

Generalized Lyndon words

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A New Word Order

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Let $(<_n)_{n \geq 1}$ be a sequence of total orders on A .

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The *generalized lexicographical order* is defined as $u < v$ if either :

- u is a proper prefix of v , or
- $u = pas$, $v = pbt$ for some $p \in A^*$, $s, t \in A^\infty$, and $a, b \in A$ s.t. $a <_{|p|+1} b$.

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Examples

- Classical order ($<$) : $a <_n b$ for all $n \geq 1$.

$$a < aa < ab < aba < baa$$

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- Classical order ($<$) : $a <_n b$ for all $n \geq 1$.
- Alternate order ($<_{alt}$) : $\begin{cases} a <_n b & \text{if } n \equiv 1 \pmod{2} \\ b <_n a & \text{otherwise.} \end{cases}$

$$a_1 a_2 a_3 \cdots <_{alt} b_1 b_2 b_3 \cdots \iff a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots}}} < b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{\dots}}}$$

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- Alternate order (\langle_{alt}) : $\begin{cases} a <_n b & \text{if } n \equiv 1 \pmod{2} \\ b <_n a & \text{otherwise.} \end{cases}$
- Prime order (\langle_π) : $\begin{cases} b <_n a & \text{if } n \text{ is prime} \\ a <_n b & \text{otherwise.} \end{cases}$

$$aba <_\pi abaa <_\pi aab <_\pi bab <_\pi baab$$

Generalized lexicographical order

inverse order

The *inverse (generalized) order* $\tilde{<}_\pi$, obtained by reversing all the orders $<_n$, is also a generalized order.

Examples

- $aba <_\pi aab <_\pi bab <_\pi baa$
- $baa \tilde{<}_\pi bab \tilde{<}_\pi aab \tilde{<}_\pi aba.$

Infinite order

$$u \prec v \quad :\Leftrightarrow \quad \left\{ \begin{array}{l} u^\omega < v^\omega \\ u^\omega = v^\omega \end{array} \right. \text{ or } \text{and } |u| > |v|.$$

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When $|u| = |v|$ one has $u < v \Leftrightarrow u^\omega < v^\omega$. In general, this is not true.

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$ab < aba$ but $(ab)^\omega > (aba)^\omega$.

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We can also consider the generalized lexicographical infinite order.

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$(ab)^\omega <_\pi a^\omega <_\pi b^\omega <_\pi (ba)^\omega$.

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$(ab)^\omega <_\pi a^\omega <_\pi b^\omega <_\pi (ba)^\omega$.

$u^\omega = v^\omega \quad \Leftrightarrow \quad u$ and v are power of a common word ($\Leftrightarrow uv = vu$).

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Let $<_g$ be a generalized order.

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Let $u = ab$ and $v = a$. Then

$$(ab.a)^\omega <_\pi (ab)^\omega <_\pi (a.ab)^\omega <_\pi a^\omega$$

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A word $w \in A^+$ is a (*classical*) *Lyndon word* if for any nontrivial factorization $w = uv$ one has $w < vu$.

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- *michel* is a $\tilde{<}$ -Lyndon word.

$$(michel)^\omega \tilde{<} (lmiche)^\omega \tilde{<} (ichelm)^\omega \tilde{<} (heliac)^\omega \tilde{<} (elmich)^\omega \tilde{<} (chelmi)^\omega$$

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- *micHEL* is a $\tilde{<}$ -Lyndon word.
- *elise* is a $<_{alt}$ -Lyndon word (*Galois word*).

$$(elise)^\omega <_{alt} (eelis)^\omega <_{alt} (iseel)^\omega <_{alt} (lisee)^\omega <_{alt} (seeli)^\omega$$

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Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

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Factorization into generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

Each word $w \in A^+$ can be factorized in a unique way as $w = l_1 l_2 \cdots l_n$, with l_i generalized Lyndon words s.t. $l_1^\omega \geq_g l_2^\omega \geq_g \cdots \geq_g l_n^\omega$.

