Inequalities for means of chords and related isoperimetric problems

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Talk overview

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- *A discrete analogue* of our problem
- *Summary and outlook*

Motivation

Isoperimetric problems are traditional in mathematical physics. Recall, e.g., the *Faber-Krahn inequality* for the Dirichlet Laplacian $-\Delta_D^M$ in a compact $M \subset \mathbb{R}^2$: among all regions with a fixed area the ground state is *uniquely minimized by the circle*,

$$\inf \sigma(-\Delta_D^M) \geq \pi\, j_{0,1}^2\, |M|^{-1};$$

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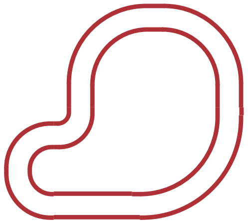
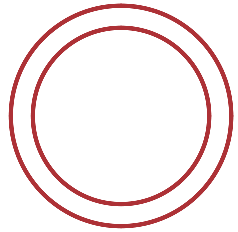
similarly *a ball* is a minimizer for a compact $M \subset \mathbb{R}^d$, $d \geq 3$

Another classical example is the *PPW conjecture* proved by *Ashbaugh* and *Benguria*: in the 2D situation we have

$$\frac{\lambda_2(M)}{\lambda_1(M)} \leq \left(\frac{j_{1,1}}{j_{0,1}}\right)^2$$

Notice that topology is important

If M is not simply connected, rotational symmetry may again lead to an extremum but its nature can be different. Recall a *a strip of fixed length and width* [E.-Harrell-Loss'99]

ground state of  $<$ ground state of 

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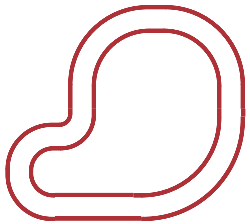
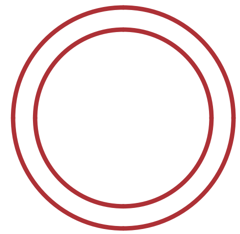
ground state of  $<$ ground state of 

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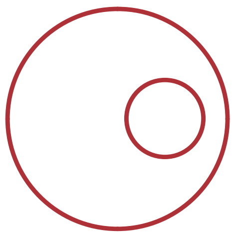
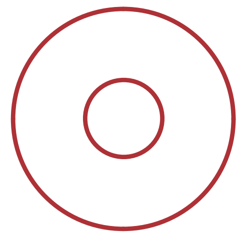
Another example is a *circular obstacle in circular cavity* [Harrell-Kröger-Kurata'01]

ground state of  $<$ ground state of 

whenever the obstacle is off center

Singular Schrödinger operators

Topology loses meaning when the confinement is due to a potential. For simplicity, we suppose is a singular one,

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \quad \alpha > 0\,,$$

in $L^2(\mathbb{R}^2)$, where Γ is a loop in the plane; we suppose that it has no *zero-angle* self-intersections

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$H_{\alpha,\Gamma}$ can be naturally associated with the quadratic form,

$$\psi \mapsto \|\nabla\psi\|^2_{L^2(\mathbb{R}^2)} - \alpha \int_\Gamma |\psi(x)|^2 \mathrm{d}x\,,$$

which is closed and below bounded in $W^{1,2}(\mathbb{R}^2)$; the second term makes sense in view of Sobolev embedding. This definition also works for various “wilder” sets Γ

Definition by boundary conditions

Since Γ is *piecewise smooth* with *no cusps* we can use an *alternative definition* by boundary conditions: $H_{\alpha,\Gamma}$ acts as $-\Delta$ on functions from $W^{2,1}_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial \psi}{\partial n}(x)\bigg|_{+} - \frac{\partial \psi}{\partial n}(x)\bigg|_{-} = -\alpha \psi(x)$$

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Remarks:

- this definition has an illustrative meaning which corresponds to a δ potential in the cross cut of Γ
- using the form associated with $H_{\alpha,\Gamma}$ one can check directly that $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma})$ *is not void* for any $\alpha > 0$; one has, of course, $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [0,\infty)$. We will ask about Γ of a fixed length which *maximizes the ground state*

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Remark: The latter has to be renormalized. The question makes sense because the divergent factor comes from the short-distance behavior being shape-independent

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We are going to show that both the mentioned problems reduce essentially to *the same geometric question*

Inequalities for L^p norms of chords

It is convenient to work in an arbitrary dimension $d \geq 2$. Let Γ be a piecewise differentiable function $\Gamma:\ [0,L] \to \mathbb{R}^d$ such that $\Gamma(0)=\Gamma(L)$ and $|\dot\Gamma(s)|=1$ for any $s\in[0,L]$. Consider *chords* corresponding to a *fixed arc length* $u \in (0, \frac12 L]$; we are interested in the inequalities

$$C_L^p(u): \quad c_\Gamma^p(u) := \int_0^L |\Gamma(s+u)-\Gamma(s)|^p\, \mathrm{d}s \leq \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L}\,, \quad p>0$$

$$C_L^{-p}(u): \quad c_\Gamma^{-p}(u) := \int_0^L |\Gamma(s+u)-\Gamma(s)|^{-p}\, \mathrm{d}s \geq \frac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}}\,, \quad p>0$$

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The right sides correspond to the maximally symmetric case, the *planar circle*. It is clear that the inequalities are invariant under scaling, so without loss of generality we may fix the length, say, to $L = 2\pi$. Notice also that for $p = 0$ the inequalities $C_L^p(u)$ and $C_L^{-p}(u)$ turn into trivial identities.

