Two-sided Rauzy Induction

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Leiden, 23th January 2014

Joint work with
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Outline

1. Interval exchange transformations
    - Interval exchange transformations
    - Regular interval exchange transformations
2. Rauzy induction
    - Right Rauzy induction
    - Two-sided Rauzy induction
3. Natural coding
    - Natural coding
    - Regular interval exchange sets
    - Return theorem

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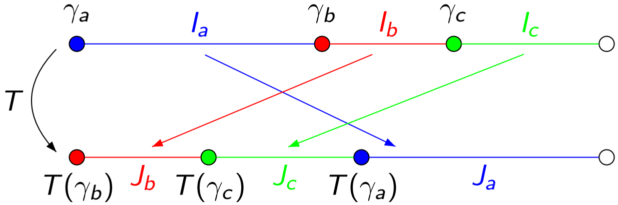
Interval exchange transformations

Let $(A, <)$ be an ordered set and let $(I_a)_{a \in A}$ be an ordered partition of $[\ell, r[$.

A *interval exchange transformation* is a function $T : [\ell, r[\to [\ell, r[$ defined by

$$T(z) = z + \alpha_z \quad \text{if } z \in I_a.$$



Regular interval exchange transformations

T is said to be *minimal* if for any $z \in [\ell, r[$ the orbit $\mathcal{O}(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $[\ell, r[$.

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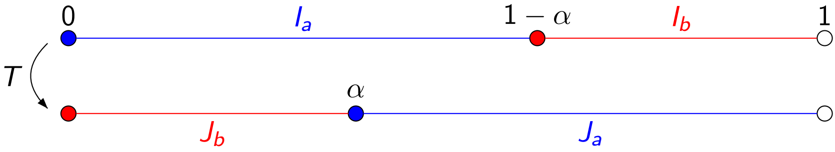

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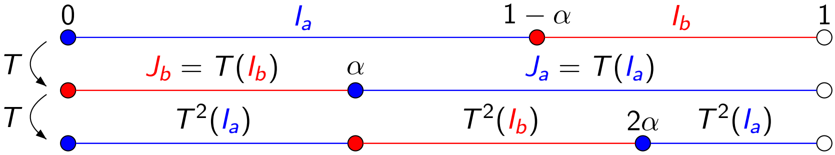

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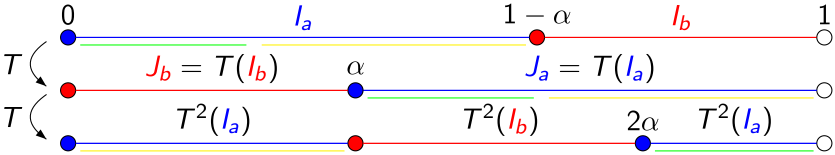

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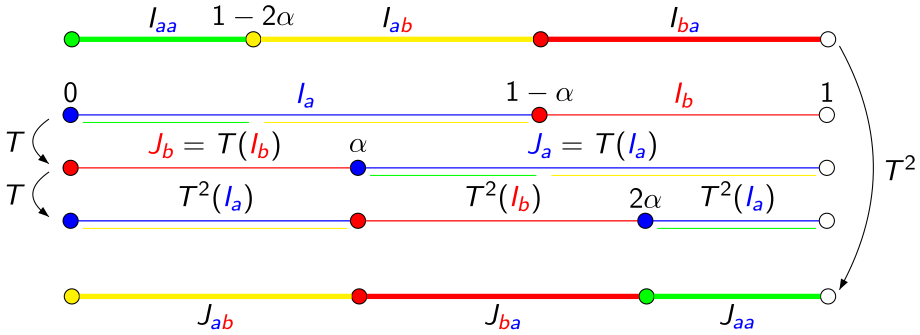

$J_{a_0 a_1 \cdots a_{m-1}} = T^m(I_{a_0}) \cap T^{m-1}(I_{a_1}) \cap \ldots \cap T(I_{b_{a-1}})$ and $I_w = T^{-|w|}(J_w)$

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Admissible semi-intervals

Let T be an interval exchange transformation on the semi-interval $[\ell, r[$.