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- The factorization in classical Lyndon word of *julien* is $(jul)(i)(en)$, since

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Moreover, ℓ_n is

- the shortest suffix s of w s.t. s^ω is minimum,
- the longest suffix of w which is a generalized Lyndon word.

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Classical Lyndon words

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Theorem [Bergman (1969)]

If $u^\omega < v^\omega$ then $u^\omega < (uv)^\omega < (vu)^\omega < v^\omega$.

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Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

Factorization into classical Lyndon words

Theorem [Ufnarovskij (1995)]

Let $w = l_1 l_2 \cdots l_n$ the unique non-increasing factorization of w in Lyndon word.

Then

- $l_1^\omega > (l_2 \cdots l_n)^\omega$
- l_1 is the shortest nontrivial prefix p s.t. $w = ps$ and $p^\omega \geq s^\omega$,
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Example

- $(jul)^\omega > ((i)(en))^\omega$,
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Galois words

The *alternating lexicographical order* $<_{alt}$ (w.r.t. an order $<$) is the generalized lexicographical order defined by the sequence $(<_n)_{n \geq 1}$ with

$$<_n = \begin{cases} < & \text{if } n \equiv 1 \pmod{2} \\ \tilde{<} & \text{otherwise.} \end{cases}$$

Example

$$(ab)^\omega <_{alt} a^\omega <_{alt} b^\omega <_{alt} (ba)^\omega.$$

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Example

$$(ab)^\omega <_{alt} a^\omega <_{alt} b^\omega <_{alt} (ba)^\omega.$$

A *Galois word* is a generalized Lyndon word for an alternating lexicographical order.

Example

The following are Galois words : b , ac , bc , aba , abb , $abaa$, $acab$.

Characterization of Galois words

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Theorem [D., Restivo, Reutenauer (2018)]

w is a Galois word if and only if for any nontrivial factorization $w = ps$ one has

$$\begin{cases} p^\omega <_{alt} w^\omega & \text{if } |p| \text{ is even,} \\ p^\omega >_{alt} w^\omega & \text{if } |p| \text{ is odd.} \end{cases}$$

Factorization into Galois words

Theorem [D., Restivo, Reutenauer (2018)]

Let $w = g_1 g_2 \cdots g_n$ with g_i Galois words s.t. $g_1^\omega \geq_{alt} g_2^\omega \geq_{alt} \cdots \geq_{alt} g_n^\omega$.

Let m be the multiplicity of g_1 .

Let p be the shortest nontrivial prefix of w s.t.

$$p^\omega \geq_{alt} w^\omega \text{ if } |p| \text{ is even} \quad \text{and} \quad p^\omega \leq_{alt} w^\omega \text{ if } |p| \text{ is odd.} \quad (\star)$$

Then

(i) if $|g_1|$ is odd, m is even, and $m < n$, then $p = g_1^2$,

(ii) otherwise, $p = g_1$.

Example

Let $w = (abb)(abb)(abaa)$.

$((abb)^2)^\omega >_{alt} w^\omega$ and each proper prefix of $(abb)^2$ does not satisfy condition (\star) .



Complete trees

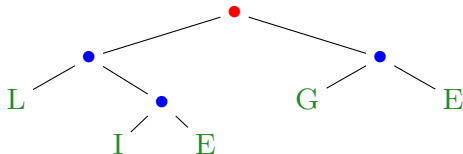
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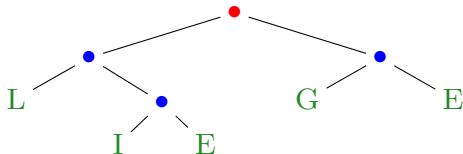


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The *foliage* $\varphi(t)$ of a tree t is defined as :

- $\varphi(a) = a$ for any $a \in A$,
- $\varphi((t_1, t_2)) = \varphi(t_1)\varphi(t_2)$ for any two trees t_1, t_2 .

Left standard factorization

Let w be a Lyndon word of length at least 2.

The *left standard factorization* of w is the factorization $w = uv$, where u is the longest nonempty proper prefix of w which is a Lyndon word.

Example

The left standard factorization of $abaacab$ is $(abaac)(ab)$.