Simple properties

Using convexity of $x \mapsto x^\alpha$ in $(0, \infty)$ for $\alpha > 1$ we get

Proposition: $C_L^p(u) \Rightarrow C_L^{p'}(u)$ if $p > p' > 0$

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The norm can be expressed through curvature of Γ. Using then a Fourier analysis, one can prove

Proposition [E'05b]: If Γ is C^2, the inequality $C_L^2(u)$, and thus also $C_L^p(u)$ for $|p| \le 2$, holds *locally*.

The global result

Theorem [Lükő'66; Abrams et al.'03; E-Harrell-Loss'05]: Let Γ be piecewise C^1 with no cusps. Then $C_L^2(u)$ is valid for any $u \in (0, \frac{1}{2}L]$, and the inequality is strict unless Γ is a planar circle.

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Proof: Without loss of generality we put $L = 2\pi$ and write

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n \, \mathrm{e}^{ins}$$

with $c_n \in \mathbb{C}^d$. Since $\Gamma(s) \in \mathbb{R}^d$ the coefficients have to satisfy $c_{-n} = \bar{c}_n$; the absence of c_0 can be always achieved by a choice of the origin of the coordinate system.

In view of the Weierstrass theorem and continuity of the functional in question, we may suppose that Γ is C^2, apart of the last part of the theorem.

Proof, continued

If Γ is C^2 its derivative is a sum of the uniformly convergent Fourier series

$$\dot{\Gamma}(s) = i \sum_{0 \neq n \in \mathbb{Z}} n c_n \, \mathrm{e}^{ins}$$

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By assumption, $|\dot{\Gamma}(s)| = 1$, and hence from the relation

$$2\pi = \int_0^{2\pi} |\dot{\Gamma}(s)|^2 \,\mathrm{d}s = \int_0^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} nm\, c_m^* \cdot c_n \,\mathrm{e}^{i(n-m)s} \,\mathrm{d}s\,,$$

where c_m^* denotes the row vector $(\bar{c}_{m,1}, \dots, \bar{c}_{m,d})$ and dot marks the inner product in $\mathbb{C}^d$, we infer that

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1$$

Proof, continued

In a similar way we can rewrite the right-hand side expression of $C^2_{2\pi}(u)$ using the Parseval relation as

$$\int_0^{2\pi} \left| \sum_{0\neq n\in\mathbb{Z}} c_n \left(\mathrm{e}^{inu} - 1\right) \mathrm{e}^{ins} \right|^2 \mathrm{d}s = 8\pi \sum_{0\neq n\in\mathbb{Z}} |c_n|^2 \left(\sin \frac{nu}{2}\right)^2$$

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Thus the sought inequality is equivalent to

$$\sum_{0\neq n\in\mathbb{Z}} n^2 |c_n|^2 \left(\frac{\sin\frac{nu}{2}}{n\sin\frac{u}{2}} \right)^2 \leq 1$$

and it is sufficient to prove that $|\sin nx| \leq n\,\sin x$ holds for all positive integers n and all $x \in (0, \frac{1}{2}\pi]$.

Proof, continued

We use induction. The claim is valid for $n = 1$ and

$$(n+1)\sin x \mp \sin(n+1)x = n\sin x \mp \sin nx\cos x + \sin x(1 \mp \cos nx)\ ,$$

so if it holds for n, the sum of the first two terms at the *rhs* is non-negative, and the same is clearly true for the last one

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We also see that if $|\sin nx| < n\,\sin x$ the inequality is strict for $n+1$ as well. Since this is true for for $n = 2$, equality can occur only for $n = 1$. Hence $C^2_{2\pi}(u)$ is strict unless $c_n = 0$ for $|n| \geq 2$, being saturated only if the jth projection of Γ equals

$$\Gamma_j(s) = 2|c_{1,j}|\cos(s + \arg c_{1,j})\,.$$

Furthermore, $|\dot{\Gamma}(s)| = 1$ can be true only if there is a basis in $\mathbb{R}^d$ where $c_{1,1} = ic_{1,2} = \frac{1}{2}$ and $c_{1,j} = 0$ for $j = 3, \dots, d$, in other words, if Γ is a planar circle

Proof, conclusion

It remains to check that the inequality cannot be saturated for a curve Γ that is not C^2, so that the sum $\sum_{0\neq n\in\mathbb{Z}} n^2|c_n|^2$ diverges

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This would require

$$\frac{\sum_{1\le n\le N} n^2|c_n|^2 \left(\frac{\sin\frac{nu}{2}}{n\sin\frac{u}{2}}\right)^2}{\sum_{1\le n\le N} n^2|c_n|^2} \to 1$$

as $N\to\infty$. This is impossible, however, because the sum in the numerator is bounded by $\sec^2\frac{u}{2}\sum_{1\le n\le N}|c_n|^2$ so it has a finite limit; this concludes the proof. $\square$

Application to charged loops

Let Γ be a closed C^2 curve in $\mathbb{R}^3$, to be compared with a planar circle. The energy cost of such a deformation is $q^2\delta(\Gamma)$, where

$$\delta(\Gamma) := 2\int_0^{L/2} \mathrm{d}u \int_0^L \mathrm{d}s\, \left[|\Gamma(s+u)-\Gamma(s)|^{-1} - \frac{\pi}{L}\csc\frac{\pi u}{L}\right]$$

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Corollary: $\delta(\Gamma)$ is finite and non-negative; it is zero if and only if $\Gamma = \mathcal{C}_L$, up to Euclidean equivalence.

