Rigidity of Substitutive Tree Words

Francesco Dolce

LaCIM UQÀM

work in progress with
Valérie Berthé, Fabien Durand,
Julien Leroy, Dominique Perrin

9 février 2018

Séminaire de Combinatoire et d'Informatique Mathématique du LaCIM

Fibonacci

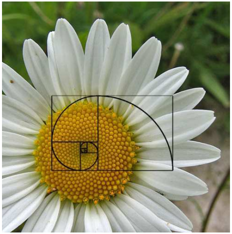
$$\mathbf{x} = 0100101001001010\cdots$$

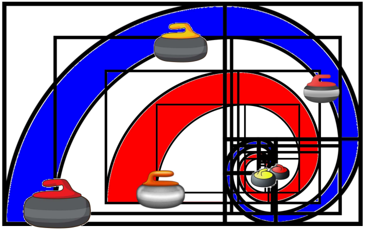
$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(0) \qquad \text{where} \qquad \varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$$

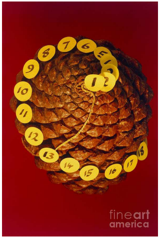
Fibonacci

$$\mathbf{x} = 0100101001001010\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(0) \qquad \text{where} \qquad \varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$$







Fibonacci

$$\mathbf{x} = 0100101001001010\cdots$$

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(0) \qquad \text{where} \qquad \varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$$

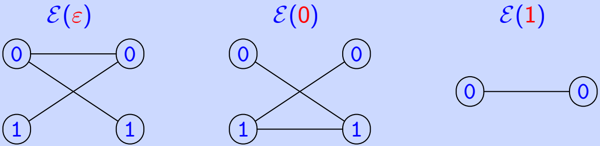
Can we describe all morphisms σ such that $\sigma(\mathbf{x}) = \mathbf{x}$

Tree sets

The *extension graph* of a word $w \in F$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$
\begin{aligned}
L(w) &= \{a \in A \mid aw \in F\}, \\
R(w) &= \{a \in A \mid wa \in F\}, \\
B(w) &= \{(a,b) \in A \mid awb \in F\}.
\end{aligned}
$$

Example (Fibonacci, $F = \{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, \ldots\}$)



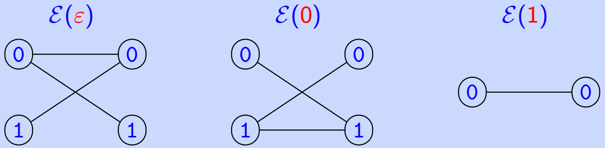
Tree sets

The *extension graph* of a word $w \in F$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

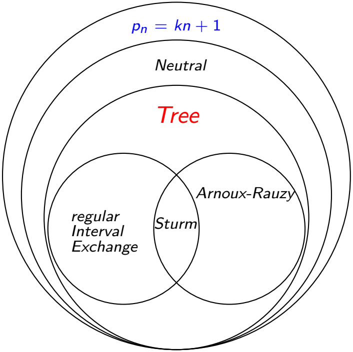
$$\begin{array}{rcl} L(w) & = & \{a \in A \mid aw \in F\}, \\ R(w) & = & \{a \in A \mid wa \in F\}, \\ B(w) & = & \{(a,b) \in A \mid awb \in F\}. \end{array}$$

A factorial set F is called a *tree set* if the graph $\mathcal{E}(w)$ is a tree for any $w \in F$.

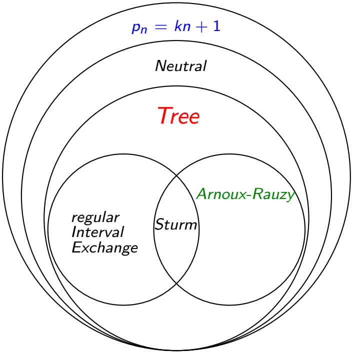
Example (Fibonacci, $F = \{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, \ldots\}$)



Tree sets


$p_n = kn + 1$
Neutral
Tree
regular Interval Exchange
Sturm
Arnoux-Rauzy

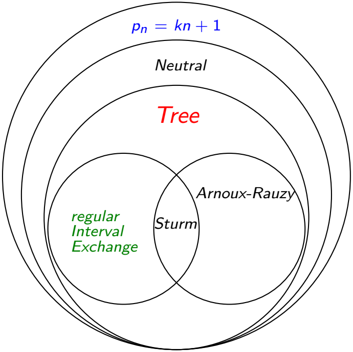
Tree sets

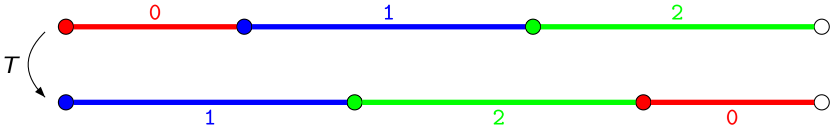


Definition

An *Arnoux-Rauzy* (or *strict episturmian*) set is a factorial set closed by reversal with $p_n = (\text{Card}\,(A) - 1)n + 1$ having a unique right special factor for each length.

