Generalized Lyndon words

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joint work with
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Generalized lexicographical order

$$(A^\infty, <)$$

$$u < v \quad :\Longleftrightarrow \quad v = us \text{ or } \begin{cases} u = pau' \\ v = pbv' \\ a < b \end{cases}$$

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- u is a proper prefix of v, or
- $u = pas$, $v = pbt$ for some $p \in A^*$, $s, t \in A^\infty$, and $a, b \in A$ s.t. $a <_{|p|+1} b$.

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Examples

- Classical order $(<)$: $\mathtt{a} <_n \mathtt{b}$ for all $n \geq 1$.

$$\mathtt{a} < \mathtt{aa} < \mathtt{ab} < \mathtt{aba} < \mathtt{baa}$$

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- Alternate order $(<_{alt})$: $\begin{cases} \mathtt{a} <_n \mathtt{b} & \text{if } n \equiv 1 \pmod 2 \\ \mathtt{b} <_n \mathtt{a} & \text{otherwise.} \end{cases}$

$$a_1 a_2 a_3 \cdots <_{alt} b_1 b_2 b_3 \cdots \quad \Longleftrightarrow \quad a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}} < b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{\cdots}}}$$

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- Prime order $(<_\pi)$: $\begin{cases} \mathtt{a} <_n \mathtt{b} & \text{if } n \text{ is prime} \\ \mathtt{b} <_n \mathtt{a} & \text{otherwise.} \end{cases}$

$$\mathtt{baab} <_\pi \mathtt{bab} <_\pi \mathtt{aba} <_\pi \mathtt{abab} <_\pi \mathtt{abaa}$$

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- Vowels First order $(<_{vf})$: $\mathtt{a} <_{vf} \cdots <_{vf} \mathtt{y} <_{vf} \mathtt{b} <_{vf} \mathtt{c} <_{vf} \cdots <_{vf} \mathtt{z}$
- Consonants First order $(<_{cf})$: $\mathtt{b} <_{cf} \mathtt{c} <_{vf} \cdots <_{vf} \mathtt{z} <_{vf} \mathtt{a} <_{vf} \cdots <_{vf} \mathtt{y}$

Generalized lexicographical order

inverse order

The *inverse (generalized) order* $\widetilde{<}$, obtained by reversing all the orders $<_n$, is also a generalized order.

Examples

- $\texttt{baa} <_\pi \texttt{bab} <_\pi \texttt{aab} <_\pi \texttt{aba}$
- $\texttt{aba} \mathrel{\widetilde{<}_\pi} \texttt{aab} \mathrel{\widetilde{<}_\pi} \texttt{bab} \mathrel{\widetilde{<}_\pi} \texttt{baa}$.

Infinite order

$$u \preceq v \quad :\Longleftrightarrow \quad u^\omega \leq v^\omega$$

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When $|u| = |v|$ one has $u < v \Leftrightarrow u^\omega < v^\omega$. In general, this is not true.

Example

$\mathtt{ab} < \mathtt{aba}$ but $\mathtt{aba} \prec \mathtt{ab}$, since $(\mathtt{aba})^\omega < (\mathtt{ab})^\omega$.

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We can also consider the generalized lexicographical infinite order.

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$(\mathtt{ba})^\omega <_\pi \mathtt{b}^\omega <_\pi \mathtt{a}^\omega <_\pi (\mathtt{ab})^\omega$.

$u^\omega = v^\omega \quad \Longleftrightarrow \quad u$ and v are power of a common word ($\Longleftrightarrow \quad uv = vu$).

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The following conditions are equivalent for nonempty words $u, v \in A^*$.

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Examples

Let $u = \mathtt{a}$ and $v = \mathtt{ab}$. Then

$$\mathtt{a}^\omega <_\pi (\mathtt{a.ab})^\omega <_\pi (\mathtt{ab.a})^\omega <_\pi (\mathtt{ab})^\omega$$

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- petr is a $<_{cf}$–Lyndon word.
- zuzana is a $\widetilde{<}$–Lyndon word.

$$(\mathtt{zuzana})^\omega \;\widetilde{<}\; (\mathtt{zana.zu})^\omega \;\widetilde{<}\; (\mathtt{uzana.z})^\omega \;\widetilde{<}\; (\mathtt{na.zuza})^\omega \;\widetilde{<}\; (\mathtt{ana.zuz})^\omega \;\widetilde{<}\; (\mathtt{a.zuzan})^\omega$$