Proof: The integrand is ≥ 0 by $C_L^{-1}(u)$, strictly so if $\Gamma \neq \mathcal{C}_L$. Moreover, we have $|\Gamma(s+u) - \Gamma(s)|^{-1} = u^{-1} + \mathcal{O}(1)$ with the error dependent on curvature and torsion of Γ but uniform in s, hence the integral converges. $\square$

Application to leaky loops

Let us turn to the singular Schrödinger operators $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)$. We have mentioned that the discrete spectrum is nonempty and finite, in particular,

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Theorem [E'05b]: Let $\Gamma : \; [0, L] \to \mathbb{R}^2$ have the indicated properties; then for any fixed $\alpha > 0$ and $L > 0$ the ground state $\epsilon_1(\alpha, \Gamma)$ is *globally uniquely maximized by the circle* of radius $L/2\pi$.

Proof is based on the generalized Birman-Schwinger principle, plus symmetry and convexity arguments

Birman-Schwinger reformulation

We employ the generalized Birman-Schwinger principle [BEKŠ'94]. One starts from the free resolvent R_0^k which is an integral operator in $L^2(\mathbb{R}^2)$ with the kernel

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Then we introduce embedding operators associated with R_0^k for measures μ, ν which are the Dirac measure m supported by Γ and the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^2$; by $R_{\nu,\mu}^k$ we denote the integral operator from $L^2(\mu)$ to $L^2(\nu)$ with the kernel G_k, i.e. we suppose that

$$R_{\nu,\mu}^k \phi = G_k * \phi\mu$$

holds ν-a.e. for all $\phi \in D(R_{\nu,\mu}^k) \subset L^2(\mu)$

BS reformulation, continued

Proposition [BEKŠ'94, Posilicano'04]: (i) There is $\kappa_0 > 0$ s.t. $I - \alpha R^{i\kappa}_{m,m}$ on $L^2(m)$ has a bounded inverse for $\kappa \geq \kappa_0$

(ii) Let $\mathrm{Im}\, k > 0$ and $I - \alpha R^k_{m,m}$ be invertible with

$$R^k := R^k_0 + \alpha R^k_{\mathrm{d}x,m}[I - \alpha R^k_{m,m}]^{-1} R^k_{m,\mathrm{d}x}$$

from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ everywhere defined. Then k^2 belongs to $\rho(H_{\alpha,\Gamma})$ and $(H_{\alpha,\Gamma} - k^2)^{-1} = R^k$

(iii) $\dim\ker(H_{\alpha,\Gamma} - k^2) = \dim\ker(I - \alpha R^k_{m,m})$ for $\mathrm{Im}\, k > 0$

(iv) an ef of $H_{\alpha,\Gamma}$ associated with k^2 can be written as

$$\psi(x) = \int_0^L R^k_{\mathrm{d}x,m}(x,s)\phi(s)\,\mathrm{d}s\,,$$

where ϕ is the corresponding ef of $\alpha R^k_{m,m}$ with the ev one

BS reformulation, continued

Putting $k = i\kappa$ with $\kappa > 0$ we look thus for solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s)-\Gamma(s')|)\,,$$

on $L^2([0,L])$. The function $\kappa \mapsto \mathcal{R}^{\kappa}_{\alpha,\Gamma}$ is strictly decreasing in $(0,\infty)$ and $\|\mathcal{R}^{\kappa}_{\alpha,\Gamma}\| \to 0$ as $\kappa \to \infty$, hence we seek the point where the *largest* ev of $\mathcal{R}^{\kappa}_{\alpha,\Gamma}$ crosses one

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We observe that this ev is *simple,* since $\mathcal{R}^{\kappa}_{\alpha,\Gamma}$ is positivity improving and ergodic. The ground state of $H_{\alpha,\Gamma}$ is, of course, also simple. Using its rotational symmetry and the claim (iv) of the Proposition we find that the respective eigenfunction of $\mathcal{R}^{\tilde{\kappa}_1}_{\alpha,\mathcal{C}}$ corresponding to the unit eigenvalue is constant; we can choose it as $\tilde{\phi}_1(s) = L^{-1/2}$.

BS reformulation, continued

Then we have

$$\max\sigma(\mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}) = (\tilde\phi_1, \mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}\tilde\phi_1) = \frac{1}{L}\int_0^L\int_0^L \mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}(s,s')\,\mathrm{d}s\mathrm{d}s'\,,$$

and on the other hand, for the same quantity referring to a general Γ a simple variational estimate gives

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Hence it is sufficient to show that

$$\int_0^L\int_0^L K_0(\kappa|\Gamma(s)-\Gamma(s')|)\,\mathrm{d}s\mathrm{d}s' \ge \int_0^L\int_0^L K_0(\kappa|\mathcal{C}(s)-\mathcal{C}(s')|)\,\mathrm{d}s\mathrm{d}s'$$

holds *for all* $\kappa>0$ and Γ in the vicinity of $\mathcal{C}$

Convexity argument

By a simple change of variables the claim is equivalent to positivity of the functional

$$F_\kappa(\Gamma) := \int_0^{L/2} \mathrm{d}u \int_0^L \mathrm{d}s \Big[K_0\big(\kappa|\Gamma(s+u)-\Gamma(s)|\big) - K_0\big(\kappa|\mathcal{C}(s+u)-\mathcal{C}(s)|\big) \Big] ;$$

the s-independent second term is equal to $K_0(\frac{\kappa L}{\pi}\sin\frac{\pi u}{L})$

