Codes bifixes, combinatoire des mots et systèmes dynamiques symboliques

Appendix

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Overview

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- Weak and strong words
- Acyclic and connected sets
- Rauzy graphs and Stallings foldings
- Specular groups
- Monoidal basis
- Doubling transducer
- Odd and even words
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- Palindromes in tree sets
- σ-palindromes
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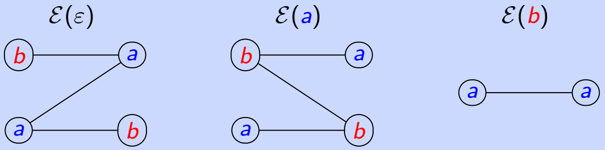
- Branching Rauzy induction
- Return Theorem for interval exchanges
- Interval exchanges over a quadratic field
- $\mathcal{S}$-adic representation

Planar tree sets

Let $<_1$ and $<_2$ be two orders on A.
For a set S and a word $w \in S$, the graph $\mathcal{E}(w)$ is *compatible* with $<_1$ and $<_2$ if for any $(a, b), (c, d) \in B(w)$, one has

$$a <_2 c \quad \Longrightarrow \quad b \leq_1 d.$$

Example (Fibonacci, $a <_1 b$ and $b <_2 a$)



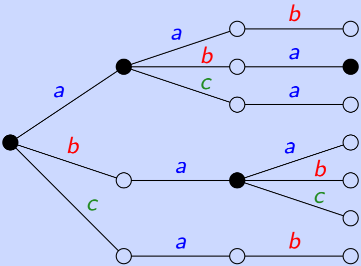
A biextendable set S is a *planar tree set* w.r.t. $<_1$ and $<_2$ on A if for any nonempty $w \in S$, the graph $\mathcal{E}(w)$ is a tree compatible with $<_1$ and $<_2$.

Planar tree sets

Example ($A = \{ a,b,c \}$)

The *Tribonacci set* is the set of factors of the Tribonacci word $f^\omega(a) = abacaba\ldots$ fixed point of the morphism

$$f : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$$



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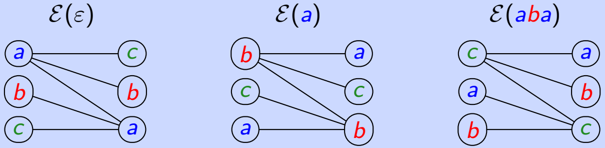
The *Tribonacci set* is not a planar tree set.
Indeed, let us consider the extension graphs of the bispecial words ε, a and aba.

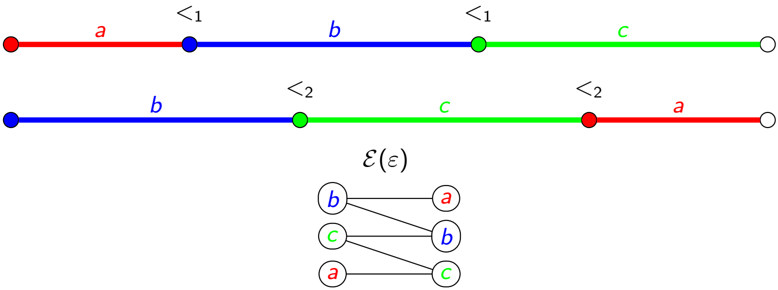


It is not possible to find two orders on A making the three graphs planar.

Planar tree sets

Theorem [S. Ferenczi, L. Zamboni (2008)]

A set S is a regular interval exchange set on A if and only if it is a recurrent tree set of characteristic 1.



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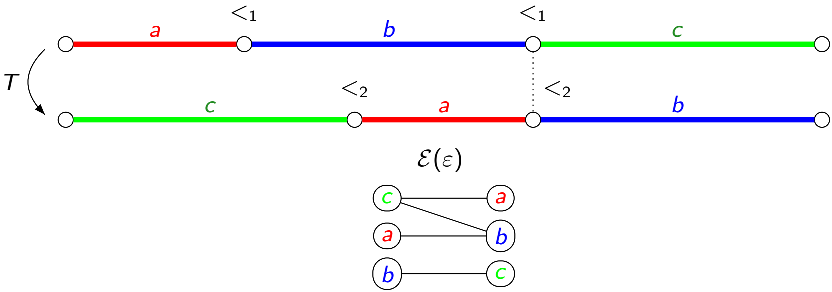
Theorem [Dolce, Perrin (2016)]

Let T be an interval exchange transformation with exactly C connections, all of length 0. Then $\mathcal{L}(T)$ is a planar tree set of characteristic $C+1$ with respect to $<_1$ and $<_2$.



