Resonance effects in transport through leaky graphs

Pavel Exner

in collaboration with *Takashi Ichinose, Sylwia Kondej,*

Kateřina Němcová and *Kazushi Yoshitomi*

exner@ujf.cas.cz

Department of Theoretical Physics, NPI, Czech Academy of Sciences

and Doppler Institute, Czech Technical University

Talk overview

- What is known about scattering and resonances in “quantum wire” systems?

Talk overview

- What is known about scattering and resonances in “quantum wire” systems?
- A model of “leaky” quantum wires and graphs, $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$

Talk overview

- What is known about scattering and resonances in "quantum wire" systems?
- A model of "leaky" quantum wires and graphs, $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$
- Geometrically induced spectral properties

Talk overview

- What is known about scattering and resonances in “quantum wire” systems?
- A model of “leaky” quantum wires and graphs, $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$
- Geometrically induced spectral properties
- How to find spectrum numerically: an approximation by point interaction Hamiltonians

Talk overview

- What is known about scattering and resonances in "quantum wire" systems?
- A model of "leaky" quantum wires and graphs, $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$
- Geometrically induced spectral properties
- How to find spectrum numerically: an approximation by point interaction Hamiltonians
- A solvable resonance model: interaction supported by a line and a family of points

Talk overview

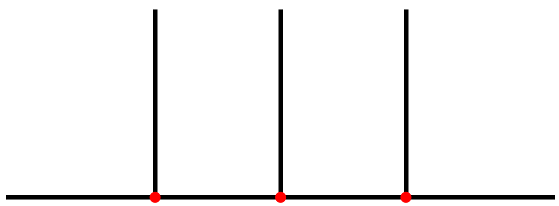
- What is known about scattering and resonances in “quantum wire” systems?
- A model of “leaky” quantum wires and graphs, $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$
- Geometrically induced spectral properties
- How to find spectrum numerically: an approximation by point interaction Hamiltonians
- A solvable resonance model: interaction supported by a line and a family of points
- Extension to higher dimension - plane and points

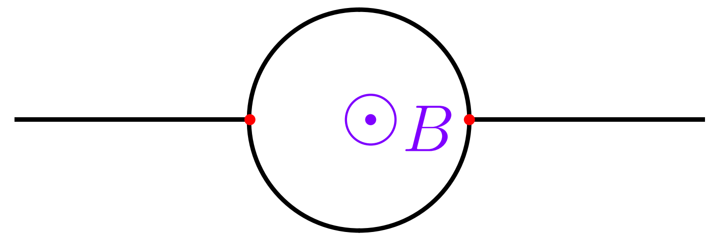
Talk overview

- What is known about scattering and resonances in "quantum wire" systems?
- A model of "leaky" quantum wires and graphs, $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$
- Geometrically induced spectral properties
- How to find spectrum numerically: an approximation by point interaction Hamiltonians
- A solvable resonance model: interaction supported by a line and a family of points
- Extension to higher dimension - plane and points
- Open questions

Usual graph resonance models

Generally known: resonances in “ideal” graphs, e.g.

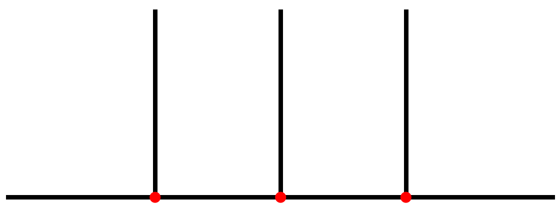


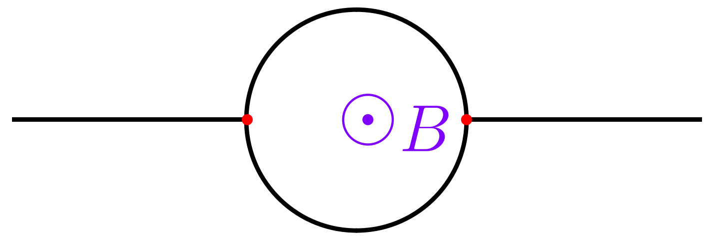


Hamiltonian in such models is a Schrödinger operator on graph, with appropriate boundary conditions at the vertices

Usual graph resonance models

Generally known: resonances in “ideal” graphs, e.g.





Hamiltonian in such models is a Schrödinger operator on graph, with appropriate boundary conditions at the vertices

Search for spectral and scattering properties is thus an *ODE problem*. Resonances typically appear if there are finite edges, which have discrete spectra when disconnected, embedded into the outer-leads continuum

Studied by many authors, for reviews see, for instance Kostrykin-Schrader, 1999; Kuchment, 2004, etc.

Finite-width effects

Less well known: A finite-width wire itself may produce resonances. Take a smoothly bent hard-wall wire Σ of width d, use natural curvilinear coordinates s, u

Rewrite the Hamiltonian $H = -\Delta_D^\Sigma$ in the curvilinear coordinates and expand it w.r.t. the transverse basis:

$$H_{jk} = -\partial_s \left[\delta_{jk} + \mathcal{O}(d)\right] \partial_s + \left(\kappa_1^2 j^2 - \frac{1}{4}\gamma(s)^2\right) \delta_{jk} + \mathcal{O}(d),$$

where $\kappa_1 := \pi/d$ and γ is the curvature of $\mathrm{bd}\,\Sigma$.

Finite-width effects

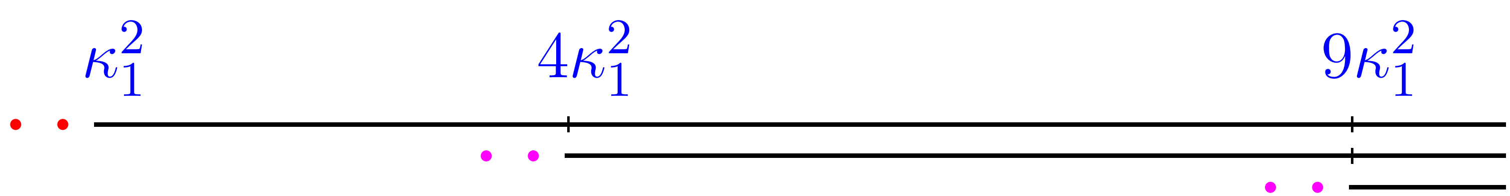
Less well known: A finite-width wire itself may produce resonances. Take a smoothly bent hard-wall wire Σ of width d, use natural curvilinear coordinates s, u

Rewrite the Hamiltonian $H = -\Delta_D^\Sigma$ in the curvilinear coordinates and expand it w.r.t. the transverse basis:

$$H_{jk} = -\partial_s \left[\delta_{jk} + \mathcal{O}(d)\right] \partial_s + \left(\kappa_1^2 j^2 - \frac{1}{4}\gamma(s)^2\right) \delta_{jk} + \mathcal{O}(d),$$

where $\kappa_1 := \pi/d$ and γ is the curvature of $\mathrm{bd}\,\Sigma$.

Thus in leading order, transverse modes are decoupled:



Finite-width effects

The mode-coupling perturbation turns the embedded ev's into resonances, exponentially narrow w.r.t. d:

Theorem [Nedelec, 1997; Duclos-E.-Meller, 1998]: Suppose that Σ is not straight and does not intersect itself. Let the curvature satisfy $|\gamma(s)| \leq c\langle s\rangle^{-1-\varepsilon}$ and extend *analytically* to a "waisted sector" $\{z \in \mathbb{C} : |\arg(\pm z)| < \alpha_0,\ |\mathrm{Im}\, z| < \eta_0\}$ for positive α_0, η_0 with the same decay property. Then

$$0 \leq -\mathrm{Im}\, \epsilon_{j,n}(d) \leq c_{j,n} \mathrm{e}^{-2\pi\eta\sqrt{2j-1}/d}$$

holds for all $\eta < \eta_0$ and d small enough.