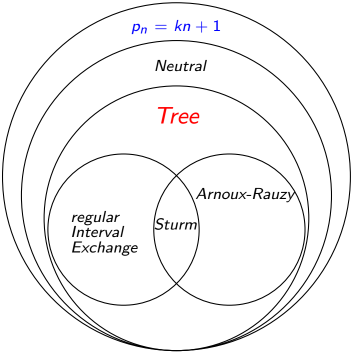
Tree sets





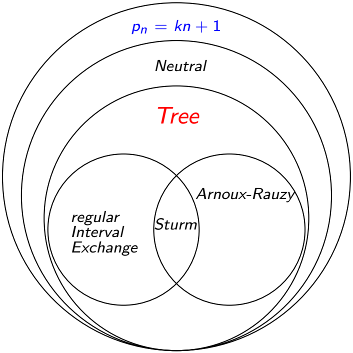
$$F = \{0, 1, 2, 02, 10, 11, 21, 22, 021, 022, 102, \ldots\}$$

Tree sets



A *tree word* is an infinite word $\mathbf{x} \in A^\omega$ such that its language $\mathcal{L}(\mathbf{x}) \subset A^*$ is a tree set.

Tree sets



A *tree word* is an infinite word $\mathbf{x} \in A^\omega$ such that its language $\mathcal{L}(\mathbf{x}) \subset A^*$ is a tree set.

- Sturmian words,
- Strict episturmian (Arnoux-Rauzy) words,
- Natural coding of regular Interval Exchanges,
- other quirky examples, . . .

Recurrence and uniformly recurrence

Definition

An infinite word $\mathbf{x}$ is *recurrent* if for every $u \in \mathcal{L}(\mathbf{x})$ there is a v such that uvu is in $\mathcal{L}(\mathbf{x})$.

It is *uniformly recurrent* if for every $u \in \mathcal{L}(\mathbf{x})$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in $\mathcal{L}(\mathbf{x})$.

Proposition

Uniform recurrence $\Longrightarrow$ recurrence.

Recurrence and uniformly recurrence

Definition

An infinite word $\mathbf{x}$ is *recurrent* if for every $u \in \mathcal{L}(\mathbf{x})$ there is a v such that uvu is in $\mathcal{L}(\mathbf{x})$.

It is *uniformly recurrent* if for every $u \in \mathcal{L}(\mathbf{x})$ there exists an $n \in \mathbb{N}$ such that u is a factor of every word of length n in $\mathcal{L}(\mathbf{x})$.

Proposition

Uniform recurrence $\implies$ recurrence.

Theorem [D., Perrin (2016)]

A recurrent tree word is uniformly recurrent.

Morphisms and substitutions

A (non-erasing) *morphism* $\sigma : A^* \to B^*$ is a map s.t. $\sigma(uv) = \sigma(u)\sigma(v)$ for all $u, v \in A^*$ (and $\sigma(u) \in B^+$ for all $u \in A^+$).

A *substitution* is a morphism from A^* to A^*.

Morphisms and substitutions

A (non-erasing) *morphism* $\sigma : A^* \to B^*$ is a map s.t. $\sigma(uv) = \sigma(u)\sigma(v)$ for all $u, v \in A^*$ (and $\sigma(u) \in B^+$ for all $u \in A^+$).

A *substitution* is a morphism from A^* to A^*.

A substitution is *primitive* if there exists a $k \in \mathbb{N}$ s.t. $b \in \mathcal{L}\left(\sigma^k(a)\right)$ for all $a, b \in A$.

An infinite word of the form $\mathbf{x} = \sigma^\omega(a) = \lim_{n\to\infty} \sigma^n(a)$, with $a \in A$, is a *fixed point* of σ, that is $\sigma(\mathbf{x}) = \mathbf{x}$.

Morphisms and substitutions

A (non-erasing) *morphism* $\sigma : A^* \to B^*$ is a map s.t. $\sigma(uv) = \sigma(u)\sigma(v)$ for all $u, v \in A^*$ (and $\sigma(u) \in B^+$ for all $u \in A^+$).

A *substitution* is a morphism from A^* to A^*.

A substitution is *primitive* if there exists a $k \in \mathbb{N}$ s.t. $b \in \mathcal{L}\left(\sigma^k(a)\right)$ for all $a, b \in A$.

An infinite word of the form $\mathbf{x} = \sigma^\omega(a) = \lim_{n \to \infty} \sigma^n(a)$, with $a \in A$, is a *fixed point* of σ, that is $\sigma(\mathbf{x}) = \mathbf{x}$.

Proposition

If σ is a primitive substitution, there exists a $k \in \mathbb{N}$ such that σ^k admits a fixed point.