Left standard factorization

Let w be a Lyndon word of length at least 2.

The *left standard factorization* of w is the factorization $w = uv$, where u is the longest nonempty proper prefix of w which is a Lyndon word.

Proposition

Both u and v are Lyndon words.

Moreover, either v is a letter or $v = v_1v_2$, and $v_1 \leq u$.

Example

The left standard factorization of $aabaacab$ is $(aabaac)(ab)$.

The left standard factorization of ab is $(a)(b)$, and $a \leq aabaac$.

Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

- $\mathcal{L}(a) = a$ for each letter $a \in A$;
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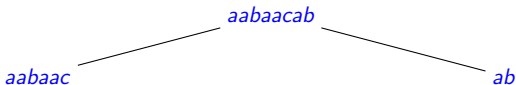
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abaacab

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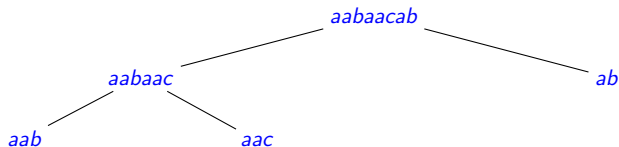
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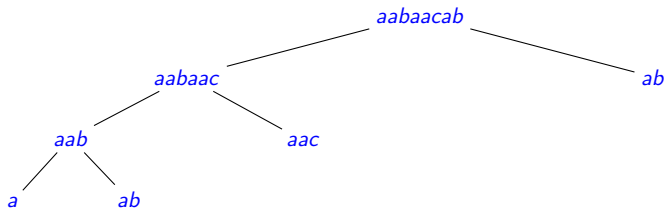
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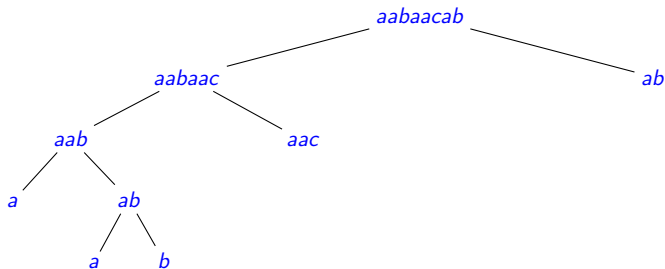
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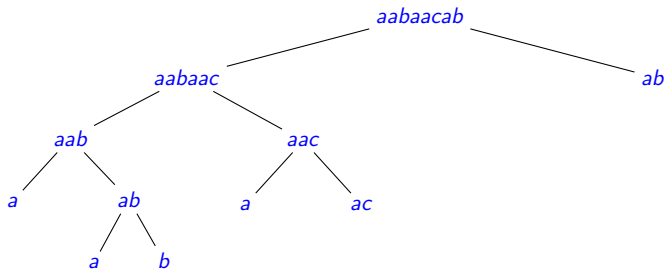
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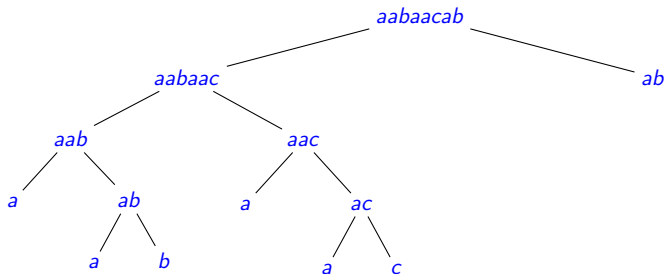
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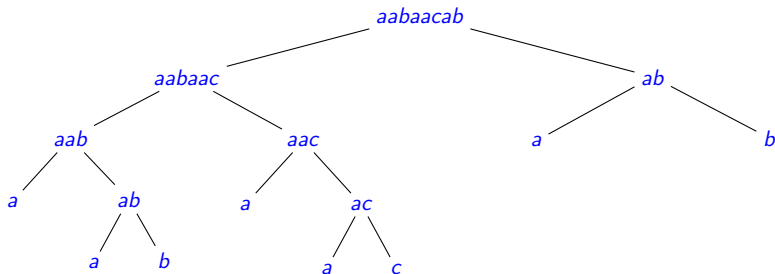
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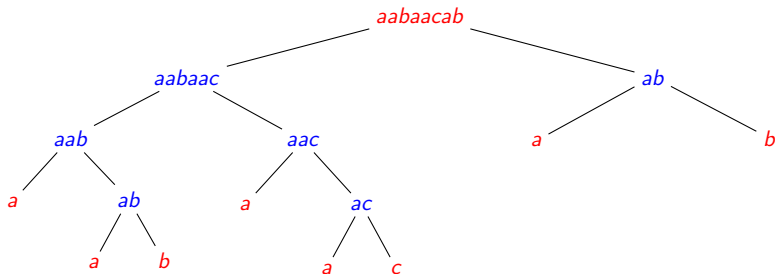
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Clearly $\varphi(\mathcal{L}(w)) = w$.