Convexity argument

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The (strict) convexity of K_0 yields by means of Jensen inequality the estimate

$$\frac{1}{L}\, F_\kappa(\Gamma) \geq \int_0^{L/2} \left[K_0\left(\frac{\kappa}{L} \int_0^L |\Gamma(s+u)-\Gamma(s)|\mathrm{d}s \right) - K_0\left(\frac{\kappa L}{\pi} \sin\frac{\pi u}{L} \right) \right] \mathrm{d}u \,,$$

where the inequality is sharp unless $\int_0^L |\Gamma(s+u)-\Gamma(s)|\mathrm{d}s$ is independent of s

Monotonicity argument

Finally, we observe that K_0 is decreasing in $(0, \infty)$, hence it is sufficient to check the inequality

$$\int_0^L |\Gamma(s+u) - \Gamma(s)| \, \mathrm{d}s \le \frac{L^2}{\pi} \sin \frac{\pi u}{L}$$

for all $u \in (0, \frac{1}{2}L]$ and furthermore, to show that it is strict unless Γ is a circle

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In this way our problem is reduced to the $C_L^1(u)$ inequality which follows from $C_L^2(u)$ proved above. $\square$

For which p do the inequalities hold?

It is natural to expect that the inequality $C_L^p(u)$ may be *invalid for large enough p*. A “stadium-perimeter” example in [E-Harrell-Loss’05] shows it is the case for $p \gtrsim 3.15295$

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To find critical p notice that for a $\mathcal{C}^2$-smooth Γ we have

$$c_\Gamma^p(u) = \int_0^L \mathrm{d}s \left[\int_s^{s+u} \mathrm{d}s' \int_s^{s+u} \mathrm{d}s'' \cos\left(\int_{s'}^{s''} \gamma(\tau)\mathrm{d}\tau \right) \right]^{p/2},$$

where $\gamma := \dot\Gamma_2\ddot\Gamma_1 - \dot\Gamma_1\ddot\Gamma_2$ is *signed curvature* of Γ. Recall that

$$\Gamma(s) = \left(\int_0^s \cos\beta(t)\,\mathrm{d}t, \int_0^s \sin\beta(t)\,\mathrm{d}t \right),$$

where $\beta(s) := \int_0^s \gamma(t)\,\mathrm{d}$ is the *arc bending*.

Variations of the circle

Let us inspect the functional $\Gamma \mapsto c^p_\Gamma(u)$ for curves

$$\gamma(s) = \frac{2\pi}{L} + \varepsilon g(s),$$

where g is continuous and L-periodic, so we can write

$$g(s) = a_0 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{2\pi n s}{L}\right) + b_n \cos\left(\frac{2\pi n s}{L}\right)$$

with $\{a\}$, $\{b\} \in \ell^2$, and ε is small, i.e. $\varepsilon\|g\|_\infty \ll 1$.

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Recall that the proof in [E'05b] used the Fourier expansion to check that circle is a local minimum for $p = 2$. Let us now compute the first and second Gâteaux derivatives of the map $\Gamma \mapsto c_\Gamma^p(u)$ at the circle for a general p

Closeness of Γ

We must make sure that Γ is closed (up to higher terms)

Proposition: The tangent to $\Gamma \in \mathcal{C}^2$ is L-periodic *iff* $a_0 = 0$. Furthermore, $\Gamma(0) = \Gamma(L) + \mathcal{O}(\varepsilon^3)$ provided $a_1 = b_1 = 0$ and

$$\sum_{n=2}^{\infty} \frac{b_n b_{n+1} + a_n a_{n+1}}{n(n+1)} = \sum_{n=2}^{\infty} \frac{a_{n+1} b_n - b_{n+1} a_n}{n(n+1)} = 0$$

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Proof is based on the mentioned "reconstruction" formula in combination with expansions using $b(s) := \int_0^s g(t)\mathrm{d}t$, namely

$$\cos\beta(s) = \left(1 - \frac{1}{2}\varepsilon^2 b^2(s)\right) \cos s - \varepsilon b(s) \sin s \, + \mathcal{O}(\varepsilon^3),$$

$$\sin\beta(s) = \left(1 - \frac{1}{2}\varepsilon^2 b^2(s)\right) \sin s + \varepsilon b(s) \cos s \, + \mathcal{O}(\varepsilon^3).$$

Gâteaux derivatives

Put $L = 2\pi$. The first derivative in the direction g is

$$\mathsf{D}_g c_\Gamma^p(u) = -\frac{p}{2}\left[4\sin^2\frac{u}{2}\right]^{p/2-1}\int\limits_0^{2\pi} \mathrm{d}s \int\limits_s^{s+u} \mathrm{d}s' \int\limits_s^{s+u} \mathrm{d}s'' \sin\left(\int\limits_{s'}^{s''} \mathrm{d}t\right)\int\limits_{s'}^{s''} g(\tau)\,\mathrm{d}\tau$$

The integrals are equal to $\left(4\sin^2 u + u\sin u\right)\int\limits_0^L g(\tau)\,\mathrm{d}\tau = 0$

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Next, the second Gâteaux derivative $\mathrm{D}_g^2 c_\Gamma^p(u)$ equals

$$\frac{p}{2}\left(\frac{p}{2}-1\right)\left[4\sin^2\frac{u}{2}\right]^{p/2-2}\int_0^{2\pi}\mathrm{d}s\left(\int_s^{s+u}\mathrm{d}s'\int_s^{s+u}\mathrm{d}s''\sin(s''-s')\int_{s'}^{s''}g(\tau)\,\mathrm{d}\tau\right)^2$$