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- `lubka` is a $\widetilde{<}_{vf}$–Lyndon word
- `vaclav` is a $<_\pi$-Lyndon word

$$(\texttt{vaclav})^\omega <_\pi (\texttt{vvacla})^\omega <_\pi (\texttt{lavvac})^\omega <_\pi (\texttt{clavva})^\omega <_\pi (\texttt{aclavv})^\omega <_\pi (\texttt{avvacl})^\omega$$

Generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

A word w is a generalized Lyndon word if and only if for any non trivial factorization $w = uv$ one has :

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Example

- $(\mathtt{zu})^\omega \stackrel{\sim}{<} (\mathtt{zana})^\omega$

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Factorization into generalized Lyndon words

Theorem [Reutenauer (2005), D., Restivo, Reutenauer (2018)]

Each word $w \in A^+$ can be factorized in a unique way as $w = \ell_1 \ell_2 \cdots \ell_n$, with ℓ_i generalized Lyndon words s.t. $\ell_1^\omega \geq_g \ell_2^\omega \geq_g \cdots \geq_g \ell_n^\omega$.

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Example

- The factorization in classical Lyndon word of $\texttt{pelantova}$ is $(\texttt{pel})(\texttt{antov})(\texttt{a})$, since

$$(\texttt{pel})^\omega > (\texttt{antov})^\omega > \texttt{a}^\omega$$

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- The factorization in $<_{alt}$–Lyndon word of baradaran is (b)(arad)(aran), since

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Moreover, ℓ_n is

- the shortest suffix s of w s.t. s^ω is minimum,
- the longest suffix of w which is a generalized Lyndon word.

Example

- The factorization in classical Lyndon word of `pelantova` is `(pel)(antov)(a)`, since
$$(\texttt{pel})^\omega > (\texttt{antov})^\omega > \texttt{a}^\omega$$
- The factorization in $<_{alt}$–Lyndon word of `baradaran` is `(b)(arad)(aran)`, since
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Classical Lyndon words

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The following conditions are equivalent for nonempty words $u, v \in A^*$.

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Theorem [Bergman (1969)]

If $u^\omega < v^\omega$ then $u^\omega \; < \; (uv)^\omega \; < \; (vu)^\omega \; < \; v^\omega$.

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Theorem [Bergman (1969)]

If $u^\omega < v^\omega$ then $u^\omega < (uv)^\omega < (vu)^\omega < v^\omega$.

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

Factorization into classical Lyndon words

Theorem [Ufnarovskij (1995)]

Let $w = \ell_1 \ell_2 \cdots \ell_n$ the unique non-increasing factorization of w in Lyndon word. Then

- $\ell_1^\omega > (\ell_2 \cdots \ell_n)^\omega$
- ℓ_1 is the shortest nontrivial prefix p s.t. $w = ps$ and $p^\omega \geq s^\omega$,
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Example

- $(\mathtt{pel})^\omega > (\mathtt{antov.a})^\omega$,
- $(\mathtt{pel})^\omega > (\mathtt{pel.antov.a})^\omega$.

Galois words

The *alternating lexicographical order* $<_{alt}$ (w.r.t. an order $<$) is the generalized lexicographical order defined by the sequence $(<_n)_{n\geq 1}$ with

$$<_n = \begin{cases} < & \text{if } n \equiv 1 \pmod 2 \\ \tilde{<} & \text{otherwise.} \end{cases}$$

Example

$(\mathtt{ab})^\omega <_{alt} \mathtt{a}^\omega <_{alt} \mathtt{b}^\omega <_{alt} (\mathtt{ba})^\omega$.

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Example

$(\mathtt{ab})^\omega <_{alt} \mathtt{a}^\omega <_{alt} \mathtt{b}^\omega <_{alt} (\mathtt{ba})^\omega$.

A *Galois word* is a generalized Lyndon word for an alternating lexicographical order.

Example

The following are Galois words : b, ac, bc, aba, abb, abaa, acab.

Characterization of Galois words

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Theorem [D., Restivo, Reutenauer (2018)]

w is a Galois word if and only if for any nontrivial factorization $w = ps$ one has

$$\begin{cases} p^\omega <_{alt} w^\omega & \text{if } |p| \text{ is even,} \\ p^\omega >_{alt} w^\omega & \text{if } |p| \text{ is odd.} \end{cases}$$

Factorization into Galois words

Theorem [D., Restivo, Reutenauer (2018)]

Let $w = g_1 g_2 \cdots g_n$ with g_i Galois words s.t. $g_1^\omega \geq_{alt} g_2^\omega \geq_{alt} \cdots \geq_{alt} g_n^\omega$.
Let m be the multiplicity of g_1.
Let p be the shortest nontrivial prefix of w s.t.

$$p^\omega \geq_{alt} w^\omega \text{ if } |p| \text{ is even} \quad \text{and} \quad p^\omega \leq_{alt} w^\omega \text{ if } |p| \text{ is odd.} \qquad (\star)$$

Then

(i) if $|g_1|$ is odd, m is even, and $m < n$, then $p = g_1^2$,

(ii) otherwise, $p = g_1$.