Complete return words

A *complete return word* to w in S is a nonempty word u such that $u \in S$ starts and ends with w but has no w as an internal factor. Formally,

$$\mathcal{CR}_S(w) = S \cap (wA^+ \cap A^+w) \setminus A^+wA^+$$

Example (Fibonacci)

$\mathcal{CR}_S(aa) = \{\underline{aa}b\underline{aa}, \underline{aa}bab\underline{aa}\}$

$$\varphi(a)^\omega = ab\underline{aa}bab\underline{aa}baababaabab\underline{aa}b\underline{aa}babaabaab\cdots$$

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A *complete return word* to a set $X \subset S$ is a nonempty word starting and ending with a word of X having no internal factor in X. Formally,

$$\mathcal{CR}_S(X) = S \cap (XA^+ \cap A^+X) \setminus A^+XA^+$$

Example (Fibonacci)

$\mathcal{CR}_S(aa, bab) = \{aabaa,\ aa\,bab,\ bab\,aa\}$

$$\varphi(a)^\omega = abaa\,babaabaabab\,aababaabaababaabaab\cdots$$

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Example (Fibonacci)

$\mathcal{CR}_S(S \cap A^n) = S \cap A^{n+1}$

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Theorem [Dolce, Perrin (2016)]

Let S be a neutral set. For any finite nonempty bifix code $X \subset S$ with empty kernel, we have

$$\text{Card}\,(\mathcal{CR}_S(X)) \leq \text{Card}\,(X) + \text{Card}\,(A) - \chi(S)$$

with equality if S is recurrent.

Weak and strong words

The *multiplicity* of a word w is the quantity

$$m(w) = \mathrm{Card}\,(B(w)) - \mathrm{Card}\,(L(w)) - \mathrm{Card}\,(R(w)) + 1.$$

A word is called *neutral* if $m(w) = 0$, *weak* if $m(w) < 0$ and *strong* if $m(w) > 0$.

Definition

A factorial set S is *neutral* if every nonempty word is neutral. It is *weak* (resp. *strong*) if every word is weak or neutral (resp. strong or neutral).

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Theorem [Dolce, Perrin (2016)]

The factor complexity of a factorial set S is given by $p_0 = 1$ and for every $n \geq 1$:

(i) $p_n = (\text{Card}\,(A) - \chi(S))\,n + \chi(S)$ if S is neutral ;

(ii) $p_n \leq (\text{Card}\,(A) - \chi(S))\,n + \chi(S)$ if S is weak ;

(iii) $p_n \geq (\text{Card}\,(A) - \chi(S))\,n + \chi(S)$ if S is strong.

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Theorem [Dolce, Perrin (2016)]

Let S be a recurrent set and $X \subset S$ a finite S-maximal bifix code. One has :

(i) $\text{Card}\,(X) = (\text{Card}\,(A) - \chi(S))\, d_S(X) + \chi(S)$ if S is neutral ;

(ii) $\text{Card}\,(X) \leq (\text{Card}\,(A) - \chi(S))\, d_S(X) + \chi(S)$ if S is weak ;

(iii) $\text{Card}\,(X) \geq (\text{Card}\,(A) - \chi(S))\, d_S(X) + \chi(S)$ if S is strong.

Acyclic and connected sets

The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in S\}, \\ R(w) &= \{a \in A \mid wa \in S\}, \\ B(w) &= \{(a, b) \in A \mid awb \in S.\} \end{aligned}$$

Definition

A factorial set S is called a *tree set* if the graph $\mathcal{E}(w)$ is a tree for all nonempty $w \in S$. It is *acyclic* (resp. *connected*) if for every $w \in S$ the graph $\mathcal{E}(w)$ is acyclic (resp. connected).