For $\ell < t < r$, the semi-interval $[\ell, t[$ is *right-admissible* for T if there is a $k \in \mathbb{Z}$ s.t. $t = T^k(\gamma_a)$ for some $a \in A$ and :

(i) if $k > 0$, then $t < T^h(\gamma_a)$ for all $0 < h < k$,

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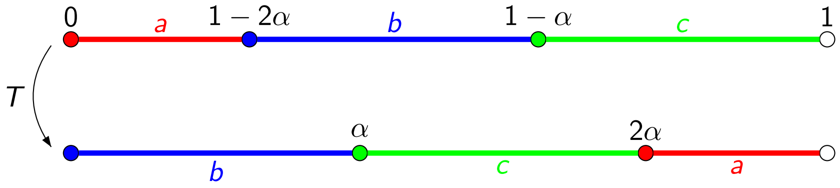

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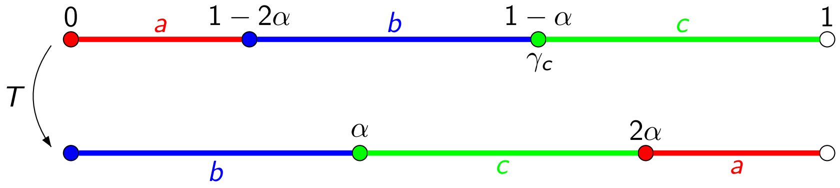

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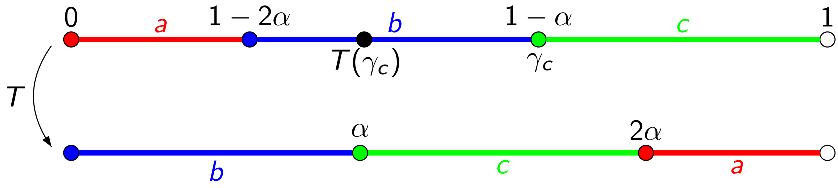

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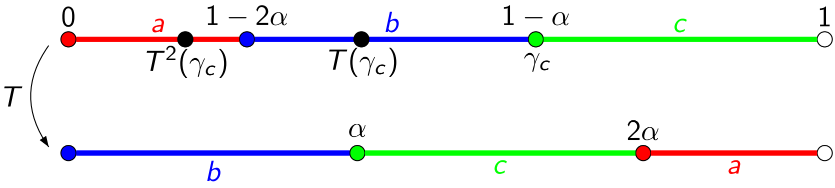

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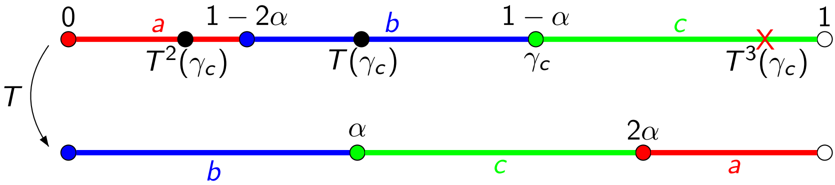

Induced transformations

Let T be a minimal interval exchange transformation and $I \subset [\ell, r[$.
The *transformation induced* by T on I is the transformation $S : I \to I$ defined by

$$S(z) = T^n(z) \qquad \text{with } n = \min\{k > 0 \mid T^k(z) \in I\}$$

The semi-interval I is called the *domain* of S, denoted $D(S)$.

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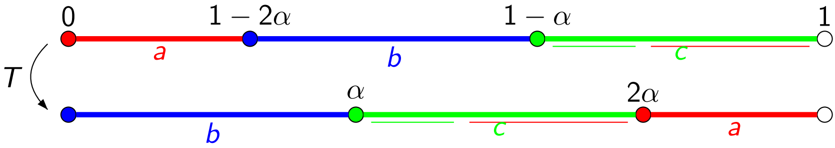
Theorem [Rauzy, 1979]

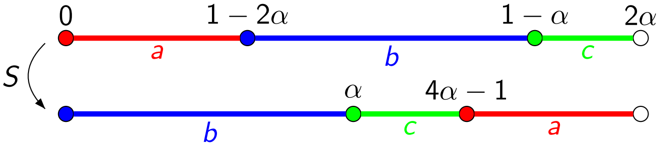
Let T be a regular interval exchange transformation and I a right-admissible interval for T. The induced transformation is a regular interval exchange transformation.