Moreover, all fixed points of σ (or some power of it) have the same language, called the *language of σ*, and this is uniformly recurrent.

Substitutive words

An infinite word $\mathbf{y} \in B^\omega$ is *substitutive* if there exist a substitution σ over B and a morphism $\tau : A^* \to B^*$ such that

$$\mathbf{y} = \tau(\sigma^\omega(b))$$

with $b \in B$. It is said *substitutive primitive* whenever σ is primitive.

Substitutive words

An infinite word $\mathbf{y} \in B^\omega$ is *substitutive* if there exist a substitution σ over B and a morphism $\tau : A^* \to B^*$ such that

$$\mathbf{y} = \tau(\sigma^\omega(b))$$

with $b \in B$. It is said *substitutive primitive* whenever σ is primitive.

Examples

- The map $\varphi : \{0,1\}^* \to \{0,1\}^*$ defined by $\varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$ is a substitution.

Substitutive words

An infinite word $\mathbf{y} \in B^\omega$ is *substitutive* if there exist a substitution σ over B and a morphism $\tau : A^* \to B^*$ such that

$$\mathbf{y} = \tau(\sigma^\omega(b))$$

with $b \in B$. It is said *substitutive primitive* whenever σ is primitive.

Examples

- The map $\varphi : \{0,1\}^* \to \{0,1\}^*$ defined by $\varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$ is a substitution.
- The map $\tau : \{0,1\}^* \to \{\alpha,\beta,\gamma\}^*$ defined by $\tau : \begin{cases} 0 \mapsto \alpha\beta\alpha \\ 1 \mapsto \gamma \end{cases}$ is a morphism.

Substitutive words

An infinite word $\mathbf{y} \in B^\omega$ is *substitutive* if there exist a substitution σ over B and a morphism $\tau : A^* \to B^*$ such that

$$\mathbf{y} = \tau(\sigma^\omega(b))$$

with $b \in B$. It is said *substitutive primitive* whenever σ is primitive.

Examples

- The map $\varphi : \{0,1\}^* \to \{0,1\}^*$ defined by $\varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$ is a substitution.
- The map $\tau : \{0,1\}^* \to \{\alpha,\beta,\gamma\}^*$ defined by $\tau : \begin{cases} 0 \mapsto \alpha\beta\alpha \\ 1 \mapsto \gamma \end{cases}$ is a morphism.
- The word $\mathbf{x} = \tau(\varphi^\omega(0)) = \tau(0100101001001010\cdots) = \alpha\beta\alpha\,\gamma\,\alpha\beta\alpha\,\alpha\beta\alpha\,\gamma\,\alpha\beta\alpha\,\gamma\,\alpha\beta\alpha\,\cdots$ is substitutive primitive.

Invertible substitutions

Given an alphabet A, the *free group* $\mathbb{F}_A$ is the set of all words over $A \cup A^{-1}$ which are *reduced* (i.e., $aa^{-1} \equiv a^{-1}a \equiv \varepsilon$ for every $a \in A$).

A substitution $\sigma : A^* \to A^*$ can be extended to a morphism of the free group by defining $\sigma(a^{-1}) = \sigma(a)^{-1}$.

Example

$$\begin{array}{rrcl} \varphi : & \mathbb{F}_{\{0,1\}} & \to & \mathbb{F}_{\{0,1\}} \\ & 0 & \mapsto & 01 \\ & 1 & \mapsto & 0 \\ & 0^{-1} & \mapsto & 1^{-1}0^{-1} \\ & 1^{-1} & \mapsto & 0^{-1} \end{array}$$

Invertible substitutions

Given an alphabet A, the *free group* $\mathbb{F}_A$ is the set of all words over $A \cup A^{-1}$ which are *reduced* (i.e., $aa^{-1} \equiv a^{-1}a \equiv \varepsilon$ for every $a \in A$).

A substitution $\sigma : A^* \to A^*$ can be extended to a morphism of the free group by defining $\sigma(a^{-1}) = \sigma(a)^{-1}$.

A morphism $\sigma : A^* \to A^*$ is *invertible* if its extension $\sigma : \mathbb{F}_A \to \mathbb{F}_A$ is a (positive) automorphism, i.e., if there exists σ^{-1} such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = Id$.

Example

$$\begin{array}{rrcl} \varphi : & \mathbb{F}_{\{0,1\}} & \to & \mathbb{F}_{\{0,1\}} \\ & 0 & \mapsto & 01 \\ & 1 & \mapsto & 0 \\ & 0^{-1} & \mapsto & 1^{-1}0^{-1} \\ & 1^{-1} & \mapsto & 0^{-1} \end{array}$$

$$\varphi^{-1} : \mathbb{F}_{\{0,1\}} \to \mathbb{F}_{\{0,1\}}$$

Invertible substitutions

Given an alphabet A, the *free group* $\mathbb{F}_A$ is the set of all words over $A \cup A^{-1}$ which are *reduced* (i.e., $aa^{-1} \equiv a^{-1}a \equiv \varepsilon$ for every $a \in A$).