Finite-width effects

The mode-coupling perturbation turns the embedded ev's into resonances, exponentially narrow w.r.t. d:

Theorem [Nedelec, 1997; Duclos-E.-Meller, 1998]: Suppose that Σ is not straight and does not intersect itself. Let the curvature satisfy $|\gamma(s)| \le c\langle s\rangle^{-1-\varepsilon}$ and extend *analytically* to a "waisted sector" $\{z \in \mathbb{C} : |\arg(\pm z)| < \alpha_0,\, |\mathrm{Im}\, z| < \eta_0\}$ for positive α_0, η_0 with the same decay property. Then

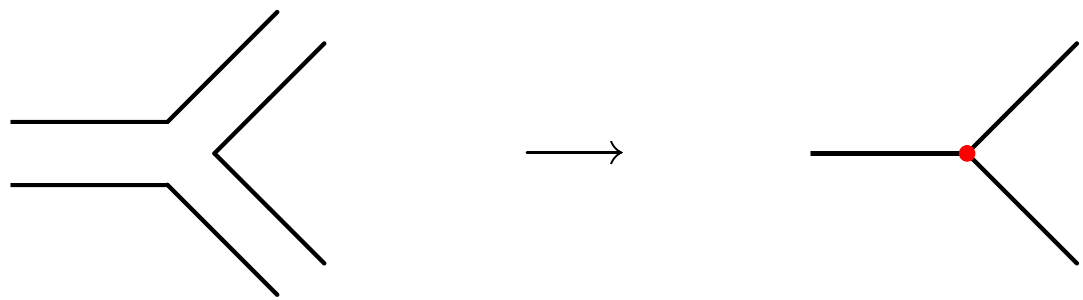
$$0 \le -\mathrm{Im}\, \epsilon_{j,n}(d) \le c_{j,n} \mathrm{e}^{-2\pi\eta\sqrt{2j-1}/d}$$

holds for all $\eta < \eta_0$ and d small enough.

Remark: The non-existence of embedded ev's which survive curvature-induced perturbation is an open question

Drawbacks of these models

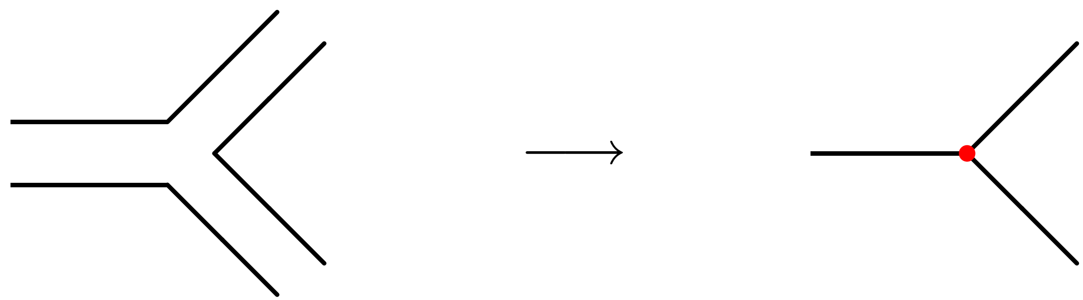
- Presence of ad hoc parameters in the b.c. describing branchings. A natural remedy: use a zero-width limit in a more realistic description



However, the answer is known so far only for Neumann-type situations [Rubinstein-Schatzman, 2001; Kuchment-Zeng, 2001; E.-Post, 2003], the Dirichlet case needed here is open (and difficult)

Drawbacks of these models

- Presence of ad hoc parameters in the b.c. describing branchings. A natural remedy: use a zero-width limit in a more realistic description



However, the answer is known so far only for Neumann-type situations [Rubinstein-Schatzman, 2001; Kuchment-Zeng, 2001; E.-Post, 2003], the Dirichlet case needed here is open (and difficult)

- Quantum tunneling is neglected: recall that a true quantum-wire boundary is a finite potential jump

Leaky quantum graphs

We consider "leaky" graphs with an *attractive interaction supported by graph edges*. Formally we have

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)\,, \quad \alpha > 0\,,$$

in $L^2(\mathbb{R}^2)$, where Γ is the graph in question.

Leaky quantum graphs

We consider "leaky" graphs with an *attractive interaction supported by graph edges*. Formally we have

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \quad \alpha > 0\,,$$

in $L^2(\mathbb{R}^2)$, where Γ is the graph in question.

A proper definition of $H_{\alpha,\Gamma}$: it can be associated naturally with the quadratic form,

$$\psi \mapsto \|\nabla\psi\|^2_{L^2(\mathbb{R}^n)} - \alpha \int_\Gamma |\psi(x)|^2 \mathrm{d}x\,,$$

which is closed and below bounded in $W^{2,1}(\mathbb{R}^n)$; the second term makes sense in view of Sobolev embedding. This definition also works for various "wilder" sets Γ

Leaky quantum-graph Hamiltonians

For Γ with locally finite number of smooth edges and *no cusps* we can use an *alternative definition* by boundary conditions: $H_{\alpha,\Gamma}$ acts as $-\Delta$ on functions from $W^{2,1}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\Big|_{+} - \frac{\partial\psi}{\partial n}(x)\Big|_{-} = -\alpha\psi(x)$$

Leaky quantum-graph Hamiltonians

For Γ with locally finite number of smooth edges and *no cusps* we can use an *alternative definition* by boundary conditions: $H_{\alpha,\Gamma}$ acts as $-\Delta$ on functions from $W^{2,1}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\Big|_{+} - \frac{\partial\psi}{\partial n}(x)\Big|_{-} = -\alpha\psi(x)$$

Remarks:

- for graphs in $\mathbb{R}^3$ we use generalized b.c. which define a two-dimensional point interaction in normal plane
- one can combine “edges” of different dimensions as long as $\operatorname{codim}\Gamma$ does not exceed three

Geometrically induced spectrum

(a) Bending means binding, i.e. it may create isolated eigenvalues of $H_{\alpha,\Gamma}$. Consider a *piecewise* C^1*-smooth* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ parameterized by its arc length, and assume:

Geometrically induced spectrum

(a) Bending means binding, i.e. it may create isolated eigenvalues of $H_{\alpha,\Gamma}$. Consider a *piecewise* C^1*-smooth* $\Gamma:\ \mathbb{R}\to\mathbb{R}^2$ parameterized by its arc length, and assume:

- $|\Gamma(s)-\Gamma(s')|\geq c|s-s'|$ holds for some $c\in(0,1)$
- Γ is asymptotically straight: there are $d>0,\ \ \mu>\frac{1}{2}$ and $\omega\in(0,1)$ such that

$$1-\frac{|\Gamma(s)-\Gamma(s')|}{|s-s'|}\leq d\left[1+|s+s'|^{2\mu}\right]^{-1/2}$$

  in the sector $S_\omega:=\left\{(s,s'):\ \omega<\frac{s}{s'}<\omega^{-1}\right\}$

- straight line is excluded, i.e. $|\Gamma(s)-\Gamma(s')|<|s-s'|$ holds for some $s,s'\in\mathbb{R}$

Bending means binding

Theorem [E.-Ichinose, 2001]: Under these assumptions, $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [-\frac{1}{4}\alpha^2, \infty)$ and $H_{\alpha,\Gamma}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$

Bending means binding

Theorem [E.-Ichinose, 2001]: Under these assumptions, $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [-\frac{1}{4}\alpha^2, \infty)$ and $H_{\alpha,\Gamma}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$

- The same for *curves in* $\mathbb{R}^3$, under stronger regularity, with $-\frac{1}{4}\alpha^2$ is replaced by the corresponding 2D p.i. ev
- For *curved surfaces* $\Gamma \subset \mathbb{R}^3$ such a result is proved in the strong coupling asymptotic regime only
- *Implications for graphs:* let $\tilde\Gamma \supset \Gamma$ in the set sense, then $H_{\alpha,\tilde\Gamma} \le H_{\alpha,\Gamma}$. If the essential spectrum threshold is the same for both graphs and Γ fits the above assumptions, we have $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma}) \ne \emptyset$ by minimax principle

More geometrically induced properties

(b) Perturbation theory for punctured manifolds:
let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be as above, C^2-*smooth*, and let Γ_ε differ by ε-long hiatus around a fixed point $x_0 \in \Gamma$. Let φ_j be the ef of $H_{\alpha,\Gamma}$ corresponding to a simple ev $\lambda_j \equiv \lambda_j(0)$ of $H_{\alpha,\Gamma}$.