Induced transformations



$I = [0, 2\alpha[$

$$S(z) = \begin{cases} T^2(z) & \text{if } 0 \leq z < 1 - 2\alpha \\ T(z) & \text{otherwise} \end{cases}$$



Right Rauzy induction

Let T be a regular interval exchange transformation on $[\ell, r[$. Set

$$Z(T) = [\ell, \max_a \{\gamma_a, T(\gamma_a)\}[.$$

We denote by $\psi(T)$ the transformation induced by T on $Z(T)$.

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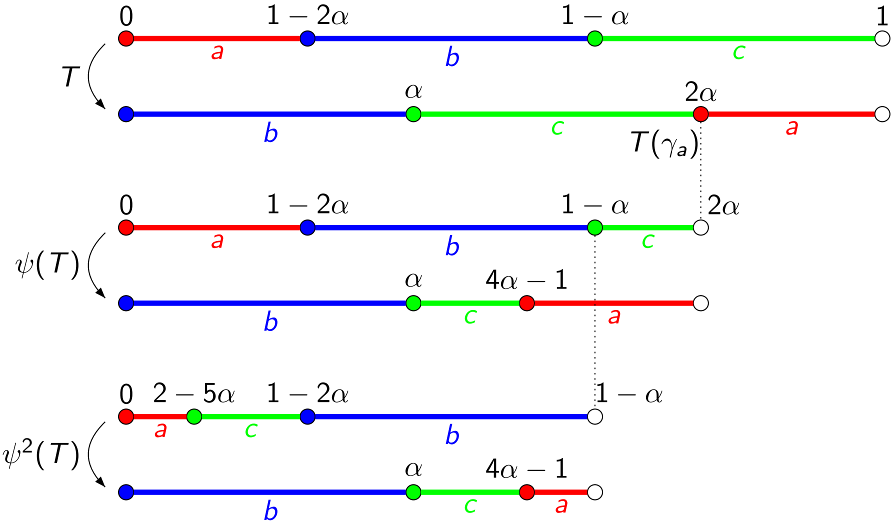
Theorem [Rauzy, 1979]

Let T be a regular interval exchange transformation.
A semi-interval I is right-admissible for $T \iff I = Z(\psi^n(T))$ for some $n > 0$.
In this case, the transformation induced by T on I is $\psi^{n+1}(T)$.

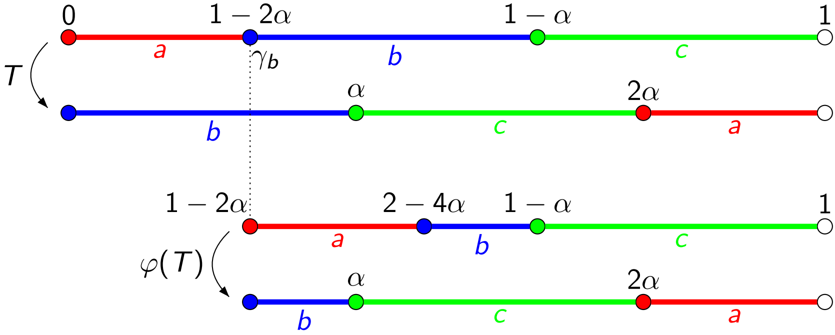
The map $T \to \psi(T)$ is called the *right Rauzy induction*.

Right Rauzy induction



Left Rauzy induction

The symmetrical notions of *left admissible semi-interval* and *left Rauzy induction*, denoted φ, are defined similary.



Two-sided Rauzy induction

Let T be a regular interval exchange transformation. For $\ell \le u < v \le r$ we say that the semi-interval $I = [u, v[$ is *admissible* for T if $u, v \in \mathrm{Div}(I, T) \cup r$ with

$$\mathrm{Div}(I, T) = \bigcup_a \left\{ T^k(\gamma_a) \;\middle|\; -\rho^-(\gamma_a) \le k < \rho^+(\gamma_a) \right\}$$

$$\rho^-(z) = \min\{n > 0 \mid T^n(z) \in]u, v[\}, \qquad \rho^+(z) = \min\{n \ge 0 \mid T^{-n}(z) \in]u, v[\}.$$

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Theorem [BDDPRR (i.e. *us*), 2013]

The transformation induced by T on I is a regular interval exchange transformation.