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Critical exponent

Inspecting the second derivative yields the following claim:

Theorem [E-Fraas-Harrell'07]: For a fixed $u \in (0, \frac{1}{2}L]$ define

$$p_c(u) := \frac{4 - \cos\left(\frac{2\pi u}{L}\right)}{1 - \cos\left(\frac{2\pi u}{L}\right)},$$

then we have the alternative: for $p > p_c(u)$ the circle is either a saddle point or a local minimum, while for $p < p_c(u)$ it is a local maximum of $\Gamma \mapsto c_\Gamma^p(u)$. In particular, $p_c(\frac{1}{2}L) = \frac{5}{2}$

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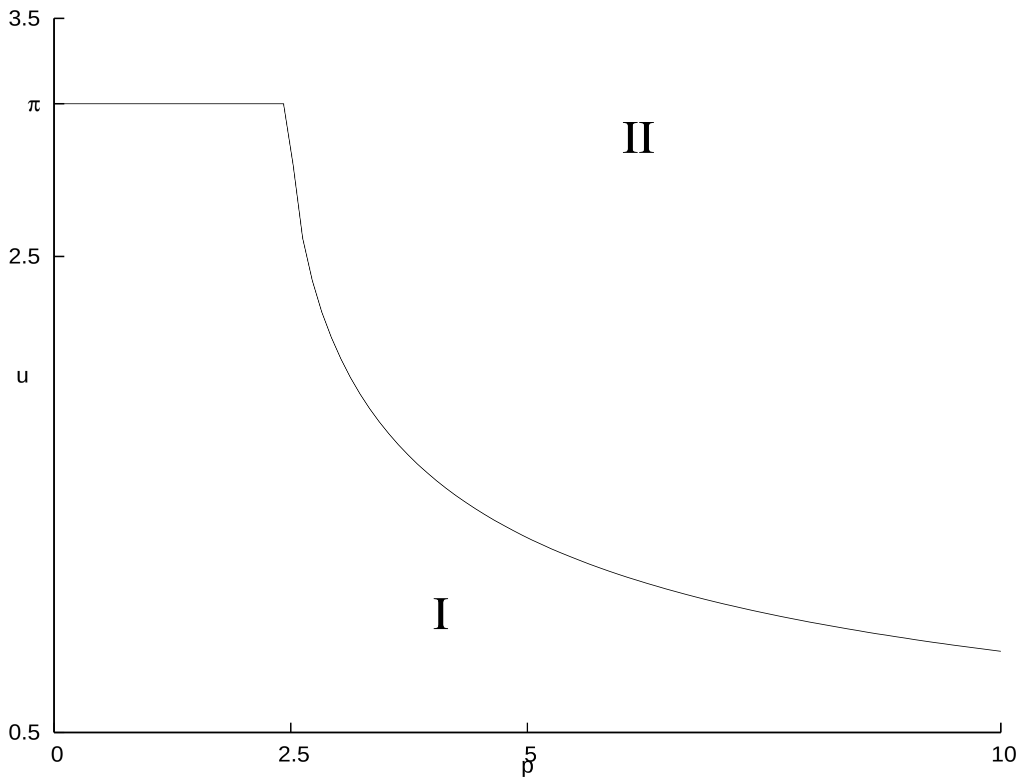
Remarks: (a) we do not discuss the critical case $p = p_c(u)$ when higher derivatives of $c_\Gamma^p(u)$ come into play

(b) It is natural to expect and easy to verify that for $p > p_c$ circle is in fact a saddle point of the functional

Critical exponent



Relation between the critical exponent p_c and the arc length u for $L = 2\pi$. The inequalities hold locally in the region I

Sketch of the proof

Put again $L = 2\pi$. Using Fourier expansion we cast the second derivative given above into the form

$$\mathsf{D}_g^2 c_\Gamma^p(u) = \sum_{n=2}^{\infty} (a_n^2 + b_n^2) \, \frac{2^p \pi \sin^{p-2}\left(\frac{u}{2}\right)}{8(n - n^3)^2} \, p \, T(n,\, u,\, p),$$

where $T(n,\, u,\, p)$ is denotes the following expression

$$-\Big(2n^4 - 6n^2 - 2(n^2-1)^2 \cos u + (n+1)^2 \cos(n-1)u + (n-1)^2 \cos(n+1)u\Big)$$
$$+2(p-2)\left(-2n \cos\left(\frac{nu}{2}\right) \sin\left(\frac{u}{2}\right) + 2\cos\left(\frac{u}{2}\right) \sin\left(\frac{nu}{2}\right)\right)^2$$

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Since $\sin(u/2) > 0$ for $u \in (0,\, \pi)$ the sign of each term is determined by that of $T(n,\, u,\, p)$. It is straightforward to check that $T(2,\, u,\, p) > 0$ for $p > p_c(u)$, hence for $p > p_c(u)$ the circle fails to be a local maximum of $\Gamma \mapsto c_\Gamma^p(u)$

Sketch of the proof

It is easy to see that $p \mapsto T(n,\, u,\, p)$ is strictly increasing, hence to prove the other part of the theorem it is sufficient to show that $T(n,\, u,\, p_c(u))$ is negative for $n \geq 3$. We define

$$S(n,\, u) = -(1 - \cos u)\, T(n,\, u,\, p_c(u))$$

and prove that this function is positive for $n \geq 3$

Sketch of the proof

It is easy to see that $p \mapsto T(n, u, p)$ is strictly increasing, hence to prove the other part of the theorem it is sufficient to show that $T(n, u, p_c(u))$ is negative for $n \geq 3$. We define