Theorem [E.-Yoshitomi, 2003]: The j-th ev of $H_{\alpha,\Gamma_\varepsilon}$ is

$$\lambda_j(\varepsilon) = \lambda_j(0) + \alpha|\varphi_j(x_0)|^2\varepsilon + o(\varepsilon^{n-1}) \quad \text{as} \quad \varepsilon \to 0$$

More geometrically induced properties

(b) Perturbation theory for punctured manifolds:
let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be as above, C^2-*smooth*, and let Γ_ε differ by ε-long hiatus around a fixed point $x_0 \in \Gamma$. Let φ_j be the ef of $H_{\alpha,\Gamma}$ corresponding to a simple ev $\lambda_j \equiv \lambda_j(0)$ of $H_{\alpha,\Gamma}$.

Theorem [E.-Yoshitomi, 2003]: The j-th ev of $H_{\alpha,\Gamma_\varepsilon}$ is

$$\lambda_j(\varepsilon) = \lambda_j(0) + \alpha|\varphi_j(x_0)|^2\varepsilon + o(\varepsilon^{n-1}) \quad \text{as} \quad \varepsilon \to 0$$

Remarks: Similarly one can express perturbed *degenerate* ev's. Analogous results hold for ev's for punctured compact, $(d-1)$-dimensional, $C^{1+[d/2]}$-smooth manifolds in $\mathbb{R}^d$. Formally a small hole acts as *repulsive* δ *interaction* with coupling α times $(d-1)$-Lebesgue measure of the hole

Strongly attractive curves

(c) Strong coupling asymptotics: let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be as above, now supposed to be C^4-*smooth*

Theorem [E.-Yoshitomi, 2001]: The j-th ev of $H_{\alpha,\Gamma}$ is

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1}\ln\alpha) \quad \text{as} \quad \alpha \to \infty\,,$$

where μ_j is the j-th ev of $S_\Gamma := -\frac{\mathrm{d}}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$ on $L^2((\mathbb{R})$ and γ is the curvature of Γ.

Strongly attractive curves

(c) Strong coupling asymptotics: let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be as above, now supposed to be C^4-*smooth*

Theorem [E.-Yoshitomi, 2001]: The j-th ev of $H_{\alpha,\Gamma}$ is

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1}\ln\alpha) \quad \text{as} \quad \alpha \to \infty\,,$$

where μ_j is the j-th ev of $S_\Gamma := -\frac{\mathrm{d}}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$ on $L^2((\mathbb{R})$ and γ is the curvature of Γ. The same holds if Γ is a loop; then we also have

$$\#\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma}) = \frac{|\Gamma|\alpha}{2\pi} + \mathcal{O}(\ln\alpha) \quad \text{as} \quad \alpha \to \infty$$

Further extensions

- $H_{\alpha,\Gamma}$ with a *periodic* Γ has a band-type spectrum, but analogous asymptotics is valid for its *Floquet components* $H_{\alpha,\Gamma}(\theta)$, with the comparison operator $S_\Gamma(\theta)$ satisfying the appropriate b.c. over the period cell. It is important that the error term is uniform w.r.t. θ

Further extensions

- $H_{\alpha,\Gamma}$ with a *periodic* Γ has a band-type spectrum, but analogous asymptotics is valid for its *Floquet components* $H_{\alpha,\Gamma}(\theta)$, with the comparison operator $S_\Gamma(\theta)$ satisfying the appropriate b.c. over the period cell. It is important that the error term is uniform w.r.t. θ
- Similar result holds for planar loops *threaded by mg field*, homogeneous, AB flux line, etc.

Further extensions

- $H_{\alpha,\Gamma}$ with a *periodic* Γ has a band-type spectrum, but analogous asymptotics is valid for its *Floquet components* $H_{\alpha,\Gamma}(\theta)$, with the comparison operator $S_\Gamma(\theta)$ satisfying the appropriate b.c. over the period cell. It is important that the error term is uniform w.r.t. θ
- Similar result holds for planar loops *threaded by mg field*, homogeneous, AB flux line, etc.
- *Higher dimensions:* the results extend to loops, infinite and periodic curves in $\mathbb{R}^3$

Further extensions

- $H_{\alpha,\Gamma}$ with a *periodic* Γ has a band-type spectrum, but analogous asymptotics is valid for its *Floquet components* $H_{\alpha,\Gamma}(\theta)$, with the comparison operator $S_\Gamma(\theta)$ satisfying the appropriate b.c. over the period cell. It is important that the error term is uniform w.r.t. θ
- Similar result holds for planar loops *threaded by mg field*, homogeneous, AB flux line, etc.
- *Higher dimensions:* the results extend to loops, infinite and periodic curves in $\mathbb{R}^3$
- and to *curved surfaces* in $\mathbb{R}^3$; then the comparison operator is $-\Delta_{\mathrm{LB}} + K - M^2$, where K, M, respectively, are the corresponding Gauss and mean curvatures

How to find the spectrum?

The above general results do not tell us how to find the spectrum for a particular Γ. There are various possibilities:

- *Direct solution* of the PDE problem $H_{\alpha,\Gamma}\psi = \lambda\psi$ is feasible in a few simple examples only

How to find the spectrum?

The above general results do not tell us how to find the spectrum for a particular Γ. There are various possibilities:

- *Direct solution* of the PDE problem $H_{\alpha,\Gamma}\psi = \lambda\psi$ is feasible in a few simple examples only
- Using trace maps of $R^k \equiv (-\Delta - k^2)^{-1}$ and the *generalized BS principle*

$$R^k := R_0^k + \alpha R^k_{\mathrm{d}x,m}[I - \alpha R^k_{m,m}]^{-1} R^k_{m,\mathrm{d}x}\,,$$

where m is δ measure on Γ, we pass to a 1D integral operator problem, $\alpha R^k_{m,m}\psi = \psi$

How to find the spectrum?

The above general results do not tell us how to find the spectrum for a particular Γ. There are various possibilities:

- *Direct solution* of the PDE problem $H_{\alpha,\Gamma}\psi = \lambda\psi$ is feasible in a few simple examples only
- Using trace maps of $R^k \equiv (-\Delta - k^2)^{-1}$ and the *generalized BS principle*

$$R^k := R_0^k + \alpha R^k_{\mathrm{d}x,m}[I - \alpha R^k_{m,m}]^{-1} R^k_{m,\mathrm{d}x}\,,$$

  where m is δ measure on Γ, we pass to a 1D integral operator problem, $\alpha R^k_{m,m}\psi = \psi$
- *discretization* of the latter which amounts to a point-interaction approximations to $H_{\alpha,\Gamma}$

2D point interactions

Such an interaction at the point a with the "coupling constant" α is defined by b.c. which change *locally* the domain of $-\Delta$: the functions behave as

$$\psi(x) = -\frac{1}{2\pi}\log|x-a|\,L_0(\psi,a) + L_1(\psi,a) + \mathcal{O}(|x-a|)\,,$$

where the generalized b.v. $L_0(\psi,a)$ and $L_1(\psi,a)$ satisfy

$$L_1(\psi,a) + 2\pi\alpha L_0(\psi,a) = 0\,, \qquad \alpha \in \mathbb{R}$$

2D point interactions

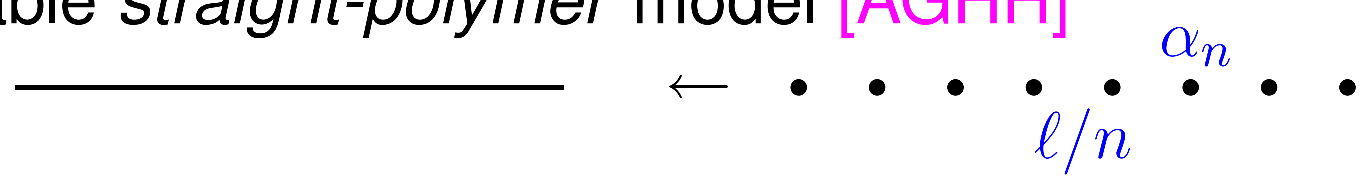
Such an interaction at the point a with the “coupling constant” α is defined by b.c. which change *locally* the domain of $-\Delta$: the functions behave as

$$\psi(x) = -\frac{1}{2\pi} \log |x-a|\, L_0(\psi, a) + L_1(\psi, a) + \mathcal{O}(|x-a|)\,,$$

where the generalized b.v. $L_0(\psi, a)$ and $L_1(\psi, a)$ satisfy

$$L_1(\psi, a) + 2\pi\alpha L_0(\psi, a) = 0\,, \qquad \alpha \in \mathbb{R}$$

For our purpose, the coupling should depend on the set Y approximating Γ. To see how compare a line Γ with the solvable *straight-polymer* model [AGHH]



2D point-interaction approximation

Spectral threshold convergence requires $\alpha_n = \alpha n$ which means that individual point interactions get *weaker*. Hence we approximate $H_{\alpha,\Gamma}$ by point-interaction Hamiltonians H_{α_n,Y_n} with $\alpha_n = \alpha|Y_n|$, where $|Y_n| := \sharp Y_n$.