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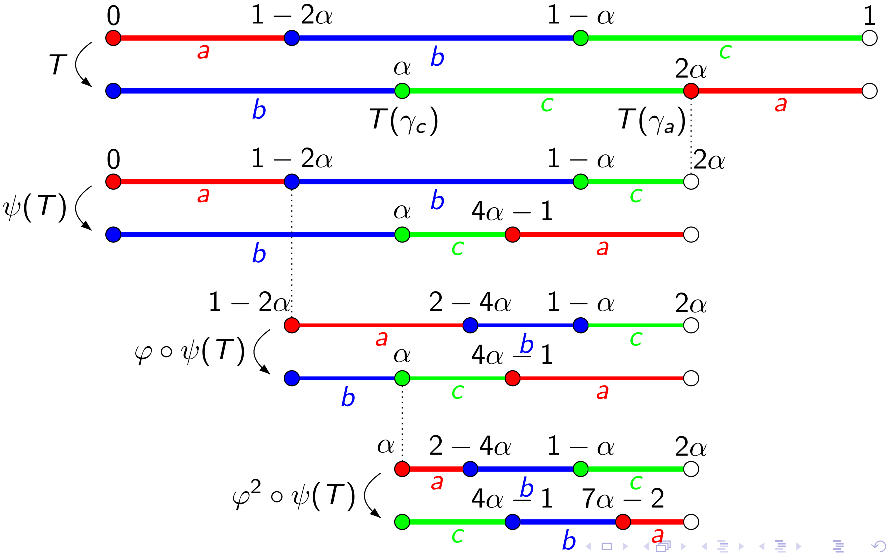
I is admissible for T $\iff$ if I is the domain of a $\chi \in \{\varphi, \psi\}^*$.
In this case, the transformation induced by T on I is $\chi(T)$.

Two-sided Rauzy induction



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Natural Coding

Let T be an interval exchange transformation relative to $(I_a)_{a \in A}$.
The *natural coding* of T relative to $z \in [\ell, r[$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si} \quad T^n(z) \in I_a.$$

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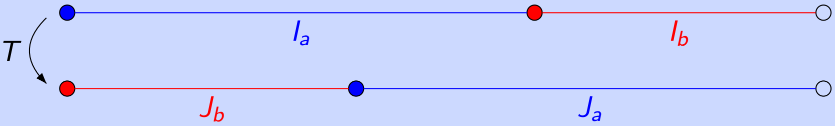
Example

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point α, i.e. $T(z) = z + \alpha \bmod 1$.



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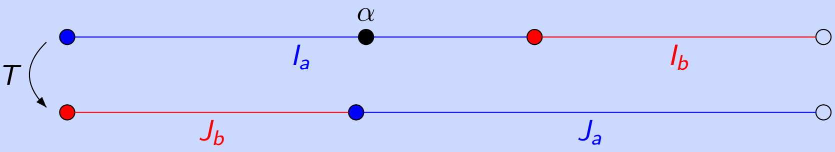

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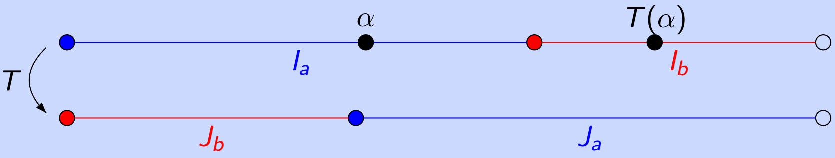

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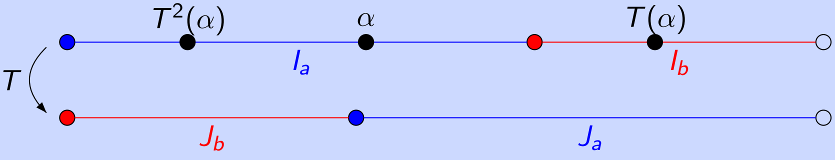

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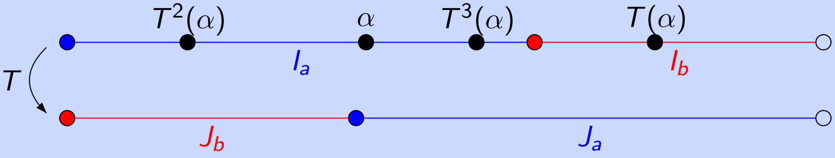

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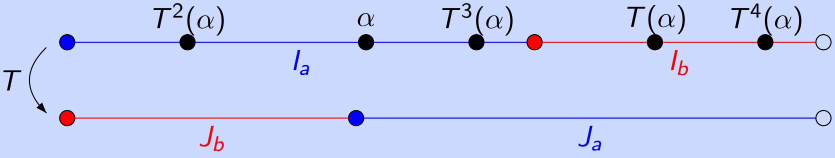

