Specular sets

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based on a joint work with
V. Berthé, C. De Felice, V. Delecroix,
J. Leroy, D. Perrin, C. Reutenauer, G. Rindone

Outline

1. Specular sets
    - Tree sets
    - Specular sets
    - Specular groups

2. Two examples
    - Linear involutions
    - Doubling maps

3. Return words and subgroups
    - Even and odd words
    - Right, complete and mixed return words

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a,b) \in A \times A \mid awb \in S\}. \end{aligned}$$

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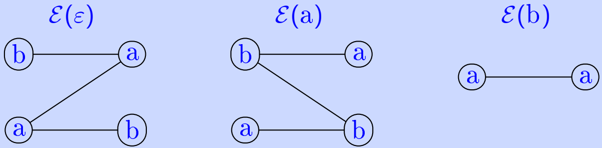
Example (Fibonacci)

$S = \{\varepsilon,\ a, b,\ aa, ab, ba,\ aab, aba, baa, bab, \ldots\}$.



A factorial set S is called a *tree set* of *characteristic* c if $\mathcal{E}(w)$ is a tree for any nonempty $w \in S$, and $\mathcal{E}(\varepsilon)$ is a union of c trees.

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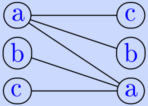
Theorem

Families of (uniformly) recurrent tree sets of characteristic 1 :

- Factors of Arnoux-Rauzy (*Sturmian*) words ;
- Natural coding of regular interval exchanges.

Example (Tribonacci)

$\mathcal{E}(\varepsilon)$



Let $\theta : \mathrm{A} \to \mathrm{A}$ be an involution (possibly with some fixed point).

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A word is *θ-reduced* if it has no factor of the form $a\theta(a)$ for $a \in A$.

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Let $\theta : a \mapsto a,\ b \mapsto d,\ c \mapsto c,\ d \mapsto b$.

The θ-reduction of the word daaacdb is dac.

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The θ-reduction of the word $d\not{a}\not{a}ac\not{d}\not{b}$ is dac.

A set is called *θ-symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric for $\theta : b \leftrightarrow d$ fixing a, c.

A *specular set* on an alphabet A (w.r.t. an involution θ) is a set

- biextendable,
- θ-symmetric,
- θ-reduced,
- tree set of characteristic 2.

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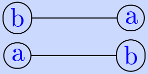

$\mathcal{E}(baba)$

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Proposition [*using* J. Cassaigne (1997)]

The factor complexity of a specular set is given by $p_n = n\,(\mathrm{Card}\,(A) - 2) + 2$ for all $n \geq 1$.

Given an involution $\theta : A \to A$ (possibly with some fixed point), let us define

$$G_\theta = \langle a \in A \mid a \cdot \theta(a) = 1 \text{ for every } a \in A \rangle.$$

$G_\theta = \mathbb{Z}^{*i} * (\mathbb{Z}/2\mathbb{Z})^{*j}$ is a *specular group* of type (i, j), and $\mathrm{Card}(A) = 2i + j$ is its *symmetric rank*.

Example

Let $\theta : b \leftrightarrow d$ fixing a, c.

$$G_\theta = \langle a, b, c, d \mid a^2 = c^2 = bd = db = 1 \rangle.$$

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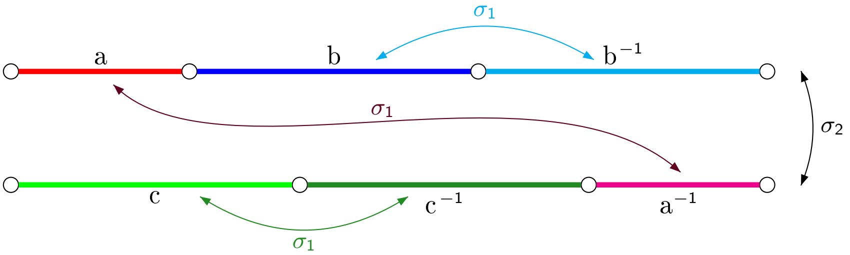
A specular set (w.r.t. θ) is thus a (biextendable, θ-symmetric, tree set of characteristic 2) subset of G_θ.