2D point-interaction approximation

Spectral threshold convergence requires $\alpha_n = \alpha n$ which means that individual point interactions get *weaker*. Hence we approximate $H_{\alpha,\Gamma}$ by point-interaction Hamiltonians H_{α_n,Y_n} with $\alpha_n = \alpha|Y_n|$, where $|Y_n| := \sharp Y_n$.

Theorem [E.-Němcová, 2003]: Let a family $\{Y_n\}$ of finite sets $Y_n \subset \Gamma \subset \mathbb{R}^2$ be such that

$$\frac{1}{|Y_n|} \sum_{y \in Y_n} f(y) \;\to\; \int_\Gamma f \, \mathrm{d}m$$

holds for any bounded continuous function $f : \Gamma \to \mathbb{C}$, *together with technical conditions*, then $H_{\alpha_n,Y_n} \to H_{\alpha,\Gamma}$ in the strong resolvent sense as $n \to \infty$.

Comments on the approximation

- A more general result is valid: Γ need not be a graph and the coupling may be non-constant

Comments on the approximation

- A more general result is valid: Γ need not be a graph and the coupling may be non-constant
- The result applies to finite graphs, however, an infinite Γ can be approximated in strong resolvent sense by a *family of cut-off graphs*

Comments on the approximation

- A more general result is valid: Γ need not be a graph and the coupling may be non-constant
- The result applies to finite graphs, however, an infinite Γ can be approximated in strong resolvent sense by a *family of cut-off graphs*
- The idea is due to Brasche, Figari and Teta, 1998, who analyzed point-interaction approximations of measure perturbations with $\operatorname{codim}\Gamma = 1$ in $\mathbb{R}^3$. There are differences, however, for instance in the 2D case we can approximate *attractive* interactions only

Scheme of the proof

Resolvent of H_{α_n,Y_n} is given *Krein's formula*. Given $k^2 \in \rho(H_{\alpha_n,Y_n})$ define $|Y_n| \times |Y_n|$ matrix by

$$\Lambda_{\alpha_n,Y_n}(k^2;x,y) \;=\; \frac{1}{2\pi}\left[2\pi|Y_n|\alpha + \ln\left(\frac{ik}{2}\right) + \gamma_E\right]\delta_{xy} \\ -G_k(x-y)\,(1-\delta_{xy})$$

for $x, y \in Y_n$, where γ_E is *Euler' constant*.

Scheme of the proof

Resolvent of H_{α_n,Y_n} is given *Krein's formula*. Given $k^2 \in \rho(H_{\alpha_n,Y_n})$ define $|Y_n| \times |Y_n|$ matrix by

$$\Lambda_{\alpha_n,Y_n}(k^2;x,y) \;=\; \frac{1}{2\pi}\left[2\pi|Y_n|\alpha + \ln\left(\frac{ik}{2}\right) + \gamma_E\right]\delta_{xy} \\ -G_k(x-y)\,(1-\delta_{xy})$$

for $x, y \in Y_n$, where γ_E is *Euler' constant*. Then

$$(H_{\alpha_n,Y_n} - k^2)^{-1}(x,y) = G_k(x-y) \\ + \sum_{x',y'\in Y_n} \left[\Lambda_{\alpha_n,Y_n}(k^2)\right]^{-1}(x',y')G_k(x-x')\,G_k(y-y')$$

Scheme of the proof

Resolvent of $H_{\alpha,\Gamma}$ is given by the *generalized BS formula* given above; one has to check directly that the difference of the two vanishes as $n \to \infty$ $\square$

Scheme of the proof

Resolvent of $H_{\alpha,\Gamma}$ is given by the *generalized BS formula* given above; one has to check directly that the difference of the two vanishes as $n \to \infty$ $\square$

Remarks:

- Spectral condition in the n-th approximation, i.e. $\det \Lambda_{\alpha_n,Y_n}(k^2) = 0$, is a discretization of the integral equation coming from the generalized BS principle
- A solution to $\Lambda_{\alpha_n,Y_n}(k^2)\eta = 0$ determines the approximating ef by $\psi(x) = \sum_{y_j \in Y_n} \eta_j G_k(x - y_j)$
- A *match with solvable models* illustrates the convergence and shows that it is *not fast,* slower than n^{-1} in the eigenvalues. This comes from singular “spikes” in the approximating functions

An interlude: scattering on leaky graphs

Let Γ be a graph with *semi-infinite "leads"*, e.g. an infinite asymptotically straight curve. What we know about scattering in such systems? Almost nothing!

- *First question:* What is the "free" operator? $-\Delta$ is not a good candidate, rather $H_{\alpha,\Gamma}$ for a straight line Γ. Recall that we are particularly interested in energy interval $(-\frac{1}{4}\alpha^2, 0)$, i.e. 1D transport of states laterally bound to Γ

An interlude: scattering on leaky graphs

Let Γ be a graph with *semi-infinite "leads"*, e.g. an infinite asymptotically straight curve. What we know about scattering in such systems? Almost nothing!

- *First question:* What is the "free" operator? $-\Delta$ is not a good candidate, rather $H_{\alpha,\Gamma}$ for a straight line Γ. Recall that we are particularly interested in energy interval $(-\frac{1}{4}\alpha^2, 0)$, i.e. 1D transport of states laterally bound to Γ
- Existence proof for the wave operators is absent

An interlude: scattering on leaky graphs

Let Γ be a graph with *semi-infinite "leads"*, e.g. an infinite asymptotically straight curve. What we know about scattering in such systems? Almost nothing!

- *First question:* What is the "free" operator? $-\Delta$ is not a good candidate, rather $H_{\alpha,\Gamma}$ for a straight line Γ. Recall that we are particularly interested in energy interval $(-\frac{1}{4}\alpha^2, 0)$, i.e. 1D transport of states laterally bound to Γ
- Existence proof for the wave operators is absent
- Conjecture: For *strong coupling*, $\alpha \to \infty$, the scattering is described in leading order by $S_\Gamma := -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$

An interlude: scattering on leaky graphs

Let Γ be a graph with *semi-infinite "leads"*, e.g. an infinite asymptotically straight curve. What we know about scattering in such systems? Almost nothing!

- *First question:* What is the "free" operator? $-\Delta$ is not a good candidate, rather $H_{\alpha,\Gamma}$ for a straight line Γ. Recall that we are particularly interested in energy interval $(-\frac{1}{4}\alpha^2, 0)$, i.e. 1D transport of states laterally bound to Γ
- Existence proof for the wave operators is absent
- Conjecture: For *strong coupling*, $\alpha \to \infty$, the scattering is described in leading order by $S_\Gamma := -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$
- On the other hand, in general, the global geometry of Γ is expected to determine the S-matrix

Finally, the resonances

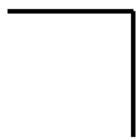
Consider infinite curves Γ, straight outside a compact, and ask for examples of resonances. Recall the L^2-approach: in 1D potential scattering one explores *spectral properties* of the problem cut to a finite length L. It is time-honored trick that scattering resonances are manifested as avoided crossings in L dependence of the spectrum – for a recent proof see Hagedorn-Meller, 2000. Try the same here:

Finally, the resonances

Consider infinite curves Γ, straight outside a compact, and ask for examples of resonances. Recall the L^2-approach: in 1D potential scattering one explores *spectral properties* of the problem cut to a finite length L. It is time-honored trick that scattering resonances are manifested as avoided crossings in L dependence of the spectrum – for a recent proof see Hagedorn-Meller, 2000. Try the same here:

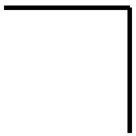
- *Broken line:* absence of "intrinsic" resonances due lack of higher transverse thresholds
- *Z-shaped* Γ*:* if a single bend has a significant reflection, a double band should exhibit resonances
- *Bottleneck curve:* a good candidate to demonstrate tunneling resonances