A substitution $\sigma : A^* \to A^*$ can be extended to a morphism of the free group by defining $\sigma(a^{-1}) = \sigma(a)^{-1}$.

Example

$$\begin{array}{rrcl} \varphi : & \mathbb{F}_{\{0,1\}} & \to & \mathbb{F}_{\{0,1\}} \\ & 0 & \mapsto & 01 \\ & 1 & \mapsto & 0 \\ & 0^{-1} & \mapsto & 1^{-1}0^{-1} \\ & 1^{-1} & \mapsto & 0^{-1} \end{array}$$

Tame substitutions

An automorphism σ is *positive* if $\sigma(a) \in A^+$ for every $a \in A$.

An automorphism is *elementary positive* if it is a permutation of A or of the form $\alpha_{a,b}$ or $\widetilde{\alpha}_{a,b}$, with $a, b \in A$ and $a \neq b$, where

$$\alpha_{a,b} : \begin{cases} a \mapsto ab & \\ c \mapsto c & \text{if } c \neq a \end{cases} \quad \text{and} \quad \widetilde{\alpha}_{a,b} : \begin{cases} a \mapsto ba & \\ c \mapsto c & \text{if } c \neq a \end{cases}$$

The set of elementary automorphisms is denoted $\mathcal{S}_e$.
A positive automorphism (resp. substitution) $\sigma \in \mathcal{S}_e^*$ is said to be *tame*.

Example

The set of elementary automorphisms over $A = \{0, 1\}$ is

$$\mathcal{S}_e = \left\{ Id, \pi_{(01)}, \ \alpha_{0,1}, \alpha_{1,0}, \ \widetilde{\alpha}_{0,1}, \widetilde{\alpha}_{1,0} \right\}.$$

The substitution $\varphi = \pi_{(01)}\widetilde{\alpha}_{0,1} : \begin{cases} 0 \mapsto 10 \mapsto 01 \\ 1 \mapsto 1 \ \ \mapsto 0 \end{cases}$ is tame.

Tame and invertible substitutions

tame substitutions	$\subset$	invertible substitutions

- Every permutations $\pi \in \mathfrak{S}_A$ is invertible.
- The inverses of

$$\alpha_{a,b} : \begin{cases} a \mapsto ab \\ c \mapsto c \quad\quad \text{if } c \neq a \end{cases} \quad \text{and} \quad \widetilde{\alpha}_{a,b} : \begin{cases} a \mapsto ba \\ c \mapsto c \quad\quad \text{if } c \neq a \end{cases}$$

are respectively

$$\alpha_{a,b}^{-1} : \begin{cases} a \mapsto ab^{-1} \\ c \mapsto c \quad\quad \text{if } c \neq a \end{cases} \quad \text{and} \quad \widetilde{\alpha}_{a,b}^{-1} : \begin{cases} a \mapsto b^{-1}a \\ c \mapsto c \quad\quad \text{if } c \neq a \end{cases} .$$

Episturmian and epistandard morphisms

epistandard substitutions $\subset$ episturmian substitutions $\subset$ tame substitutions $\subset$ invertible substitutions

The monoid of *episturmian* (or *Arnoux-Rauzy*) *substitutions* is generated by permutations of A and morphisms of the form ψ_a and $\widetilde{\psi}_a$, with $a \in A$, where

$$\psi_a : \begin{cases} a \mapsto a & \\ b \mapsto ab & \text{if } b \neq a \end{cases} \qquad \text{and} \qquad \widetilde{\psi}_a : \begin{cases} a \mapsto a & \\ b \mapsto ba & \text{if } b \neq a \end{cases}$$

The monoid of *epistandard substitutions* is generated by permutations of A and morphisms of the form ψ_a, with $a \in A$ (i.e., no $\widetilde{\psi}_b$).

Episturmian and epistandard morphisms

epistandard substitutions	$\subset$	episturmian substitutions	$\subset$	tame substitutions	$\subset$	invertible substitutions

The monoid of *episturmian* (or *Arnoux-Rauzy*) *substitutions* is generated by permutations of A and morphisms of the form ψ_a and $\widetilde{\psi}_a$, with $a \in A$, where

$$\psi_a : \begin{cases} a \mapsto a & \\ b \mapsto ab & \text{if } b \neq a \end{cases} \quad \text{and} \quad \widetilde{\psi}_a : \begin{cases} a \mapsto a & \\ b \mapsto ba & \text{if } b \neq a \end{cases}$$

The monoid of *epistandard substitutions* is generated by permutations of A and morphisms of the form ψ_a, with $a \in A$ (i.e., no $\widetilde{\psi}_b$).