A *symmetric basis* of G_θ is a (monoidal) basis for G_θ that is θ-symmetric.

Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The natural coding of a linear involution without connections is a specular set.

$$T = \sigma_2 \circ \sigma_1$$



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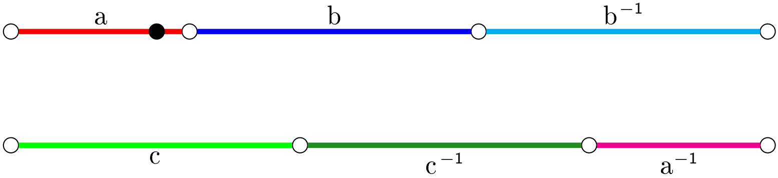

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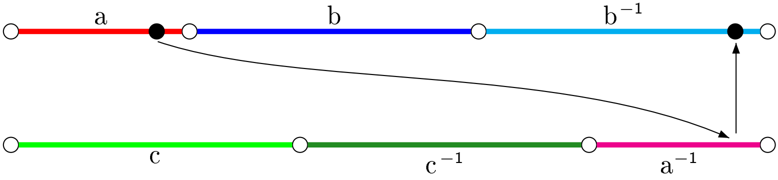

$\Sigma(z) = ab^{-1}$

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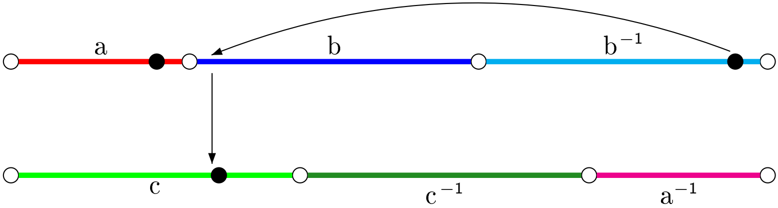

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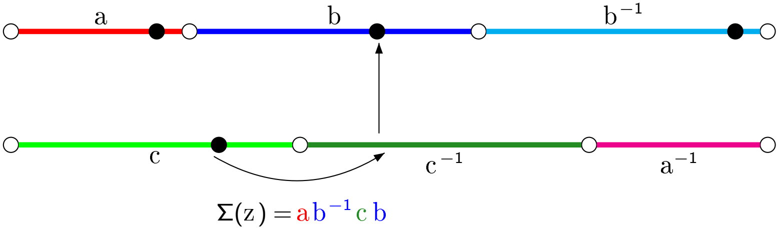

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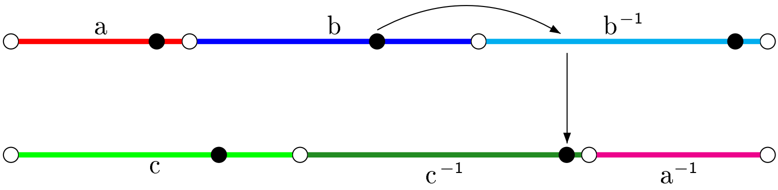

$$\Sigma(z) = a\, b^{-1}\, c\, b\, c^{-1} \cdots$$

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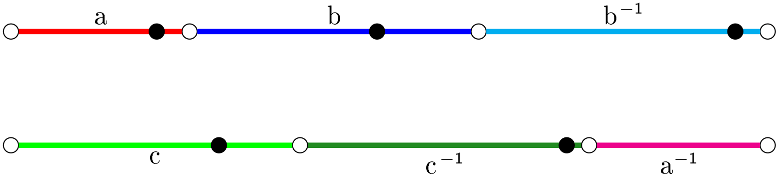


$$\Sigma(z) = a\, b^{-1}\, c\, b\, c^{-1} \cdots$$

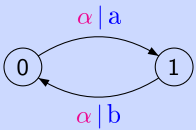
$$\mathcal{L}(T) = \bigcup_{z} \mathrm{Fac}(\Sigma(z))$$

A *doubling transducer* is a transducer with set of states $\{0, 1\}$ such that :

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



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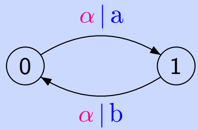

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The *image* of a set T is $\delta(T) = \delta_0(T) \cup \delta_1(T)$.