Broken line

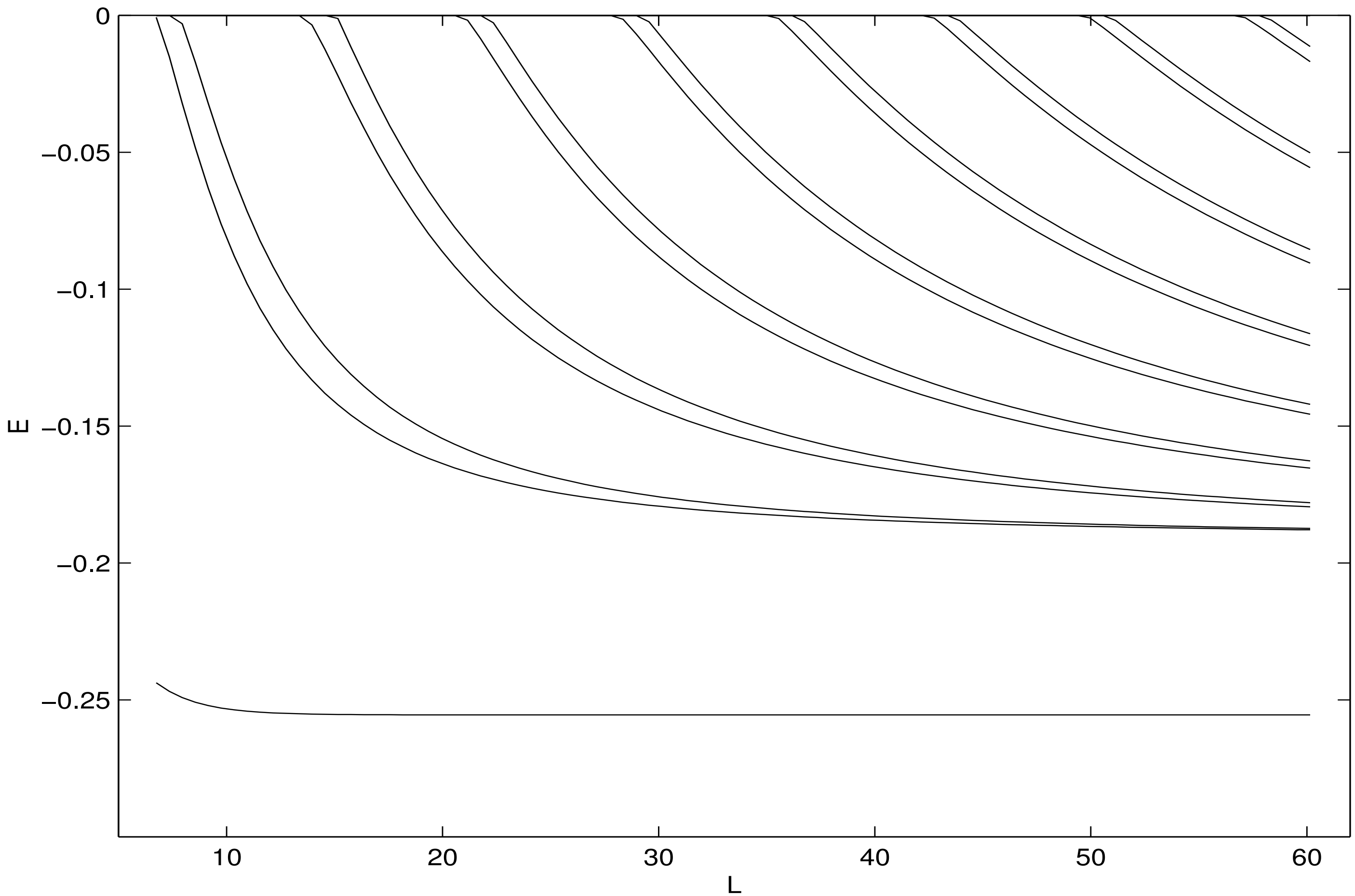


$\alpha = 1$

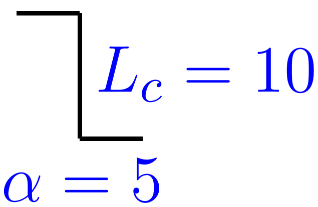
Broken line



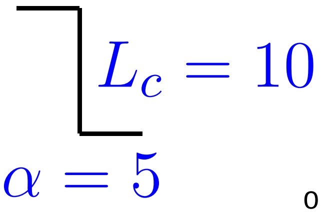
$\alpha = 1$

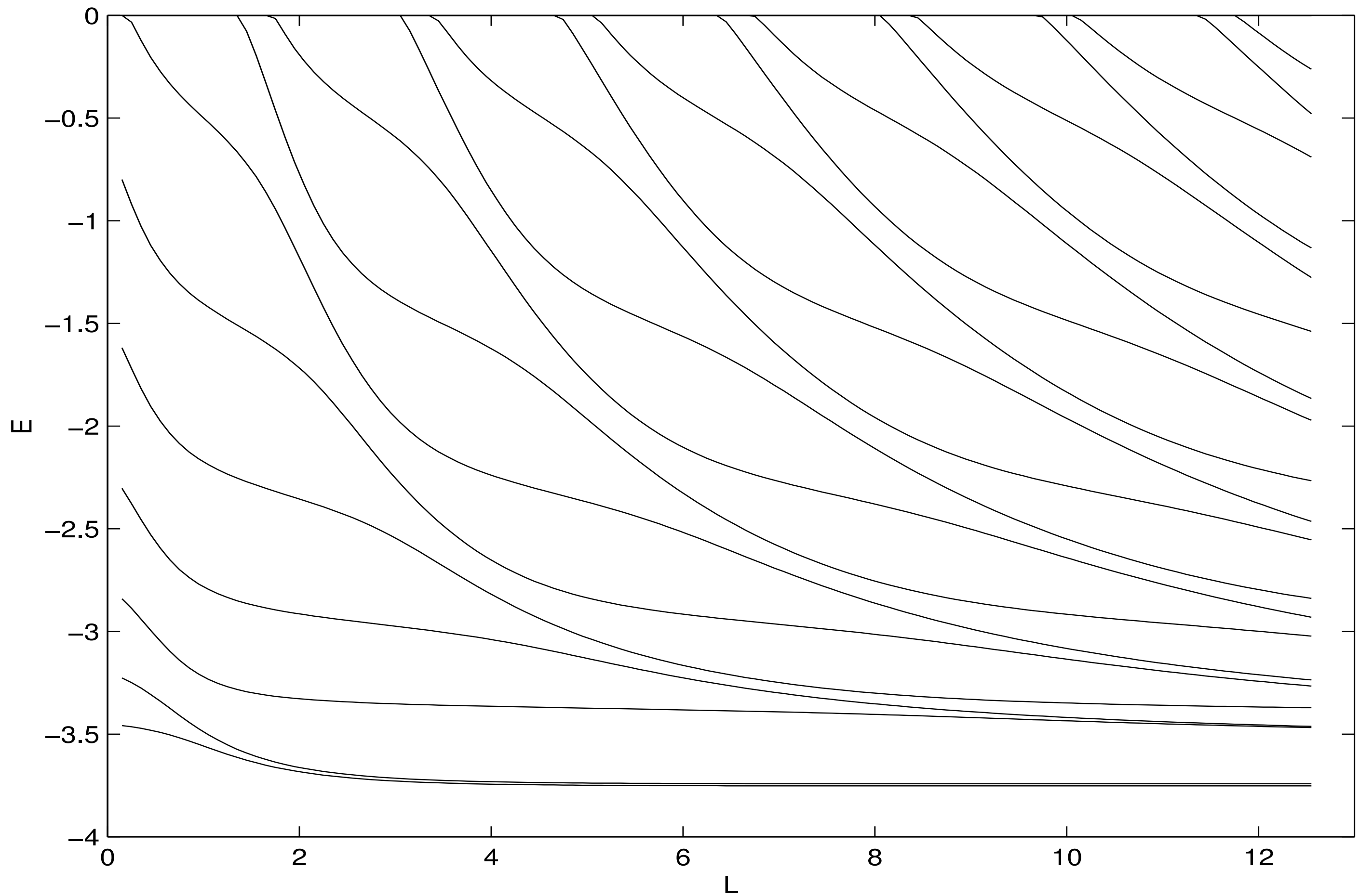


Z shape with $\theta = \frac{\pi}{2}$



Z shape with $\theta = \frac{\pi}{2}$





Z shape with $\theta = 0.32\pi$