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Proposition

If S is connected then it is strong.
If S is acyclic then it is weak. Moreover, in that case, $c(w) = 1 - m(w)$ is the number of connected components of $\mathcal{E}(w)$.

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Maximal Bifix Decoding Theorem [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone (2015)]

The family of biextendable acyclic sets is closed under maximal bifix decoding.

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Freeness Theorem [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone (2015)]

A set $S \subset A^+$ is acyclic if and only if any bifix code $X \subset S$ is a *free* subset of the free group on A (i.e. X is a basis of $\langle X \rangle$).

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Saturation Theorem [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone (2015)]

Let S be an acyclic set. Then any bifix code $X \subset S$ is *saturated* in S (i.e. $X^* \cap S = \langle X \rangle \cap S$).

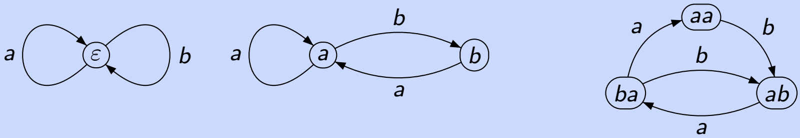
Rauzy graphs

Theorem [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone (2014)]

Let S be a recurrent connected set.
The group described by a Rauzy graph w.r.t. any vertex is the free group on A.

Example (Fibonacci)

$G_0(S)$ $G_1(S)$ $G_2(S)$



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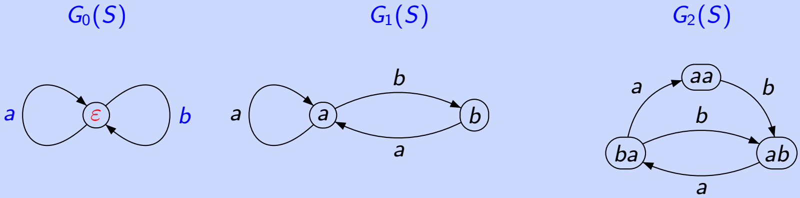

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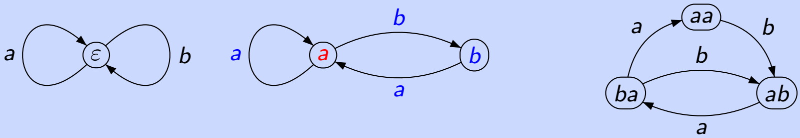

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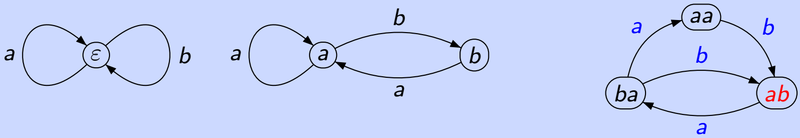

$\langle a(ba)^* ab \rangle$

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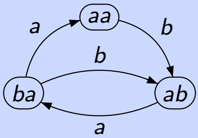
$G_2(S)$



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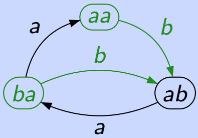
Example (Fibonacci)

$$G_2(S) \rightsquigarrow G_1(S)$$



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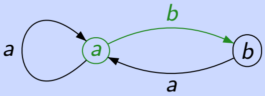

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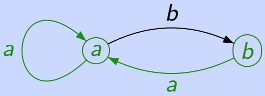

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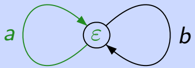

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Example (Fibonacci)

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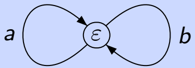


Specular groups

Given an involution $\theta : A \to A$ (possibly with some fixed point), let us define

$$G_\theta = \langle a \in A \mid a \cdot \theta(a) = 1 \text{ for every } a \in A \rangle.$$

$G_\theta = \mathbb{Z}^i * (\mathbb{Z}/2\mathbb{Z})^j$ is a *specular group* of type (i, j), and $\text{Card}\,(A) = 2i + j$ is its *symmetric rank*.

Example

Let $A = \{a, b, c, d\}$ and let θ be the involution which exchanges b, d and fixes a, c, i.e.,

$$G_\theta = \langle a, b, c, d \mid a^2 = c^2 = bd = db = 1 \rangle.$$

$G_\theta = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^2$ is a specular group of type $(1, 2)$ and symmetric rank 4.