Example (Fibonacci and Tribonacci)

- The substitution $\varphi = \psi_0\, \pi_{(01)} : \begin{cases} 0 \mapsto 1 \mapsto 01 \\ 1 \mapsto 0 \mapsto 0 \end{cases}$ is epistandard.
- The substitution $\eta = \psi_0\, \pi_{(012)} : \begin{cases} 0 \mapsto 1 \mapsto 01 \\ 1 \mapsto 2 \mapsto 02 \\ 2 \mapsto 0 \mapsto 0 \end{cases}$ is epistandard.

Episturmian and epistandard morphisms

epistandard substitutions	=	episturmian substitutions	=	tame substitutions	=	invertible substitutions

$$A = \{0, 1\}$$

Theorem [Mignosi, Séébold (1993) ; Wen, Wen (1994)]

In the binary case (*Sturmian substitutions*) the four monoids coincide.

Proof. (of the first two inequalities)

- For every $a \in \{0, 1\}$, one has $\widetilde{\psi}_a = \pi_{(0,1)}\, \psi_a\, \pi_{(0,1)}$.
- $\alpha_{0,1} = \pi_{(0,1)}\, \psi_0$, $\alpha_{1,0} = \pi_{(0,1)}\, \psi_1\, \pi_{(0,1)}$, $\widetilde{\alpha}_{0,1} = \psi_1$, $\widetilde{\alpha}_{1,0} = \psi_0$.

Epistūrmian and epistandard morphisms

epistandard substitutions	=	episturmian substitutions	=	tame substitutions	=	invertible substitutions

$A = \{0, 1\}$

Theorem [Mignosi, Séébold (1993) ; Wen, Wen (1994)]

In the binary case (*Sturmian substitutions*) the four monoids coincide.

Proof. (of the first two inequalities)

- For every $a \in \{0, 1\}$, one has $\widetilde{\psi}_a = \pi_{(0,1)}\, \psi_a\, \pi_{(0,1)}$.
- $\alpha_{0,1} = \pi_{(0,1)}\, \psi_0$, $\alpha_{1,0} = \pi_{(0,1)}\, \psi_1\, \pi_{(0,1)}$, $\widetilde{\alpha}_{0,1} = \psi_1$, $\widetilde{\alpha}_{1,0} = \psi_0$.

Corollary

The monoid of positive automorphisms over a binary alphabet is finitely generated.

Episturmian and epistandard morphisms

epistandard substitutions	$\subsetneq$	episturmian substitutions	$\subsetneq$	tame substitutions	$\subsetneq$	invertible substitutions

$\mathrm{Card}\,(A) \geq 3$

Theorem [Wen, Zhang (1999) ; Richomme (2003)]

The monoid of invertible substitutions over a ternary alphabet is not finitely generated.

Fixed point of substitutions

Theorem

Every Sturmian substitution generates a Sturmian word.

Example

The substitution φ generates the *Fibonacci word*

$$\varphi^{\omega}(0) = 0100101001001010\cdots$$

which is Sturmian.

Fixed point of substitutions

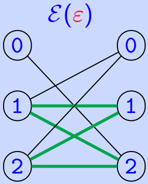
Theorem

Every Sturmian substitution generates a Sturmian word.

BUT not every tame substitution admits as a fixed point a tree word.

Example

$\xi = \alpha_{0,2}\,\alpha_{2,1}\,\alpha_{1,0} : \begin{cases} 0 \mapsto 02 \\ 1 \mapsto 102 \\ 2 \mapsto 21 \end{cases}$ is tame but its fixed point $\xi^\omega(0)$ is not a tree word.



QUESTION : *Can we characterize among substitutive tree words the fixed points?*

Stabilizer

The *stabilizer* of an infinite word $\mathbf{x} \in A^\omega$ is the submonoid of substitutions

$$\mathrm{Stab}(\mathbf{x}) = \{\sigma : A^* \to A^* \mid \sigma(\mathbf{x}) = \mathbf{x}\}$$

A word $\mathbf{x}$ such that $\mathrm{Stab}(\mathbf{x})$ is cyclic is said to be *rigid*.

Stabilizer

The *stabilizer* of an infinite word $\mathbf{x} \in A^\omega$ is the submonoid of substitutions

$$\mathrm{Stab}(\mathbf{x}) = \{\sigma : A^* \to A^* \mid \sigma(\mathbf{x}) = \mathbf{x}\}$$

A word $\mathbf{x}$ such that $\mathrm{Stab}(\mathbf{x})$ is cyclic is said to be *rigid*.

Theorem [Séébold (1998)]

Words generated by Sturmian substitutions are rigid.

Example (Fibonacci)

The stabilizer of the Fibonacci word $\mathbf{x}$ is $\mathrm{Stab}(\mathbf{x}) = \{\varphi^i \mid i \in \mathbb{N}\}$.

Stabilizer

The *stabilizer* of an infinite word $\mathbf{x} \in A^\omega$ is the submonoid of substitutions

$$\mathrm{Stab}(\mathbf{x}) = \{\sigma : A^* \to A^* \mid \sigma(\mathbf{x}) = \mathbf{x}\}$$

A word $\mathbf{x}$ such that $\mathrm{Stab}(\mathbf{x})$ is cyclic is said to be *rigid*.