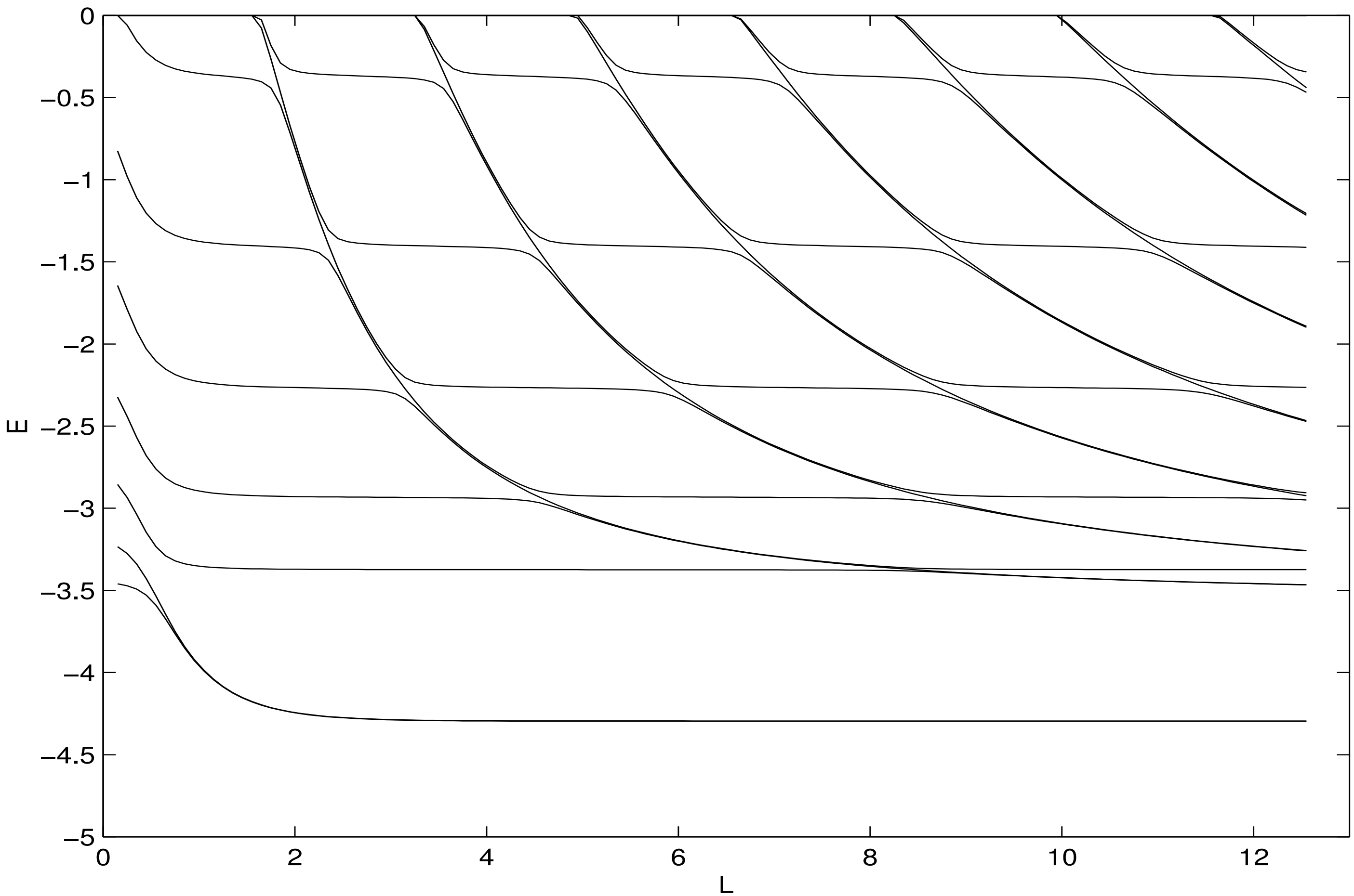
$L_c = 10$

$\alpha = 5$

Z shape with $\theta = 0.32\pi$

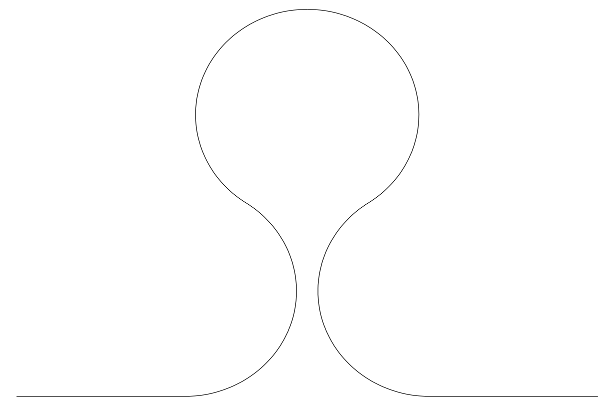
$\sum L_c = 10$

$\alpha = 5$



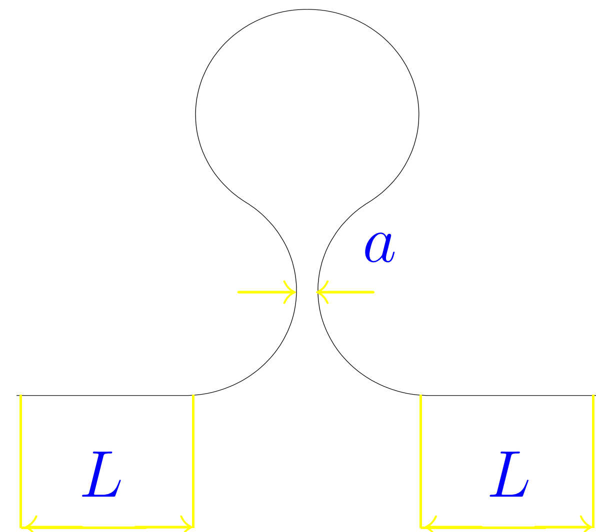
A bottleneck curve

Consider a straight line deformation which shaped as an open loop with a bottleneck the width a of which we will vary



