 2154376$

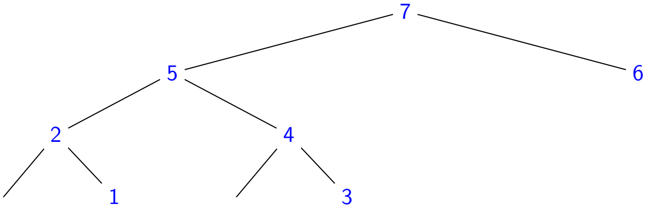


Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$$aabaacab \quad \longleftrightarrow \quad 2154376$$

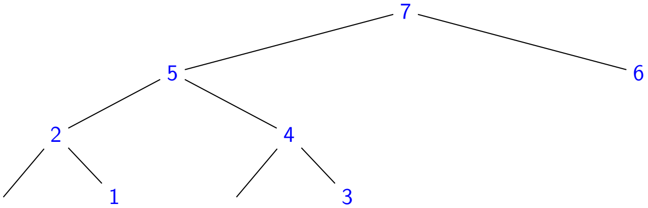


Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$aabaacab \quad \longleftrightarrow \quad 2154376$

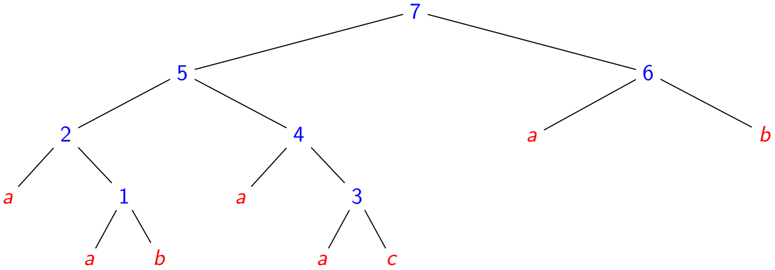


Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

$aabaacab \quad \longleftrightarrow \quad 2154376$



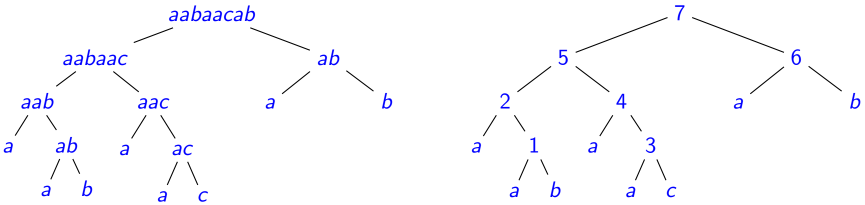
We complete in such a way that $\varphi(\mathcal{C}(w)) = w$.

Equivalence of trees

Theorem [D., Restivo, Reutenauer (2019)]

Let w be a Lyndon word. Then $\mathcal{L}(w) = \mathcal{C}(w)$.

$$aabaacab \quad \longleftrightarrow \quad 2154376$$



Open problems

The non-increasing factorization in classical Lyndon words is the factorization in Lyndon words with minimal number of factors. This is not true for generalized Lyndon words.

Example

The nonincreasing factorization in Galois words of $w = ababab$ is $(ab)(ab)(ab)$.
The word admits also the factorization $w = (ababa)(b)$.

Open problems

The non-increasing factorization in classical Lyndon words is the factorization in Lyndon words with minimal number of factors. This is not true for generalized Lyndon words.

Example

The nonincreasing factorization in Galois words of $w = ababab$ is $(ab)(ab)(ab)$.
The word admits also the factorization $w = (ababa)(b)$.

Theorem [Duval (1983)]

It is possible to compute the non-increasing factorization of a word into classical Lyndon words in linear time.

Open problems

The non-increasing factorization in classical Lyndon words is the factorization in Lyndon words with minimal number of factors. This is not true for generalized Lyndon words.

Example

The nonincreasing factorization in Galois words of $w = ababab$ is $(ab)(ab)(ab)$.
The word admits also the factorization $w = (ababa)(b)$.

Theorem [Duval (1983)]

It is possible to compute the non-increasing factorization of a word into classical Lyndon words in linear time.

Open Problem 1

Generalize Duval's algorithm to generalized Lyndon words.

Open problems

Theorem [Duval (1983)]

The number of classical Lyndon words of length at most n is equal to the number of words of length n that are prefixes of a Lyndon word, $+1$.

Open problems

Theorem [Duval (1983)]

The number of classical Lyndon words of length at most n is equal to the number of words of length n that are prefixes of a Lyndon word, $+1$.

This property is no more true for a generalized order.

Open problems

Theorem [Duval (1983)]

The number of classical Lyndon words of length at most n is equal to the number of words of length n that are prefixes of a Lyndon word, $+1$.

This property is no more true for a generalized order.

Example

There are exactly 6 classical Lyndon words of length at most 2 (namely a, b, c, ab, aca, bc) and 5 words of length 2 prefixes of a Lyndon word (namely aa, ab, ac, bb, bc.)

Open problems

Theorem [Duval (1983)]

The number of classical Lyndon words of length at most n is equal to the number of words of length n that are prefixes of a Lyndon word, $+1$.