Example

Let $w = (\mathtt{abb})(\mathtt{abb})(\mathtt{abaa})$.
$((\mathtt{abb})^2)^\omega >_{alt} w^\omega$ and each proper prefix of $(\mathtt{abb})^2$ does not satisfy condition $(\star)$.

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- The word $\mathtt{aba}^\omega$ is a Galois word.

Factorization of infinite words

An infinite word $w \in A^\omega$ is a (generalized) *Lyndon* word if $w < v$ for each suffix v of w.

Theorem [Siromoney, Mathew, Dare, Subramanian (2005)]

Each infinite word $w \in A^\omega$ can be factorized in a unique was either as

- an infinite product of non-increasing finite classical Lyndon words

$$w = \ell_1 \ell_2 \ell_3 \cdots \quad \text{with} \quad \ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots$$

- a finite product of non-increasing classical Lyndon words

$$w = \ell_1 \cdots \ell_{n-1} \ell_n \quad \text{with} \quad \ell_1 \geq \cdots \geq \ell_{n-1} \geq \ell_n, \quad |\ell_1 \cdots \ell_n| < \infty \quad \text{and} \quad \ell_n \ \text{infinite}.$$

Factorization of infinite words

An infinite word $w \in A^\omega$ is a (generalized) *Lyndon* word if $w < v$ for each suffix v of w.

Theorem [Burcroff, Winsor (2019) ; Postic, Zamboni (2019)]

Let $<_g$ be a generalized order.
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- a finite product of non-increasing classical Lyndon words

$$w = \ell_1 \cdots \ell_{n-1} \ell_n \quad \text{with} \quad \ell_1^\omega \geq_g \cdots \geq_g \ell_{n-1}^\omega \geq_g \ell_n, \quad |\ell_1 \cdots \ell_n| < \infty, \quad \ell_n \text{ infinite}.$$

Complete trees

The set of *complete trees* over A is defined recursively as follows :

- each letter $a \in A$ is a tree ;
- if $\mathfrak{t}_1, \mathfrak{t}_2$ are trees, then $(\mathfrak{t}_1, \mathfrak{t}_2)$ is a tree.

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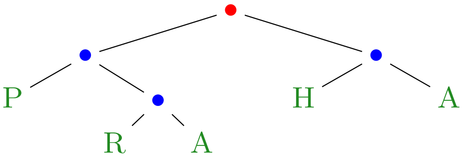

We will use the classical notions of *root*, *internal node* and *leaf* for a tree.

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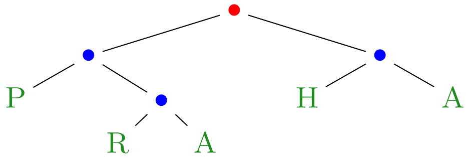

We will use the classical notions of *root*, *internal node* and *leaf* for a tree.
The *foliage* $\varphi(\mathfrak{t})$ of a tree $\mathfrak{t}$ is defined as :

- $\varphi(a) = a$ for any $a \in A$,
- $\varphi((\mathfrak{t}_1, \mathfrak{t}_2)) = \varphi(\mathfrak{t}_1)\varphi(\mathfrak{t}_2)$ for any two trees $\mathfrak{t}_1, \mathfrak{t}_2$.

Left standard factorization

Let w be a Lyndon word of length at least 2.
The *left standard factorization* of w is the factorization $w = uv$, where u is the longest nonempty proper prefix of w which is a Lyndon word.

Example

The left standard factorization of $aabaacab$ is $(aabaac)(ab)$.

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Proposition

Both u and v are Lyndon words.
Moreover, either v is a letter or $v = v_1 v_2$, and $v_1 \leq u$.

Example

The left standard factorization of $aabaacab$ is $(aabaac)(ab)$.
The left standard factorization of ab is $(a)(b)$, and $a \leq aabaac$.