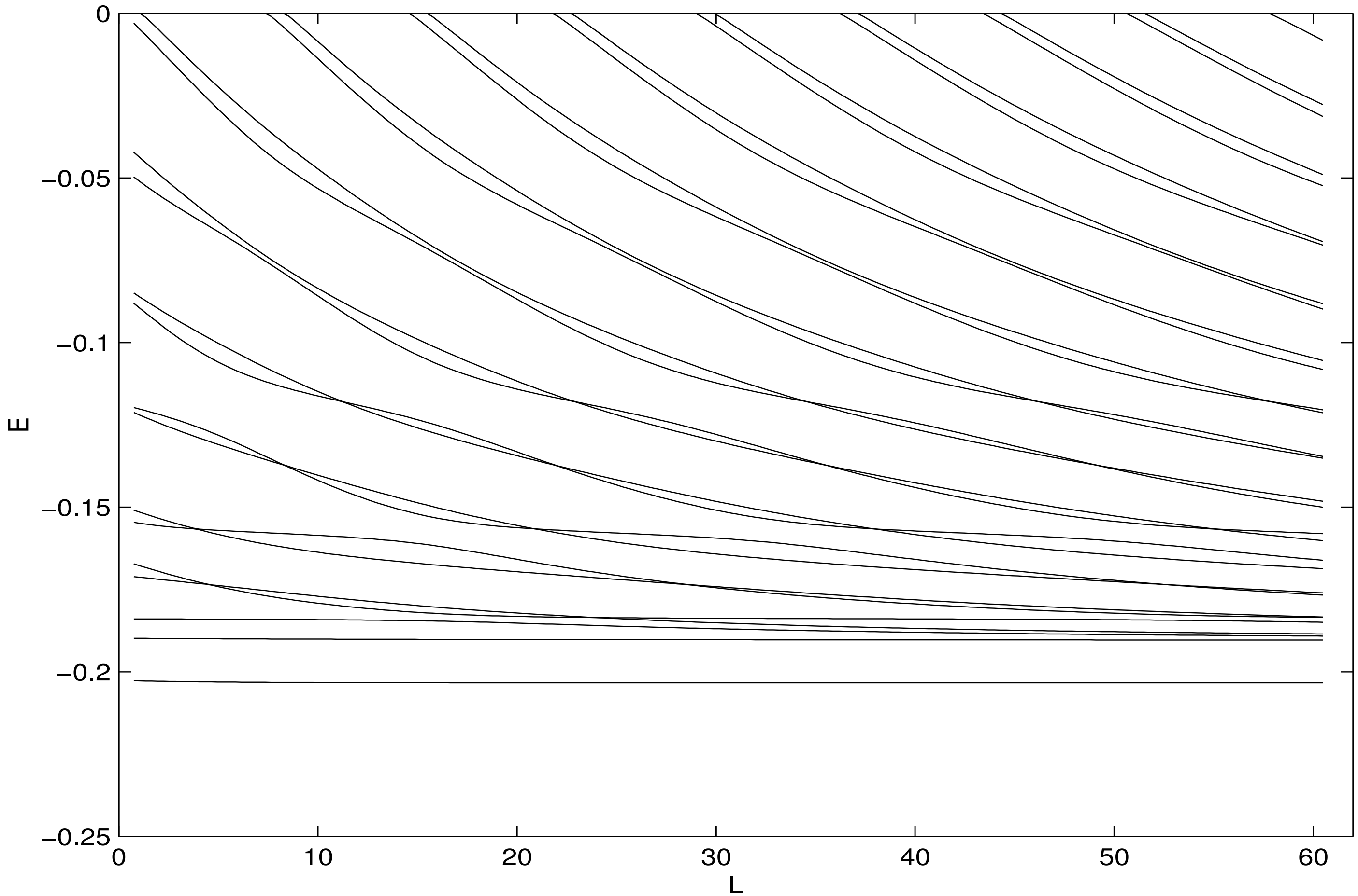
A bottleneck curve

Consider a straight line deformation which shaped as an open loop with a bottleneck the width a of which we will vary

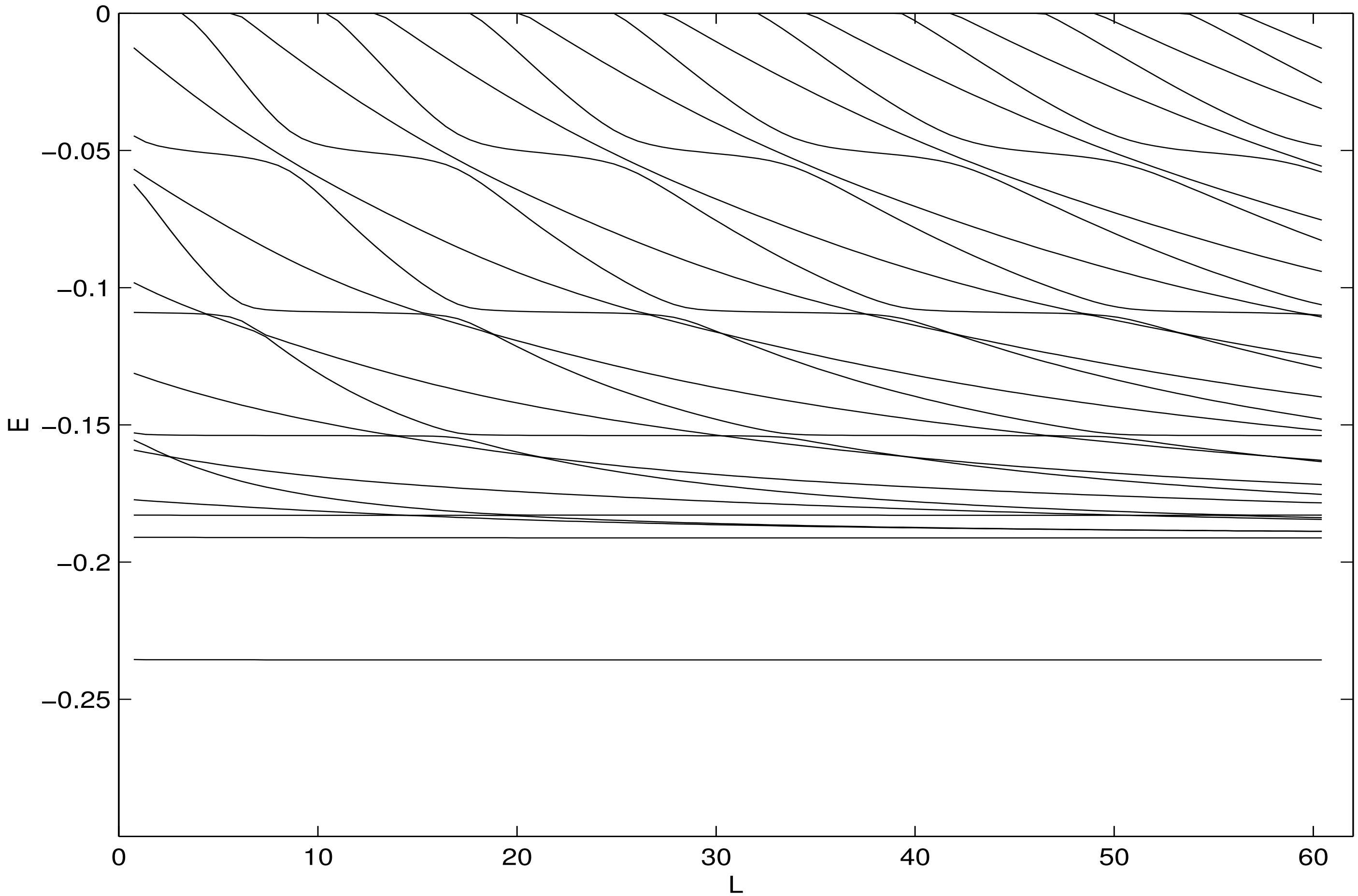


If Γ is a straight line, the transverse eigenfunction is $\mathrm{e}^{-\alpha|y|/2}$, hence the distance at which tunneling becomes significant is $\approx 4\alpha^{-1}$. In the example, we choose $\alpha = 1$