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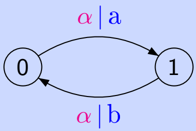

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$\delta(\mathrm{Fac}(\alpha^\omega)) = \mathrm{Fac}((ab)^\omega)$

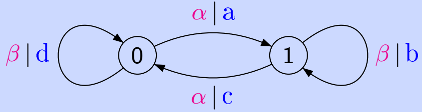
Proposition

The image by a doubling map of a tree set of characteristic 1 closed under reversal is a specular set.

Example (two doublings of Fibonacci on $\Sigma = \{\alpha, \beta\}$)

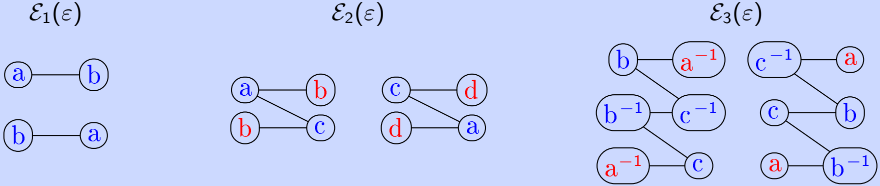
- the set of factors of the two infinite sequences abaababa$\cdots$ and cdccdcdc$\cdots$

- the set of factors of the two infinite sequences abcabcda$\cdots$ and cdacdabc$\cdots$.



A letter is *even* if its two occurences (as a element of $\mathrm{L}(\varepsilon)$ and of $\mathrm{R}(\varepsilon)$) appear in the same tree of $\mathcal{E}(\varepsilon)$. Otherwise it is *odd*.

Three examples



The letters in red are even, while the ones in blue are odd.

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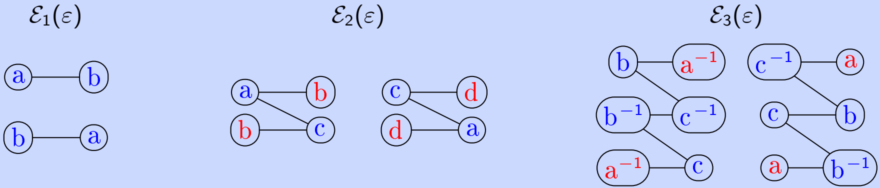


The letters in red are even, while the ones in blue are odd.

A word is *even* if it has an even number of odd letters. Otherwise it is *odd*.

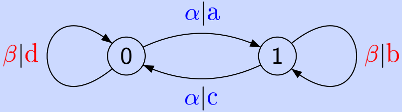
Example (S_2)

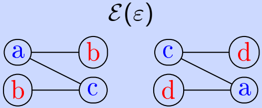
b, ca, dabc are even, while a, ab, abca are odd.

The set of even words in a specular set S has the form $X^* \cap S$, where $X \subset S$ is a *bifix code* (it does not contain any prefix or suffix of its elements) called the *even code*.

Thus, the set X is the set of even words without a nonempty even prefix (or suffix).

Example (doubling of Fibonacci)





The even code is $X = \{abc, ac, b, ca, cda, d\}$.

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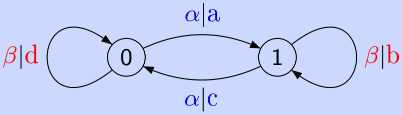

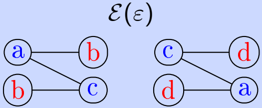

The even code is $X = \{abc, ac, b, ca, cda, d\}$.

Theorem

Let S be a recurrent specular set.
The even code of S is a symmetric basis of a subgroup (of index 2) of G_θ called the *even subgroup*.