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and prove that this function is positive for $n \geq 3$

In the case $n = 3$ the positivity of $S(n, u)$ is rather easily established since

$$S(3, u) = 2\left(2 \sin \frac{u}{2}\right)^8,$$

while for $n \geq 4$ the same result follows from a series of simple if somewhat tedious estimates. □

A discrete analogue: polymer loops

Consider a problem related to the above one; following [AGHH'88, 05] we can call it a *polymer loop*

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It is an extension of the "discrete" problem to a more general class of curves: we take a closed loop Γ and consider a class of singular Schrödinger operators in $L^2(\mathbb{R}^d),\ d = 2, 3,$ given formally by the expression

$$H^N_{\alpha,\Gamma} = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta\left(x - \Gamma\left(\frac{jL}{N}\right)\right)$$

We are interested in the shape of Γ which *maximizes* the ground state energy provided, of course, that the discrete spectrum of $H^N_{\alpha,\Gamma}$ is non-empty.

A reminder: 2D point interactions

Fixing the site y_j and “coupling constant” α we define them by b.c. which change *locally* the domain of $-\Delta$: we require

$$\psi(x) = -\frac{1}{2\pi} \log|x-y_j|\, L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x-y_j|)\,,$$

where the generalized b.v. $L_0(\psi, y_j)$ and $L_1(\psi, y_j)$ satisfy

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0\,, \quad \alpha \in \mathbb{R}$$

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For $Y_\Gamma := \{y_j := \Gamma\left(\frac{jL}{N}\right) :\, j = 0, \dots, N-1\}$ we define in this way $-\Delta_{\alpha, Y_\Gamma}$ in $L^2(\mathbb{R}^2)$. It holds $\sigma_{\mathrm{disc}}\left(-\Delta_{\alpha, Y_\Gamma}\right) \neq \emptyset$, i.e.

$$\epsilon_1 \equiv \epsilon_1(\alpha, Y_\Gamma) := \inf \sigma\left(-\Delta_{\alpha, Y_\Gamma}\right) < 0\,,$$

which is always true in two dimensions – cf. [AGHH'88, 05]

A reminder: 3D point interactions

Similarly, for y_j and “coupling” α we define them by b.c. which change locally the domain of $-\Delta$: we require

$$\psi(x) = \frac{1}{4\pi|x-y_j|} L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x-y_j|)\,,$$

where the b.v. $L_0(\psi, y_j)$ and $L_1(\psi, y_j)$ satisfy again

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where the b.v. $L_0(\psi, y_j)$ and $L_1(\psi, y_j)$ satisfy again

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0\,, \quad \alpha \in \mathbb{R},$$

giving $-\Delta_{\alpha,Y_\Gamma}$ in $L^2(\mathbb{R}^3)$. However, $\sigma_{\mathrm{disc}}\left(-\Delta_{\alpha,Y_\Gamma}\right) \neq \emptyset$, i.e.

$$\epsilon_1 \equiv \epsilon_1(\alpha, Y_\Gamma) := \inf \sigma\left(-\Delta_{\alpha,Y_\Gamma}\right) < 0\,,$$

is now a nontrivial requirement; it holds only for α below some critical value α_0 – cf. [AGHH'88, 05]

A geometric reformulation

By Krein's formula, the spectral condition is reduced to an algebraic problem. Using $k = i\kappa$ with $\kappa > 0$, we find the ev's $-\kappa^2$ of our operator from

$$\det \Gamma_k = 0 \quad \text{with} \quad (\Gamma_k)_{ij} := (\alpha - \xi^k)\delta_{ij} - (1-\delta_{ij})g^k_{ij} \,,$$

where the off-diagonal elements are $g^k_{ij} := G_k(y_i - y_j)$, or equivalently

$$g^k_{ij} = \frac{1}{2\pi} K_0(\kappa|y_i - y_j|)$$

and the regularized Green's function at the interaction site is

$$\xi^k = -\frac{1}{2\pi}\left(\ln\frac{\kappa}{2} + \gamma_E\right)$$

Geometric reformulation, continued

The ground state refers to the point where the *lowest* ev of $\Gamma_{i\kappa}$ vanishes. Using smoothness and monotonicity of the κ-dependence we have to check that

$$\min \sigma(\Gamma_{i\tilde{\kappa}_1}) < \min \sigma(\tilde{\Gamma}_{i\tilde{\kappa}_1})$$

holds locally for $\Gamma \neq \tilde{\mathcal{P}}_N$, where $-\tilde{\kappa}_1^2 := \epsilon_1(\alpha, \tilde{\mathcal{P}}_N)$

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There is a *one-to-one relation* between an ef $c = (c_1, \dots, c_N)$ of $\Gamma_{i\kappa}$ at that point and the corresponding ef of $-\Delta_{\alpha,\Gamma}$ given by $c \leftrightarrow \sum_{j=1}^N c_j G_{i\kappa}(\cdot - y_j)$, up to normalization. In particular, the lowest ev of $\tilde\Gamma_{i\tilde\kappa_1}$ corresponds to the eigenvector $\tilde\phi_1 = N^{-1/2}(1, \dots, 1)$; hence the spectral threshold is

$$\min \sigma(\tilde\Gamma_{i\tilde\kappa_1}) = (\tilde\phi_1, \tilde\Gamma_{i\tilde\kappa_1} \tilde\phi_1) = \alpha - \xi^{i\tilde\kappa_1} - \frac{2}{N} \sum_{i<j} \tilde g_{ij}^{i\tilde\kappa_1}$$

Geometric reformulation, continued

On the other hand, we have $\min \sigma(\Gamma_{i\tilde{\kappa}_1}) \leq (\tilde{\phi}_1, \Gamma_{i\tilde{\kappa}_1}\tilde{\phi}_1)$, and therefore it is sufficient to check that

$$\sum_{i<j} G_{i\kappa}(y_i - y_j) > \sum_{i<j} G_{i\kappa}(\tilde{y}_i - \tilde{y}_j)$$

holds *for all* $\kappa > 0$ and $\Gamma \neq \tilde{\mathcal{P}}_N$.