This property is no more true for a generalized order.

Example

There are exactly 6 classical Lyndon words of length at most 2 (namely a, b, c, ab, aca, bc) and 5 words of length 2 prefixes of a Lyndon word (namely aa, ab, ac, bb, bc.)

There are exactly 6 Galois words of length at most 2 (namely a, b, c, ab, aca, bc) but only 3 words of length 2 prefixes of a Galois word (namely ab, ac, bc.)

Open problems

Theorem [Duval (1983)]

The number of classical Lyndon words of length at most n is equal to the number of words of length n that are prefixes of a Lyndon word, $+1$.

This property is no more true for a generalized order.

Example

There are exactly 6 classical Lyndon words of length at most 2 (namely a, b, c, ab, aca, bc) and 5 words of length 2 prefixes of a Lyndon word (namely aa, ab, ac, bb, bc.)

There are exactly 6 Galois words of length at most 2 (namely a, b, c, ab, aca, bc) but only 3 words of length 2 prefixes of a Galois word (namely ab, ac, bc.)

Open Problem 2

Find a formula to count generalized Lyndon words of a given length.

Open problems

An *infinite Lyndon word* is an infinite word which has infinitely many prefixes that are (finite) Lyndon words.

Open problems

An *infinite Lyndon word* is an infinite word which has infinitely many prefixes that are (finite) Lyndon words.

Theorem [Siromoney, Mathew, Dare, Subramanian (1994)]

$\mathbf{x}$ is an infinite Lyndon word if and only if $\mathbf{x}$ is smaller than any of its nontrivial suffixes.

Open problems

An *infinite Lyndon word* is an infinite word which has infinitely many prefixes that are (finite) Lyndon words.

Theorem [Siromoney, Mathew, Dare, Subramanian (1994)]

$\mathbf{x}$ is an infinite Lyndon word if and only if $\mathbf{x}$ is smaller than any of its nontrivial suffixes.

Moreover, each infinite word $\mathbf{w}$ could be factorized as either :

- $\mathbf{w} = \ell_1 \ell_2 \cdots$, with ℓ_i finite Lyndon words, and $\ell_1^\omega \geq \ell_2^\omega \geq \cdots$
- $\mathbf{w} = \ell_1 \cdots \ell_n \mathbf{s}$, with ℓ_i finite Lyndon words, $\mathbf{s}$ infinite Lyndon words, and $\ell_1^\omega \geq \cdots \geq \ell_n^\omega \geq \mathbf{s}$.

Open problems

An *infinite Lyndon word* is an infinite word which has infinitely many prefixes that are (finite) Lyndon words.

Theorem [Siromoney, Mathew, Dare, Subramanian (1994)]

$\mathbf{x}$ is an infinite Lyndon word if and only if $\mathbf{x}$ is smaller than any of its nontrivial suffixes.

Moreover, each infinite word $\mathbf{w}$ could be factorized as either :

- $\mathbf{w} = \ell_1 \ell_2 \cdots$, with ℓ_i finite Lyndon words, and $\ell_1^\omega \geq \ell_2^\omega \geq \cdots$
- $\mathbf{w} = \ell_1 \cdots \ell_n \mathbf{s}$, with ℓ_i finite Lyndon words, $\mathbf{s}$ infinite Lyndon words, and $\ell_1^\omega \geq \cdots \geq \ell_n^\omega \geq \mathbf{s}$.

A *generalized infinite Lyndon word* is an infinite word $\mathbf{x}$ which is smaller that any of its nontrivial suffixes.

Open problems

An *infinite Lyndon word* is an infinite word which has infinitely many prefixes that are (finite) Lyndon words.

Theorem [Siromoney, Mathew, Dare, Subramanian (1994)]

$\mathbf{x}$ is an infinite Lyndon word if and only if $\mathbf{x}$ is smaller than any of its nontrivial suffixes.

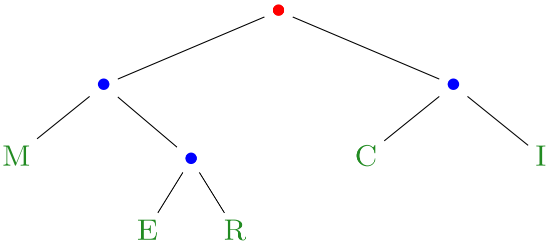
Moreover, each infinite word $\mathbf{w}$ could be factorized as either :

- $\mathbf{w} = \ell_1 \ell_2 \cdots$, with ℓ_i finite Lyndon words, and $\ell_1^\omega \geq \ell_2^\omega \geq \cdots$
- $\mathbf{w} = \ell_1 \cdots \ell_n \mathbf{s}$, with ℓ_i finite Lyndon words, $\mathbf{s}$ infinite Lyndon words, and $\ell_1^\omega \geq \cdots \geq \ell_n^\omega \geq \mathbf{s}$.

A *generalized infinite Lyndon word* is an infinite word $\mathbf{x}$ which is smaller that any of its nontrivial suffixes.

Open Problem 3

Prove that each infinite word can be factorized in a unique way as a nonincresing product of finite and infinite generalized Lyndon words.


M
E
R
C
I