A *right return word* to w in S is a nonempty word u such that $wu \in S$, starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{R}(w) = \{u \in A^+ \mid wu \in (A^+w \setminus A^+wA^+) \cap S\}.$$

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Right Return Theorem

For any w in a recurrent specular set, the set $\mathcal{R}(w)$ is a basis of the even subgroup. In particular,

$$\text{Card}\,(\mathcal{R}(w)) = \text{Card}\,(A) - 1.$$

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Example (doubling of Fibonacci)

The even code is $X = \{abc, ac, b, ca, cda, d\}$, while $\mathcal{R}(a) = \{bca, bcda, cda\}$. One has $\langle \mathcal{R}(a) \rangle = \langle X \rangle$, indeed

$$\begin{cases} cda = cda & ca = (b)^{-1}(bca) \\ abc = (cda)^{-1} & ac = (ca)^{-1} \\ b = (bcda)(abc) & d = b^{-1} \end{cases}$$

A *complete return word* to a set $X \subset S$ is a word starting and ending with a word of X but having no internal factor in X. Formally,

$$\mathcal{CR}(X) = S \cap (XA^+ \cap A^+X) \setminus A^+XA^+.$$

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Cardinality Theorem for Complete Return Words

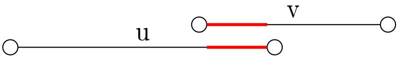
Let S be a recurrent specular set and $X \subset S$ be a finite *bifix code*[1] with empty *kernel*[2]. Then,

$$\text{Card}\,(\mathcal{CR}(X)) = \text{Card}\,(X) + \text{Card}\,(A) - 2.$$

In particular, $\text{Card}\,(\mathcal{CR}(\{w\})) = \text{Card}\,(\mathcal{R}(w)) = \text{Card}\,(A) - 1$.

1. *bifix code* : set that does not contain any proper prefix or suffix of its elements.
2. *kernel* : set of words of X which are also internal factors of X.

Two words u, v *overlap* if a nonempty suffix of one of them is a prefix of the other.



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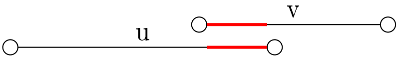

Consider a word w not overlapping with w^{-1} (i.e. $\theta(w)$).

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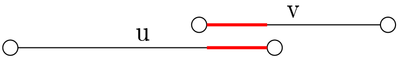
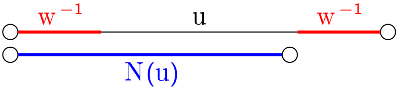


Consider a word w not overlapping with w^{-1} (i.e. $\theta(w)$).

A *mixed return word* to w is the word $N(u)$ obtained from $u \in \mathcal{CR}(\{w, w^{-1}\})$ erasing the prefix if it is w and the suffix if it is w^{-1}.



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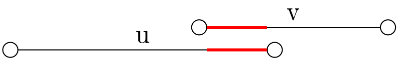

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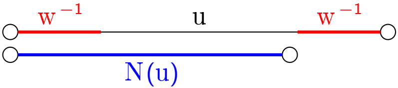

Mixed Return Theorem

Let S be a recurrent specular set and $w \in S$ such that w, w^{-1} do not overlap. Then, $\mathcal{MR}(w)$ is a symmetric basis of G_θ. In particular,

$$\text{Card}(\mathcal{MR}(w)) = \text{Card}(A).$$

Conclusions

Summing up

- Tree and specular sets

  Linear involutions and Doubling maps

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- Cardinality Theorems

$$\begin{aligned} \text{Card}(\mathcal{R}(w)) &= \text{Card}(A) - 1 \\ \text{Card}(\mathcal{CR}(X)) &= \text{Card}(X) + \text{Card}(A) - 2 \\ \text{Card}(\mathcal{MR}(w)) &= \text{Card}(A) \end{aligned}$$

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- Return Theorems

  $\mathcal{R}(\mathrm{w})$ basis of the even subgroup

  $\mathcal{MR}(\mathrm{w})$ symmetric basis of G_θ (provided that $\mathrm{w}, \mathrm{w}^{-1}$ does not overlap)

Further Research Directions

and other works in progress

- Decidability of the tree condition

  [work in progress with Revekka Kyriakoglou and Julien Leroy]

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  Recurrent tree set of characteristic $1 \Longrightarrow \mathcal{R}(w)$ a basis of the *free group*

  Conjecture : $\Longleftarrow$

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  Recurrent tree set of characteristic $1 \Longrightarrow \mathcal{R}(w)$ a basis of the *free group*

  Conjecture : $\Longleftarrow$

- Tree sets and palindromes

  Tree sets of characteristic 1 closed under reversal are rich

  Specular sets obtained by doubling maps are G-rich

Merci