Theorem [Séébold (1998)]

Words generated by Sturmian substitutions are rigid.

Theorem [Krieger (2008)]

Fixed points of strict epistandard morphisms are rigid.

Example (Tribonacci)

The stabilizer of the Tribonacci word $\mathbf{y} = \eta^\omega(0)$ is $\mathrm{Stab}(\mathbf{y}) = \{\eta^i \mid i \in \mathbb{N}\}$.

Stabilizers of tree words

QUESTION : *Are tree words rigid?*

Stabilizers of tree words

QUESTION : *Are tree words rigid?* ANSWER : *Dunno!*

Stabilizers of tree words

QUESTION : *Are tree words rigid?* ANSWER : *Dunno*! But...

Theorem [Berthé, D., Durand, Leroy, Perrin (2018)]

Let $\mathbf{x}$ be a tree word and $\sigma, \tau \in \mathrm{Stab}(\mathbf{x})$ primitive substitutions.
Then, there exist $i, j \geq 1$ such that $\sigma^i = \tau^j$.

Stabilizers of tree words

QUESTION : *Are tree words rigid?* ANSWER : *Dunno*! But...

Theorem [Berthé, D., Durand, Leroy, Perrin (2018)]

Let $\mathbf{x}$ be a tree word and $\sigma, \tau \in \mathsf{Stab}(\mathbf{x})$ primitive substitutions.
Then, there exist $i, j \geq 1$ such that $\sigma^i = \tau^j$.

Let $\mathbf{x}$ be a recurrent tree word.
There exists a *primitive tame* substitution θ such that for any primitive $\sigma \in \mathsf{Stab}(\mathbf{x})$, one can find a *positive tame automorphism* τ and integers $i, j \geq 1$ such that $\sigma^i = \tau\theta^j\tau^{-1}$.

Stabilizers of tree words

QUESTION : *Are tree words rigid?* ANSWER : *Dunno*! But...

Theorem [Berthé, D., Durand, Leroy, Perrin (2018)]

Let $\mathbf{x}$ be a tree word and $\sigma, \tau \in \mathsf{Stab}(\mathbf{x})$ primitive substitutions.
Then, there exist $i, j \geq 1$ such that $\sigma^i = \tau^j$.

Let $\mathbf{x}$ be a recurrent tree word.
There exists a *primitive tame* substitution θ such that for any primitive $\sigma \in \mathsf{Stab}(\mathbf{x})$, one can find a *positive tame automorphism* τ and integers $i, j \geq 1$ such that $\sigma^i = \tau\theta^j\tau^{-1}$.

Corollary

If $\mathbf{x}$ is a recurrent tree word, then any primitive $\sigma \in \mathsf{Stab}(\mathbf{x})$ is invertible (and thus tame).

Stabilizers of tree words

QUESTION : *Are tree words rigid?* ANSWER : *Dunno*! But...

Theorem [Berthé, D., Durand, Leroy, Perrin (2018)]

Let $\mathbf{x}$ be a tree word and $\sigma, \tau \in \mathrm{Stab}(\mathbf{x})$ primitive substitutions.
Then, there exist $i, j \geq 1$ such that $\sigma^i = \tau^j$.

Let $\mathbf{x}$ be a recurrent tree word.
There exists a *primitive tame* substitution θ such that for any primitive $\sigma \in \mathrm{Stab}(\mathbf{x})$, one can find a *positive tame automorphism* τ and integers $i, j \geq 1$ such that $\sigma^i = \tau\theta^j\tau^{-1}$.

Corollary

If $\mathbf{x}$ is a recurrent tree word, then any primitive $\sigma \in \mathrm{Stab}(\mathbf{x})$ is invertible (and thus tame).

QUESTION : *Is any non-trivial element of* $\mathrm{Stab}(\mathbf{x})$ *primitive when* $\mathbf{x}$ *is recurrent tree?*

Return words

A *left return word* to w in an infinite word $\mathbf{x}$ is a nonempty word u such that $uw \in \mathcal{L}(\mathbf{x})$ starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in A^+ \mid uw \in \mathcal{L}(\mathbf{x}) \cap (wA^+ \setminus A^+wA^+)\}$$

Example (Fibonacci)

$\mathcal{R}(0) = \{0, 01\}$

$$\mathbf{x} = 010010100100101001010010010100100 1 \cdots$$

$\mathcal{R}(00) = \{001, 00101\}$

$$\mathbf{x} = 0100101001001010010100100101001001 \cdots$$

Derived sequence

Let us decode with respect to the first letter of the infinite word.

Example

$$\mathbf{z} \quad = \quad 0110100110010110100101100110100110010110$$

Derived sequence

Let us decode with respect to the first letter of the infinite word.