Monoidal basis

A subset of a group G is called *symmetric* if it is closed under taking inverses (under θ).

Example

The set $X = \{a, adc, b, cba, d\}$ is symmetric, for $\theta : b \leftrightarrow d$ fixing a, c.

A set X in a specular group G is called a *monoidal basis* of G if :

- it is symmetric ;
- the monoid that it generates is G ;
- any product $x_1 x_2 \cdots x_m$ such that $x_k x_{k+1} \neq 1$ for every k is distinct of 1.

Example

The alphabet A is a monoidal basis of G_θ.

The *symmetric rank* of a specular group is the cardinality of any monoidal basis.

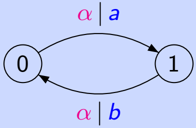
Doubling transducers

A *doubling transducer* is a transducer with set of states $Q = \{0, 1\}$ such that :

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

Example

$\Sigma = \{\alpha\}$
$A = \{a, b\}$



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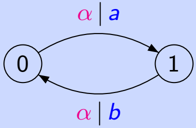

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The *image* of a set T is $\delta(T) = \delta_0(T) \cup \delta_1(T)$

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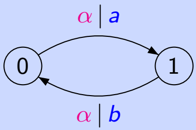

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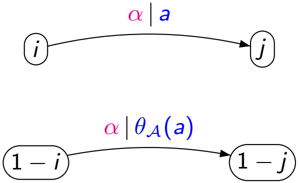
$\delta(\mathrm{Fac}(\alpha^\omega)) = (ab)^\omega \cup (ba)^\omega$

Doubling transducers

Proposition [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone (2015)]

The image of a tree set of characteristic 1 closed under reversal by a doubling map is a specular set (and, in particular, a tree set of characteristic 2).

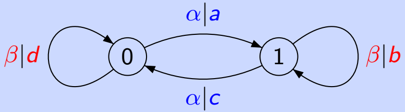


Doubling transducers

Example (two possible doublings of Fibonacci)

- $\mathrm{Fac}(abaababa\cdots) \cup \mathrm{Fac}(cdccdcdc\cdots)$

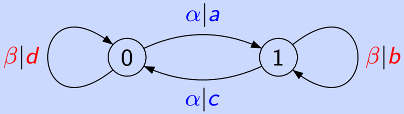
- $\mathrm{Fac}(abcabcda\cdots) \cup \mathrm{Fac}(cdacdabc\cdots)$

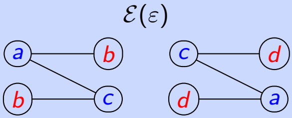


Odd and even words

A letter is said to be *even* if its two occurences (as a element of $L(\varepsilon)$ and of $R(\varepsilon)$) appear in the same tree of $\mathcal{E}(\varepsilon)$. Otherwise it is said to be *odd*.

Example (doubling of Fibonacci)





The letters b, d are even, while the letters a, c are odd.

A word is said to be *even* if it has an even number of odd letters. Otherwise it is said to be *odd*.

Return words in specular sets

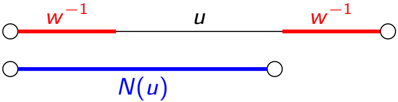
Theorem [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone (2015)]

Let S be a recurrent specular set. One has

- $\text{Card}\,(\mathcal{R}_S(w)) = \text{Card}\,(A) - 1$ for any $w \in S$;
- $\text{Card}\,(\mathcal{CR}_S(X)) = \text{Card}\,(X) + \text{Card}\,(A) - 2$ for any finite bifix $X \subset S$ code with empty kernel ;
- $\text{Card}\,(\mathcal{MR}_S(w)) = \text{Card}\,(A)$ for any $w \in S$ s.t. w, w^{-1} do not overlap.

Definition

A *mixed return word* to w (not overlapping with w^{-1}) is the word $N(u)$ obtained from $u \in \mathcal{CR}_S(\{w, w^{-1}\})$ erasing the prefix if it is w and the suffix if it is w^{-1}



Palindromes in tree sets

Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]

A word of length n has at most $n+1$ palindromes factors.

A word with maximal number of palindromes is *rich* (or *full*).
A factorial set is *rich* if all its elements are rich.