Prefix standardization

$$u \prec v \quad :\Leftrightarrow \quad \begin{cases} u^\omega < v^\omega \\ u^\omega = v^\omega \end{cases} \quad \text{or} \\ \text{and } |u| > |v|.$$

Example

$aa \prec a \prec ab \prec ba \prec b.$

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$w = \text{aabaacab}$

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Example

$w =$ **a**a**b**a**a**c**a**b
21

$aa \prec a$

Prefix standardization

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2154376

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Example

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$aa \prec a \prec aabaa \prec aaba \prec aab \prec aabaaca \prec aabac \prec w$

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Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

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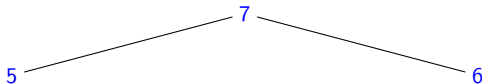
7

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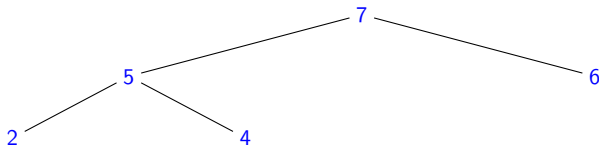


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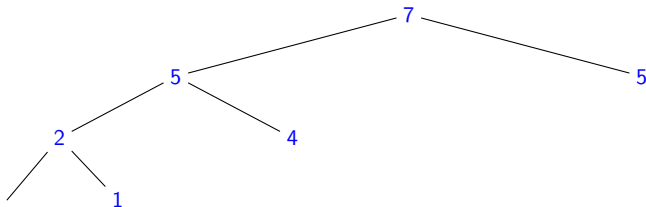


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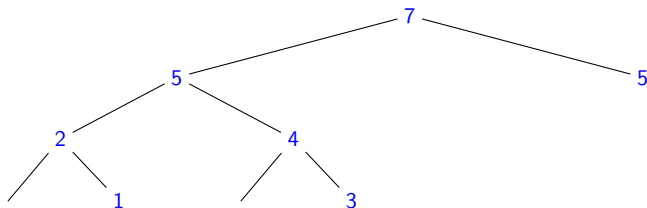


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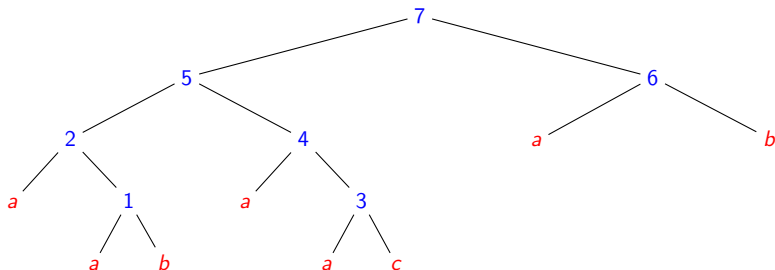


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We complete in such a way that $\varphi(\mathcal{C}(w)) = w$.

Equivalence of trees

Theorem [D., Restivo, Reutenauer (2019)]

Let w be a Lyndon word. Then $\mathcal{L}(w) = \mathcal{C}(w)$.

$aabaacab \longleftrightarrow 2154376$