Example

$\mathcal{R}(0) = \{011, 01, 0\} \quad \longleftrightarrow \quad \{0, 1, 2\}$

$$\mathbf{z} = 0110100110010110100101100110100110010110\cdots$$

$$\mathcal{D}(\mathbf{z}) = 0120210121020120210201210120210121020121\cdots$$

Derived sequence

Let us decode with respect to the first letter of the infinite word.

Example

$\mathcal{R}(0) = \{012, 021, 0121, 02\} \quad \longleftrightarrow \quad \{0, 1, 2, 3\}$

$$\mathbf{z} = 0110100110010110100101100110100110010110\cdots$$

$$\mathcal{D}(\mathbf{z}) = 0120210121020120210201210120210121020121\cdots$$

$$\mathcal{D}^2(\mathbf{z}) = 0123013201232013012301320130123201230132\cdots$$

Derived sequence

Let us decode with respect to the first letter of the infinite word.

Example

$$
\begin{array}{rcl}
\mathbf{z} & = & 0110100110010110100101100110100110010110\cdots \\
\mathcal{D}(\mathbf{z}) & = & 0120210121020120210201210120210121020121\cdots \\
\mathcal{D}^2(\mathbf{z}) & = & 0123013201232013012301320130123201230132\cdots \\
\mathcal{D}^3(\mathbf{z}) & = & 0123013201232013012301320130123201230132\cdots \\
\mathcal{D}^4(\mathbf{z}) & = & 0123013201232013012301320130123201230132\cdots \\
\cdots & & \cdots
\end{array}
$$

The sequence $(\mathcal{D}^n(\mathbf{z}))_{n\in\mathbb{N}}$ is called *derived sequence* of $\mathbf{z}$.

Derived sequence

Let us decode with respect to the first letter of the infinite word.

Example

$$
\begin{array}{rcll}
\mathbf{z} & = & 0110100110010110100101100110100110010110 \cdots & \in \{0,1\}^\infty \\
\mathcal{D}(\mathbf{z}) & = & 0120210121020120210201210120210121020121 \cdots & \in \{0,1,2\}^\infty \\
\mathcal{D}^2(\mathbf{z}) & = & 0123013201232013012301320130123201230132 \cdots & \in \{0,1,2,3\}^\infty \\
\mathcal{D}^3(\mathbf{z}) & = & 0123013201232013012301320130123201230132 \cdots & \in \{0,1,2,3\}^\infty \\
\mathcal{D}^4(\mathbf{z}) & = & 0123013201232013012301320130123201230132 \cdots & \in \{0,1,2,3\}^\infty \\
\cdots & & \cdots &
\end{array}
$$

The sequence $(\mathcal{D}^n(\mathbf{z}))_{n\in\mathbb{N}}$ is called *derived sequence* of $\mathbf{z}$.

REMARK : The alphabets are, in general, different.

Return Theorem

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

Let $\mathbf{x}$ be a recurrent tree word.
For any $w \in \mathcal{L}(\mathbf{x})$, the set $\mathcal{R}(w)$ is a basis of the free group $\mathbb{F}_A$.

Example (Fibonacci)

The set $\mathcal{R}(00) = \{001, 00101\}$ is a basis of the free group. Indeed,

$$\begin{aligned} 0 &= 001\,(00101)^{-1}\,001 \\ 1 &= 0^{-1}\,0^{-1}\,001 \end{aligned}$$

Return Theorem

Theorem [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

Let $\mathbf{x}$ be a recurrent tree word.
For any $w \in \mathcal{L}(\mathbf{x})$, the set $\mathcal{R}(w)$ is a basis of the free group $\mathbb{F}_A$.

Example (Fibonacci)

The set $\mathcal{R}(00) = \{001, 00101\}$ is a basis of the free group. Indeed,

$$\begin{aligned} 0 &= 001\,(00101)^{-1}\,001 \\ 1 &= 0^{-1}\,0^{-1}\,001 \end{aligned}$$

Corollary

For a recurrent tree word $\mathbf{x}$ one has $\mathrm{Card}\,(\mathcal{R}(w)) = \mathrm{Card}\,(A)$ for any $w \in \mathcal{L}(\mathbf{x})$.
Thus all $\mathcal{D}^n(\mathbf{x})$ are in A^ω.

Number of derived sequence

Theorem [Durand (1998)]

A uniformly recurrent word $\mathbf{x} \in A^\omega$ is primitive substitutive if and only if the set of its derived sequences $\{\mathcal{D}^n(\mathbf{x}) \mid n \in \mathbb{N}\}$ is finite.

Example (Fibonacci)

$\mathrm{Card}\left(\{\mathcal{D}^n(\mathbf{x})\}_{n\in\mathbb{N}}\right) = 1$, since

$$\mathbf{x} = 0100101001001010010100100101001001\cdots$$

$$\mathcal{D}(\mathbf{x}) = 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\cdots$$

Number of derived sequence

Theorem [Durand (1998)]

A uniformly recurrent word $\mathbf{x} \in A^{\omega}$ is primitive substitutive if and only if the set of its derived sequences $\{\mathcal{D}^n(\mathbf{x}) \mid n \in \mathbb{N}\}$ is finite.