Example (Fibonacci)

$\mathrm{Pal}(abaab) = \{\varepsilon, a, b, aa, aba, baab\}$

Theorem [Berthé, De Felice, Delecroix, Dolce, Leroy, Perrin, Reutenauer, Rindone (2016)]

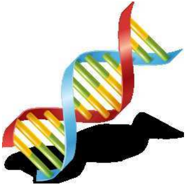
Recurrent tree sets of characteristic 1 closed under reversal are rich.

σ-palindromes

Let σ be an antimorphism.
A word w is a *σ-palindrome* if $w = \sigma(w)$.

Example

Let $\sigma : \mathtt{A} \leftrightarrow \mathtt{T},\ \mathtt{C} \leftrightarrow \mathtt{G}$.
The word $\mathtt{CTTAAG}$ is a σ-palindrome.



Theorem [Starosta (2001) ; Blondin Massé, Brlek (?)]

Let $\gamma_\sigma(w)$ be the number of transpositions of σ affecting w. Then

$$\mathrm{Card}\,(\mathrm{Pal}_\sigma(w)) = |w| + 1 - \gamma_\sigma(w)$$

A word (resp. set) is *σ-rich* if the equality holds (resp. for all its elements).

G-palindromes

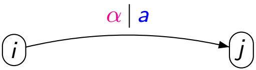
Let G be a group of morphisms and antimorphisms, containing at least one antimorphism.
A word w is a *G-palindrome* if there exists a nontrivial $g \in G$ s.t. $w = g(w)$.

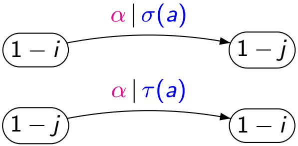
Theorem [Berthé, De Felice, Delecroix, Dolce, Leroy, Perrin, Reutenauer, Rindone (2016)]

Specular sets obtained as image under a doubling transducer $\mathcal{A}$ are $G_{\mathcal{A}}$-rich.

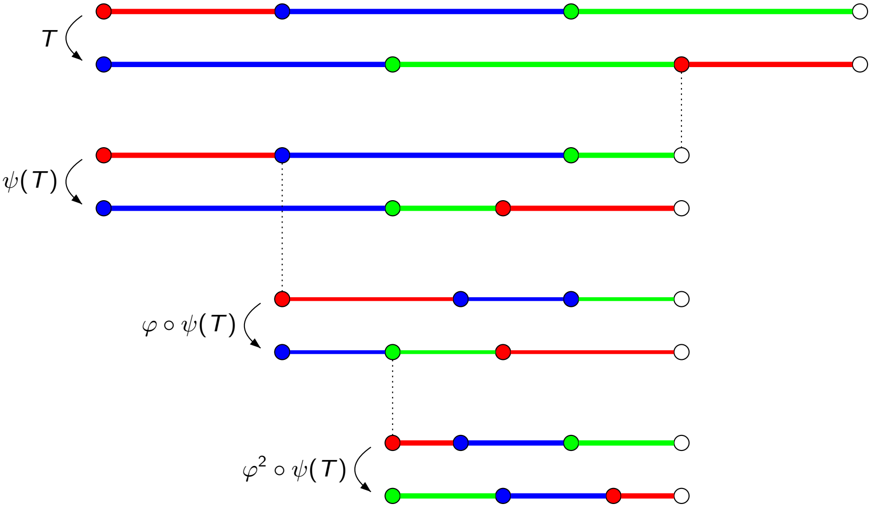
$$G_{\mathcal{A}} = \{\mathrm{id}, \sigma, \tau, \sigma\tau\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

with σ an antimorphism and τ a morphism.