Left Lyndon tree

Let $w \in A^+$ be a Lyndon word. Its *left Lyndon tree* $\mathcal{L}(w)$ is defined as :

- $\mathcal{L}(a) = a$ for each letter $a \in A$;
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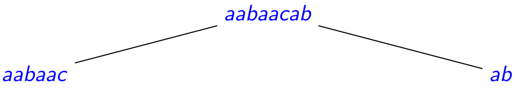

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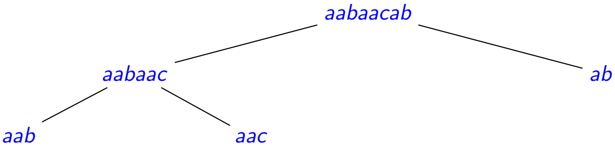

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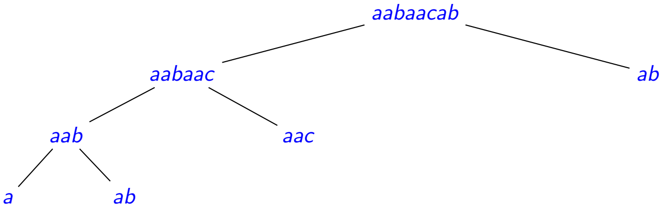

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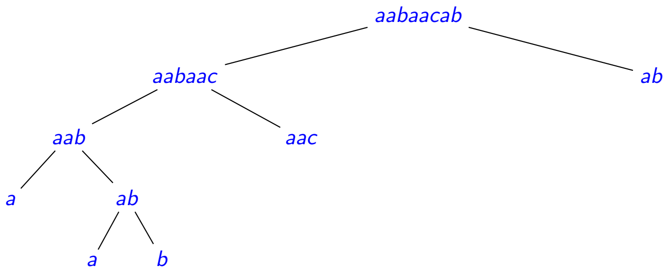

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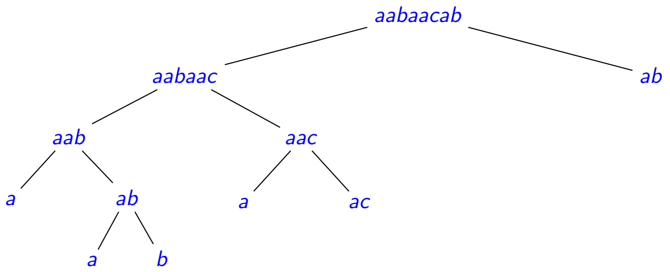

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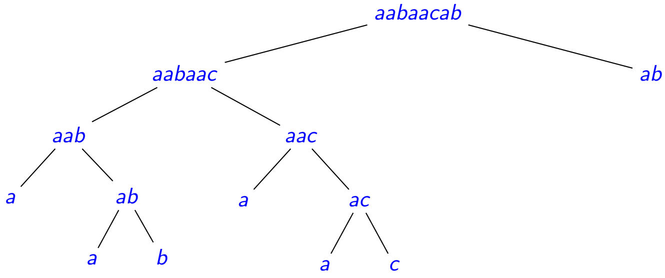

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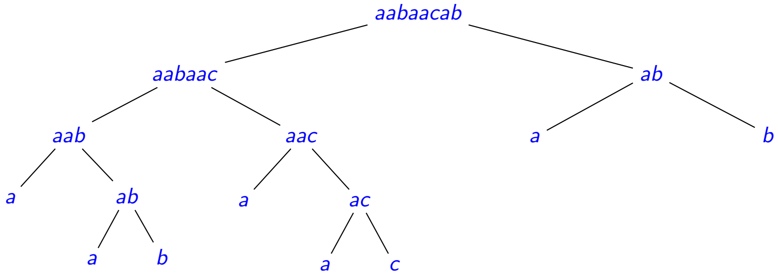

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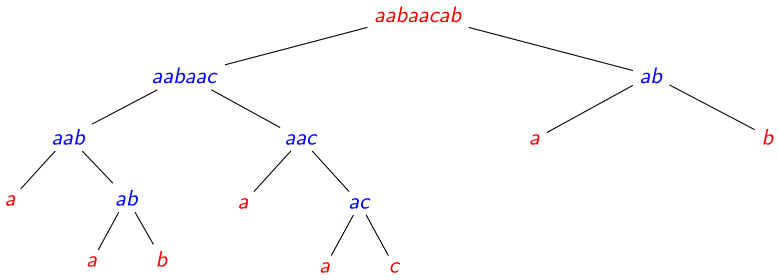

Clearly $\varphi(\mathcal{L}(w)) = w$.