Theorem [Klouda, Medková, Pelantová, Starosta (2018)]

Let $\mathbf{x}$ be a fixed point of a Sturmian substitution $\sigma = \sigma_1\sigma_2 \cdots \sigma_q \pi$, with $\sigma_i \in (\mathcal{S}_e \setminus \mathfrak{S}_A)^*$ and $\pi \in \mathfrak{S}_A$ (decomposition in a *normal* form). Then

$$1 \leq \mathrm{Card}\left(\{\mathcal{D}^n(\mathbf{x})\}_{n\in\mathbb{N}}\right) \leq 3q - 4.$$

Number of derived sequence

Theorem [Durand (1998)]

A uniformly recurrent word $\mathbf{x} \in A^\omega$ is primitive substitutive if and only if the set of its derived sequences $\{\mathcal{D}^n(\mathbf{x}) \mid n \in \mathbb{N}\}$ is finite.

Theorem [Klouda, Medková, Pelantová, Starosta (2018)]

Let $\mathbf{x}$ be a fixed point of a Sturmian substitution $\sigma = \sigma_1\sigma_2 \cdots \sigma_q\pi$, with $\sigma_i \in (\mathcal{S}_e \setminus \mathfrak{S}_A)^*$ and $\pi \in \mathfrak{S}_A$ (decomposition in a *normal* form). Then

$$1 \leq \text{Card}\left(\{\mathcal{D}^n(\mathbf{x})\}_{n\in\mathbb{N}}\right) \leq 3q - 4.$$

QUESTION : *Can we bound* $\text{Card}\left(\{\mathcal{D}^n(\mathbf{x})\}_{n\in\mathbb{N}}\right)$ *when* $\mathbf{x}$ *is recurrent tree?*

$\mathcal{S}$-adic representation

Let's conclude with some $\mathcal{S}$-adic notions...

Let $\mathcal{S}$ be a set of morphisms. An infinite word $\mathbf{x}$ is said *$\mathcal{S}$-adic* if

$$\mathbf{x} = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a_{n+1})$$

with $\sigma_n : A_{n+1}^* \to A_n^* \in \mathcal{S}$ and $a_n \in A_n$ for all $n \in \mathbb{N}$.

$\mathcal{S}$-adic representation

Let's conclude with some $\mathcal{S}$-adic notions...

Let $\mathcal{S}$ be a set of morphisms. An infinite word $\mathbf{x}$ is said *$\mathcal{S}$-adic* if

$$\mathbf{x} = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a_{n+1})$$

with $\sigma_n : A_{n+1}^* \to A_n^* \in \mathcal{S}$ and $a_n \in A_n$ for all $n \in \mathbb{N}$.

An *$\mathcal{S}$-adic representation* $\big((\sigma_n)_n, (a_n)_n\big)$ of $\mathbf{x}$ is

- *eventually periodic*, if there exist n_0, p s.t. $(\sigma_{m+p}, a_{m+p}) = (\sigma_m, a_m)$ for all $m \geq n_0$
- *primitive*, if for all m there exists k s.t. $A_m \subset \mathcal{L}\left(\sigma_m\sigma_{m+1}\cdots\sigma_k(a)\right)$ for all $a \in A_{k+1}$.

$\mathcal{S}$-adic representation

Let's conclude with some $\mathcal{S}$-adic notions...

Let $\mathcal{S}$ be a set of morphisms. An infinite word $\mathbf{x}$ is said *$\mathcal{S}$-adic* if

$$\mathbf{x} = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_n(a_{n+1})$$

with $\sigma_n : A_{n+1}^* \to A_n^* \in \mathcal{S}$ and $a_n \in A_n$ for all $n \in \mathbb{N}$.

An *$\mathcal{S}$-adic representation* $((\sigma_n)_n, (a_n)_n)$ of $\mathbf{x}$ is

- *eventually periodic*, if there exist n_0, p s.t. $(\sigma_{m+p}, a_{m+p}) = (\sigma_m, a_m)$ for all $m \geq n_0$
- *primitive*, if for all m there exists k s.t. $A_m \subset \mathcal{L}(\sigma_m\sigma_{m+1}\cdots\sigma_k(a))$ for all $a \in A_{k+1}$.

Theorem [Berthé, D., Durand, Leroy, Perrin (2018)]

A recurrent tree word is primitive substitutive if and only if it has an eventually periodic primitive $\mathcal{S}_e$-adic representation.

$$\mathcal{S}_e = \mathfrak{S}_A \sqcup \{\alpha_{a,b}\}_{a\neq b} \sqcup \{\widetilde{\alpha}_{a,b}\}_{a\neq b}$$


Merci