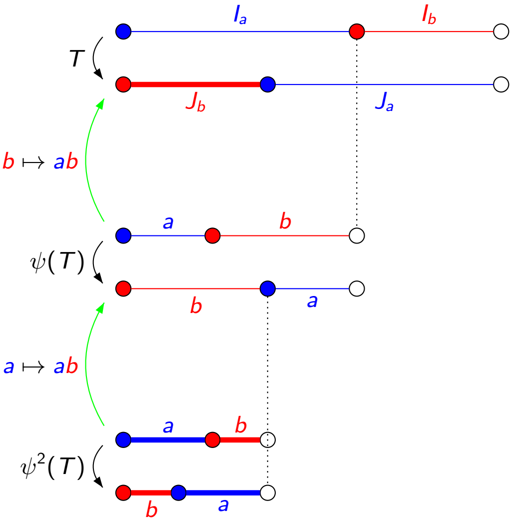




Two-sided Rauzy induction



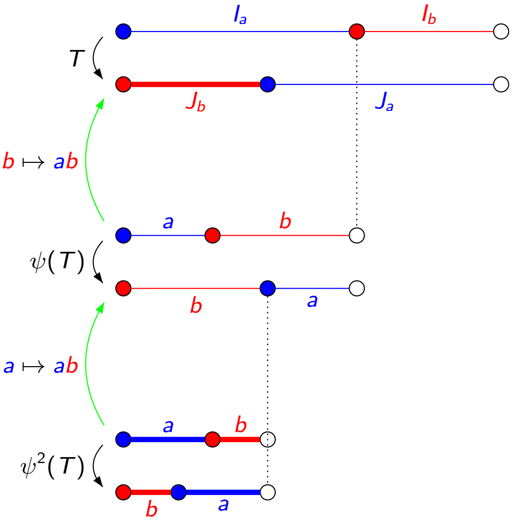
Return Theorem for interval exchanges



$$\mathcal{R}_S(b) = \{aab, ab\}$$

$$\theta : \begin{cases} a & \mapsto & ab & \mapsto & aab \\ b & \mapsto & b & \mapsto & ab \end{cases}$$

Return Theorem for interval exchanges



$$\mathcal{R}_S(b) = \{aab, ab\}$$

$$\theta : \begin{cases} a & \mapsto & ab & \mapsto & aab \\ b & \mapsto & b & \mapsto & ab \end{cases}$$

For $w = b_0 b_1 \cdots b_{m-1}$ one has :

$$I_w = I_{b_0} \cap T^{-1}(I_{b_1}) \cap \ldots \cap T^{-m+1}(I_{b_{m-1}})$$

and $J_w = T^m(I_w)$, that is :

$$J_w = T^m(I_{b_0}) \cap T^{m-1}(I_{b_1}) \cap \ldots \cap T(I_{b_{m-1}})$$

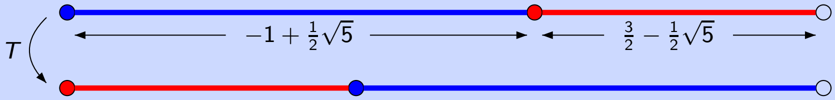
Interval exchange over a quadratic field

Theorem [*folklore*, Dolce (2014)]

Let T be a regular interval exchange transformation defined over a quadratic field. Then $\mathcal{L}(T)$ is a primitive morphic set.

Example (Fibonacci, $\alpha = (3-\sqrt{5})/2$)

$$|I_a|, |I_b| \in \mathbb{Q}[\sqrt{5}]$$



$\mathcal{S}$-adic representation

Definition

Let $\mathcal{S}$ be a set of morphisms. A set T is called *$\mathcal{S}$-adic* if $T = \bigcap_{n \in \mathbb{N}} \mathrm{Fac}\left(\sigma_0 \cdots \sigma_n \left(A_{n+1}^*\right)\right)$ where $\sigma_n : A_{n+1}^* \to A_n^*$ is a morphism of $\mathcal{S}$.
The sequence $(\sigma_0, \sigma_1, \ldots)$ is called an *$\mathcal{S}$-representation* of T.

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Theorem [S. Ferenczi (1996)]

If T is an aperiodic set. T is uniformly recurrent $\Leftrightarrow$ it has a primitive $\mathcal{S}$-adic representation.

Theorem [Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone (2015)]

If T is a recurrent tree set of characteristic 1, then it has a primitive $\mathcal{S}_e$-adic representation.

$\mathcal{S}_e$ formed by permutations and

$$\alpha_{a,b}(c) = \begin{cases} ab & \text{if } c = a, \\ c & \text{otherwise} \end{cases} \qquad \tilde{\alpha}_{a,b}(c) = \begin{cases} ba & \text{if } c = a, \\ c & \text{otherwise.} \end{cases}$$

The end


DRINK ME


EAT ME