Prefix standardization

$$u \prec v \quad :\Longleftrightarrow \quad \begin{cases} u^\omega < v^\omega & \text{or} \\ u^\omega = v^\omega & \text{and } |u| > |v|. \end{cases}$$

Example

$aa \prec a \prec ab \prec ba \prec b$.

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$$1$$

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$$21 \quad 3$$

$aa \prec a \prec aabaa$

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$$\begin{array}{rl} w = & aabaacab \\ & 21543\ \ 6 \end{array}$$

$$aa \prec a \prec aabaa \prec aaba \prec aab \prec aabaaca$$

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Example

$$w = \begin{array}{l} aabaacab \\ 2154376 \end{array}$$

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Example

$$w = \begin{matrix} aabaacab \\ 21543768 \end{matrix}$$

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Left Cartesian tree

Theorem [Ufnarovskij (1995)]

w is a Lyndon word if and only if for any nontrivial factorization $w = ps$ one has $p^\omega < w^\omega$.

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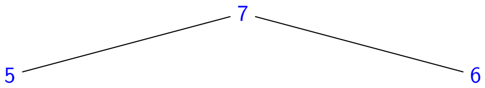

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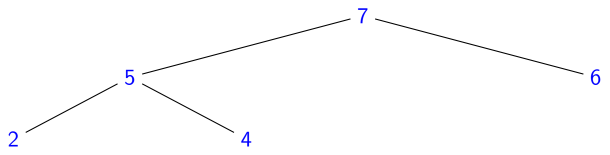

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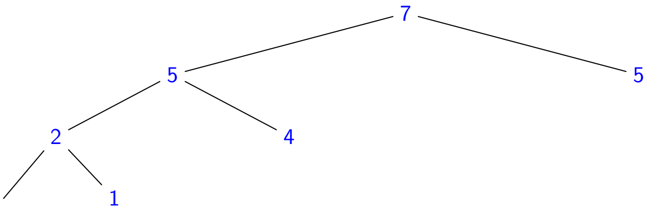

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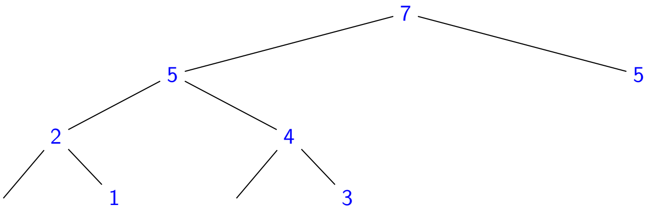

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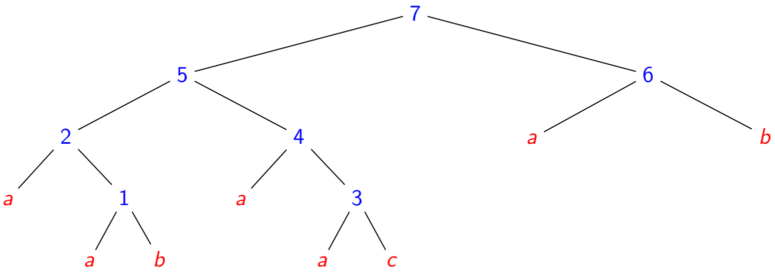


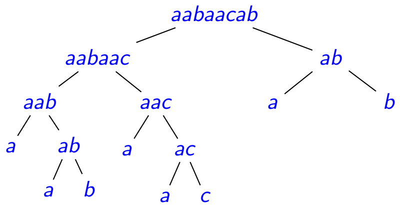
We complete in such a way that $\varphi(\mathcal{C}(w)) = w$.

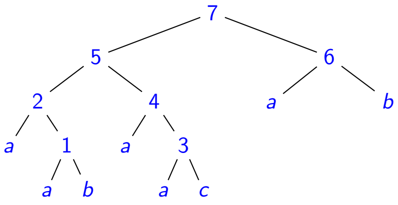
Equivalence of trees

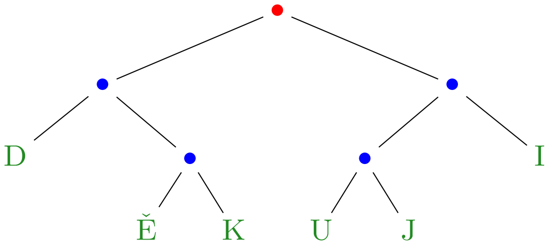
Theorem [D., Restivo, Reutenauer (2019)]

Let w be a Lyndon word. Then $\mathcal{L}(w) = \mathcal{C}(w)$.

$$aabaacab \quad \longleftrightarrow \quad 2154376$$



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