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holds *for all* $\kappa > 0$ and $\Gamma \neq \tilde{\mathcal{P}}_N$. Call $\ell_{ij} := |y_i - y_j|$ and $\tilde\ell_{ij} := |\tilde y_i - \tilde y_j|$ and define $F:\ (\mathbb{R}_+)^{N(N-3)/2} \to \mathbb{R}$ by

$$F(\{\ell_{ij}\}) := \sum_{m=2}^{[N/2]} \sum_{|i-j|=m} \left[G_{i\kappa}(\ell_{ij}) - G_{i\kappa}(\tilde\ell_{ij})\right];$$

Using the *convexity* of $G_{i\kappa}(\cdot)$ for a fixed $\kappa > 0$ we get

$$F(\{\ell_{ij}\}) \ge \sum_{m=2}^{[N/2]} \nu_m \left[G_{i\kappa}\left(\frac{1}{\nu_m}\sum_{|i-j|=m} \ell_{ij}\right) - G_{i\kappa}(\tilde\ell_{1,1+m})\right],$$

where ν_n is the number of the appropriate chords

Geometric reformulation, continued

It is easy to see that

$$\nu_m := \begin{cases} N & \dots \qquad m = 1, \dots, \left[\frac{1}{2}(N-1)\right] \\ \frac{1}{2}N & \dots \qquad m = \frac{1}{2}N \quad \text{for } N \text{ even} \end{cases}$$

since for an even N one has to prevent double counting

Geometric reformulation, continued

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Since $G_{i\kappa}(\cdot)$ is also *monotonously decreasing* in $(0,\infty)$, we thus need only to demonstrate that

$$\tilde{\ell}_{1,m+1} \ge \frac{1}{\nu_n} \sum_{|i-j|=m} \ell_{ij}$$

with the sharp inequality for at least one m if $\mathcal{P}_N \ne \tilde{\mathcal{P}}_N$. In this way the problem becomes again purely geometric

“Discrete” chord inequalities

Recall that for $\Gamma : [0, L] \to \mathbb{R}^2$ we have used the notation

$$y_j := \Gamma\left(\frac{jL}{N}\right), \quad j = 0, 1, \ldots, N-1;$$

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For fixed $L > 0$, N and $m = 1, \ldots, [\frac{1}{2}N]$ we consider the following inequalities for ℓ^p norms related to the chord lengths, that is, the quantities $\Gamma\left(\cdot + \frac{jL}{N}\right) - \Gamma(\cdot)$

$$D^p_{L,N}(m): \quad \sum_{n=1}^N |y_{n+m} - y_n|^p \le \frac{N^{1-p} L^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}}, \quad p > 0,$$

$$D^{-p}_{L,N}(m): \quad \sum_{n=1}^N |y_{n+m} - y_n|^{-p} \ge \frac{N^{1+p} \sin^p \frac{\pi}{N}}{L^p \sin^p \frac{\pi m}{N}}, \quad p > 0.$$

The RHS’s correspond to regular planar polygon $\tilde{\mathcal{P}}_N$

More on the "discrete" inequalities

In general, the inequalities *are not valid for* $p > 2$ as the example of a rhomboid shows: $D^p_{L,4}(2)$ is equivalent to $\sin^p \phi + \cos^p \phi \leq 2^{1-(p/2)}$ for $0 < \phi < \pi$ which obviously holds for $p \leq 2$ only

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Proposition: $D^p_{L,N}(m) \Rightarrow D^{p'}_{L,N}(m)$ if $p > p' > 0$ and $D^p_{L,N}(m) \Rightarrow D^{-p}_{L,N}(m)$ for any $p > 0$

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Theorem [E'05c]: The inequality $D^2_{L,N}(m)$ is valid

Remark: By $D^{-1}_{L,N}(m)$ this implies that the unique minimizers of the “discrete” electrostatic problem is the regular planar polygon $\tilde{\mathcal{P}}_N$

Global validity of $D^2_{L,N}(m)$

Let us to adapt the above proof the "discrete" case. We put $L = 2\pi$ and express Γ through its Fourier series,

$$\Gamma(s) = \sum_{0\neq n\in\mathbb{Z}} c_n\, \mathrm{e}^{ins}$$

with $c_n \in \mathbb{C}^d$; since $\Gamma(s) \in \mathbb{R}^d$ one has to require $c_{-n} = \bar{c}_n$. Again, we can choose $c_0 = 0$ and the normalization condition $\sum_{0\neq n\in\mathbb{Z}} n^2 |c_n|^2 = 1$ follows from $|\dot{\Gamma}(s)| = 1$