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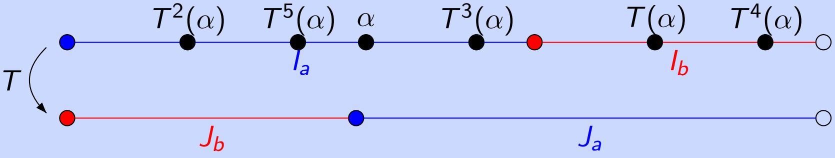

$$\Sigma_T(z) = a\,b\,a\,a\,b\,a \cdots$$

Regular interval exchange sets

Proposition

If T is minimal, $\mathcal{L}(\Sigma_T(z))$ does not depend on z.

When T is regular (minimal), $\mathcal{L}(T) = \mathcal{L}(\Sigma_T(z))$ is said a *regular* (*minimal*) *interval exchange set* (*linear complexity, neutrality, tree set*[1], *finite index basis, etc.*).

1. See Valérie Berthé's talk next week.

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Proposition

If T is minimal, $w \in \mathcal{L}(T) \iff J_w \neq 0$.

Proposition

If T is regular, J_w is admissible for every $w \in \mathcal{L}(T)$.

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Return Theorem

The set of *right return words* to a word w (w.r.t. $\mathcal{L}(T)$) is

$$\Gamma_{\mathcal{L}(T)} = \{u \in \mathcal{L}(T) \mid wu \in A^+ \cap \mathcal{L}(T)\}$$

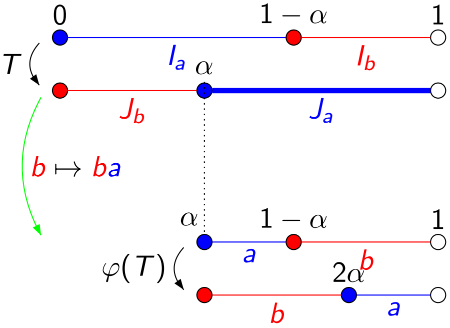
while the set of *first right return words* is

$$\mathcal{R}_{\mathcal{L}(T)} = \Gamma_{\mathcal{L}(T)} \setminus \Gamma_{\mathcal{L}(T)} A^+$$

Theorem [BDDPRR (i.e. *us*), 2013]

Let T be a regular interval exchange transformation on A. For any $w \in \mathcal{L}(T)$, the set of first right return words to w is a basis of the free group on A.

Return Theorem



The set of first right return words to a is

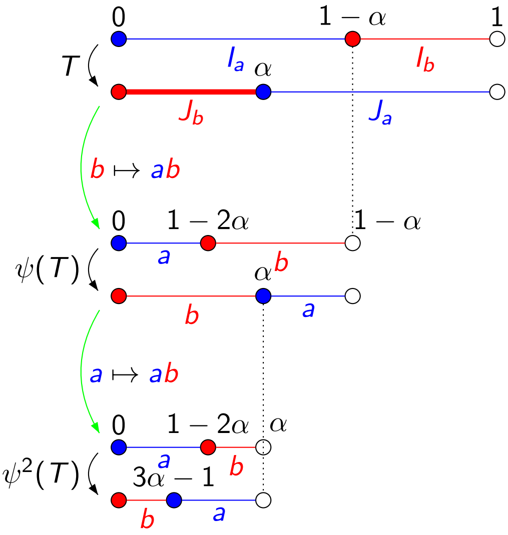
$$\mathcal{R}_{\mathcal{L}(T)}(a) = \{a, ba\}$$

$$x = a\underline{ba}ababa\underline{a}baababaab\cdots$$

And we can find it via the automorphism

$$\theta : \begin{cases} a & \mapsto & a \\ b & \mapsto & ba \end{cases}$$

Return Theorem



The set of first right return words to b is

$$\mathcal{R}_{\mathcal{L}(T)}(b) = \{aab, ab\}$$

$$x = ab\underline{aab}abaabaab\underline{ab}aab\cdots$$

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Conclusions

- Two-sided version of :
  - admissible semi-intervals,
  - Rauzy induction,
  - two Rauzy's theorems ;
- Regular interval exchange set ;
- Return theorem.

Questions