Global validity of $D^2_{L,N}(m)$

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On the other hand, the left-hand side of $D^2_{2\pi,N}(m)$ equals

$$\sum_{n=1}^{N} \sum_{0 \neq j,k \in \mathbb{Z}} c_j^* \cdot c_k \left(\mathrm{e}^{-2\pi imj/N} - 1\right) \left(\mathrm{e}^{2\pi imk/N} - 1\right) \mathrm{e}^{2\pi in(k-j)/N}$$

Global validity, continued

Next we change the order of summation and observe that $\sum_{n=1}^{N} \mathrm{e}^{2\pi i n(k-j)/N} = N$ if $j = k \pmod N$ and zero otherwise; this allows us to write the last expression as

$$4N \sum_{l\in\mathbb{Z}} \sum_{\substack{0 \neq j,k \in \mathbb{Z} \\ j-k=lN}} |j|c_j^* \cdot |k|c_k \left| j^{-1} \sin \frac{\pi m j}{N} \right| \left| k^{-1} \sin \frac{\pi m k}{N} \right| .$$

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Hence the sought inequality $D^2_{2\pi,N}(m)$ is equivalent to

$$\left(d, (A^{(N,m)} \otimes I)d\right) \leq \left(\frac{\pi \sin \frac{\pi m}{N}}{N \sin \frac{\pi}{N}} \right)^2$$

Global validity, continued

Here the vector $d \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^d$ has the components $d_j := |j|c_j$ and the operator $A^{(N,m)}$ on $\ell^2(\mathbb{Z})$ is defined as

$$A_{jk}^{(N,m)} := \begin{cases} \left|j^{-1} \sin \frac{\pi m j}{N}\right| \left|k^{-1} \sin \frac{\pi m k}{N}\right| & \text{if} \quad 0 \neq j, k \in \mathbb{Z},\ j-k = lN \\ 0 & \text{otherwise} \end{cases}$$

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Remark: The “continuous” case corresponds formally to $N = \infty$. Then $A^{(N,m)}$ is *a multiple of* I and it is only necessary to employ $|\sin jx| \leq j\, \sin x$ for any $j \in \mathbb{N}$ and $x \in (0, \frac{1}{2}\pi]$. Here due to *infinitely many side diagonals* such a simple estimate yields an unbounded Toeplitz-type operator, and one has use the *matrix-element decay*

Global validity, continued

For a given $j \neq 0$ and $d \in \ell^2(\mathbb{Z})$ we have

$$\left(A^{(N,m)}d\right)_j = \left|j^{-1}\sin\frac{\pi m j}{N}\right| \sum_{\substack{0 \neq k \in \mathbb{Z} \\ k = j (\mathrm{mod}\, N)}} \left|k^{-1}\sin\frac{\pi m k}{N}\right| d_k$$

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The norm $\|A^{(N,m)}d\|$ is then easily estimated by means of Schwarz inequality,

$$\begin{aligned} \|A^{(N,m)}d\|^2 &= \sum_{0 \neq j \in \mathbb{Z}} j^{-2} \sin^2 \frac{\pi m j}{N} \left| \sum_{\substack{0 \neq k \in \mathbb{Z} \\ k = j (\mathrm{mod}\, N)}} \left| k^{-1} \sin \frac{\pi m k}{N} \right| d_k \right|^2 \\ &\leq \sum_{n=0}^{N-1} \sin^4 \frac{\pi m n}{N} S_n^2 \sum_{\substack{n + lN \neq 0 \\ l \in \mathbb{Z}}} |d_{n+lN}|^2 \end{aligned}$$

Global validity, concluded

Here we have introduced

$$S_n := \sum_{\substack{n+lN\neq 0 \\ l\in\mathbb{Z}}} \frac{1}{(n+lN)^2} = \sum_{l=1}^{\infty} \left\{ \frac{1}{(lN-n)^2} + \frac{1}{(lN-N+n)^2} \right\}$$

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The sought claim, the validity of $D^2_{L,N}(m)$, then follows from

$$\sin\frac{\pi m}{N}\sin\frac{\pi r}{N} > \left|\sin\frac{\pi}{N}\sin\frac{\pi mr}{N}\right|, \quad 2\le r<m\le\left[\frac{1}{2}N\right]$$

This can be also equivalently written as the inequalities $U_{m-1}\left(\cos\frac{\pi}{N}\right) > \left|U_{m-1}\left(\cos\frac{\pi r}{N}\right)\right|$ for Chebyshev polynomials of the second kind; they are verified directly □

Summary and outlook

We have analyzed a class of inequalities with applications to physically interesting isoperimetric problems. Various questions remains open, for instance

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- on the mathematical side, what is generally the *critical value* of p in the *discrete case*, etc.

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[E05a] P.E.: An isoperimetric problem for point interactions, *J. Phys. A: Math. Gen.* **A38** (2005), 4795-4802

[E05b] P.E.: An isoperimetric problem for leaky loops and related mean-chord inequalities, *J. Math. Phys.* **46** (2005), 062105

[E05c] P.E.: Necklaces with interacting beads: isoperimetric problems, *Proceedings of the "International Conference on Differential Equations and Mathematical Physics" (Birmingham 2005),* AMS "Contemporary Mathematics" Series, vol. 412, Providence, R.I., 2003; pp. 141–149.

[EHL05] P.E., E. Harrell, M. Loss: Global mean-chord inequalities with application to isoperimetric problems, *Lett.Math.Phys.* **75** (2006), 225–233; addendum **77** (2006), 219

[EFH07] P.E., M. Fraas, E.M. Harrell: On the critical exponent in an isoperimetric inequality for chords, *Phys. Lett.* **A** (2007), to appear

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