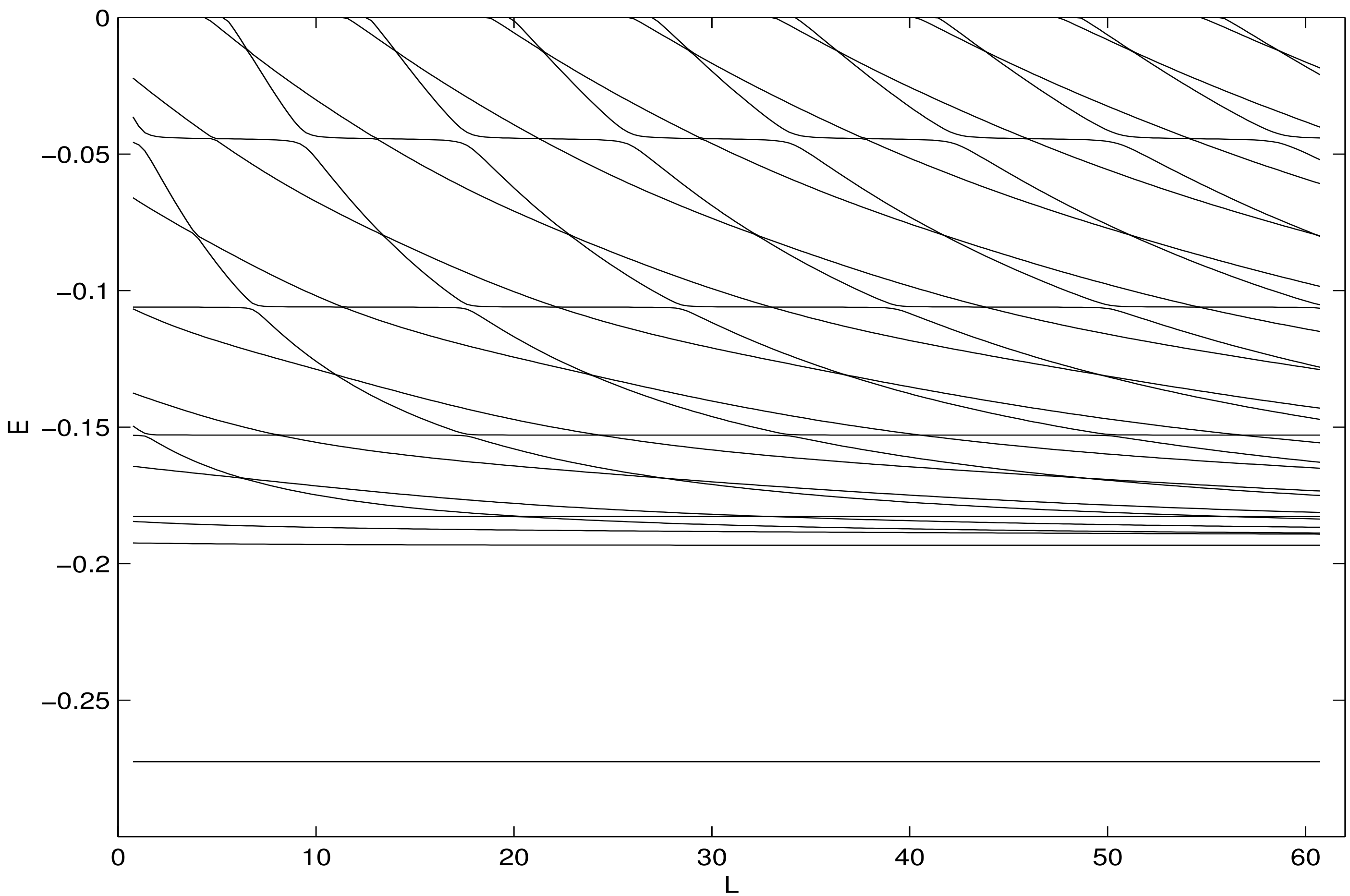
Bottleneck with $a = 5.2$



Bottleneck with $a = 2.9$



Bottleneck with $a = 1.9$



Line and points – a solvable model

Let us pass to a simple model in which existence of resonances can be proved: a straight *leaky wire* and a family of *leaky dots*.

Line and points – a solvable model

Let us pass to a simple model in which existence of resonances can be proved: a straight *leaky wire* and a family of *leaky dots*. Formal Hamiltonian

$$-\Delta - \alpha\delta(x-\Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)})$$

in $L^2(\mathbb{R}^2)$ with $\alpha > 0$. The 2D point interactions at $\Pi = \{y^{(i)}\}$ with couplings $\beta = \{\beta_1, \dots, \beta_n\}$ are properly introduced through b.c. mentioned above, giving Hamiltonian $H_{\alpha,\beta}$

Line and points – a solvable model

Let us pass to a simple model in which existence of resonances can be proved: a straight *leaky wire* and a family of *leaky dots*. Formal Hamiltonian

$$-\Delta - \alpha\delta(x-\Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)})$$

in $L^2(\mathbb{R})$ with $\alpha > 0$. The 2D point interactions at $\Pi = \{y^{(i)}\}$ with couplings $\beta = \{\beta_1, \dots, \beta_n\}$ are properly introduced through b.c. mentioned above, giving Hamiltonian $H_{\alpha,\beta}$

Resolvent by Krein-type formula: given $z \in \mathbb{C} \setminus [0,\infty)$ we start from the free resolvent $R(z) := (-\Delta - z)^{-1}$, also interpreted as unitary $\mathbf{R}(z)$ acting from L^2 to $W^{2,2}$. Then

Resolvent by Krein-type formula

- we introduce auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and trace maps $\tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j$ defined by $\tau_0 f := f \restriction_\Sigma$ and $\tau_1 f := f \restriction_\Pi$,

Resolvent by Krein-type formula

- we introduce auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and trace maps $\tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j$ defined by $\tau_0 f := f \restriction_\Sigma$ and $\tau_1 f := f \restriction_\Pi$,
- then we define canonical embeddings of $\mathbf{R}(z)$ to $\mathcal{H}_i$ by $\mathbf{R}_{i,L}(z) := \tau_i R(z) : \ L^2 \to \mathcal{H}_i$, $\mathbf{R}_{L,i}(z) := [\mathbf{R}_{i,L}(z)]^*$, and $\mathbf{R}_{j,i}(z) := \tau_j \mathbf{R}_{L,i}(z) : \mathcal{H}_i \to \mathcal{H}_j$, and

Resolvent by Krein-type formula

- we introduce auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and trace maps $\tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j$ defined by $\tau_0 f := f \upharpoonright_\Sigma$ and $\tau_1 f := f \upharpoonright_\Pi$,
- then we define canonical embeddings of $\mathbf{R}(z)$ to $\mathcal{H}_i$ by $\mathbf{R}_{i,L}(z) := \tau_i R(z) : \ L^2 \to \mathcal{H}_i$, $\mathbf{R}_{L,i}(z) := [\mathbf{R}_{i,L}(z)]^*$, and $\mathbf{R}_{j,i}(z) := \tau_j \mathbf{R}_{L,i}(z) : \mathcal{H}_i \to \mathcal{H}_j$, and
- operator-valued matrix $\Gamma(z) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$ by

$$\begin{aligned} \Gamma_{ij}(z)g &:= -\mathbf{R}_{i,j}(z)g \quad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j \,, \\ \Gamma_{00}(z)f &:= \left[\alpha^{-1} - \mathbf{R}_{0,0}(z)\right] f \quad \text{if} \quad f \in \mathcal{H}_0 \,, \\ \Gamma_{11}(z)\varphi &:= \Big(s_\beta(z)\delta_{kl} - G_z(y^{(k)}, y^{(l)})(1-\delta_{kl}) \Big) \varphi \,, \end{aligned}$$

with $s_\beta(z) := \beta + s(z) := \beta + \frac{1}{2\pi}(\ln \frac{\sqrt{z}}{2i} - \psi(1))$

Resolvent by Krein-type formula

To invert it we define the "reduced determinant"

$$D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) \;:\; \mathcal{H}_1 \to \mathcal{H}_1\,,$$

Resolvent by Krein-type formula

To invert it we define the "reduced determinant"

$$D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) \;:\; \mathcal{H}_1 \to \mathcal{H}_1 \,,$$

then an easy algebra yields expressions for "blocks" of $[\Gamma(z)]^{-1}$ in the form

$$\begin{aligned}
[\Gamma(z)]^{-1}_{11} &= D(z)^{-1}\,, \\
[\Gamma(z)]^{-1}_{00} &= \Gamma_{10}(z)^{-1}\Gamma_{11}(z)D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1}\,, \\
[\Gamma(z)]^{-1}_{01} &= -\Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1}\,, \\
[\Gamma(z)]^{-1}_{10} &= -D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1}\,;
\end{aligned}$$

thus to determine singularities of $[\Gamma(z)]^{-1}$ one has to find the null space of $D(z)$

Resolvent by Krein-type formula

With this notation we can state the sought formula:

Theorem [E.-Kondej, 2003]: For $z \in \rho(H_{\alpha,\beta})$ with $\mathrm{Im}\, z > 0$ the resolvent $R_{\alpha,\beta}(z) := (H_{\alpha,\beta} - z)^{-1}$ equals

$$R_{\alpha,\beta}(z) = R(z) + \sum_{i,j=0}^{1} \mathbf{R}_{L,i}(z)[\Gamma(z)]_{ij}^{-1} \mathbf{R}_{j,L}(z)$$

Resolvent by Krein-type formula

With this notation we can state the sought formula:

Theorem [E.-Kondej, 2003]: For $z \in \rho(H_{\alpha,\beta})$ with $\operatorname{Im} z > 0$ the resolvent $R_{\alpha,\beta}(z) := (H_{\alpha,\beta} - z)^{-1}$ equals

$$R_{\alpha,\beta}(z) = R(z) + \sum_{i,j=0}^{1} \mathbf{R}_{L,i}(z)[\Gamma(z)]_{ij}^{-1}\mathbf{R}_{j,L}(z)$$

Remark: One can also compare resolvent of $H_{\alpha,\beta}$ to that of $H_\alpha \equiv H_{\alpha,\Sigma}$ using trace maps of the latter,

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \mathbf{R}_{\alpha;L1}(z)D(z)^{-1}\mathbf{R}_{\alpha;1L}(z)$$

Spectral properties of $H_{\alpha,\beta}$

It is easy to check that

$$\sigma_{\rm ess}(H_{\alpha,\beta}) = \sigma_{\rm ac}(H_{\alpha,\beta}) = [-\frac{1}{4}\alpha^2, \infty)$$

Spectral properties of $H_{\alpha,\beta}$

It is easy to check that

$$\sigma_{\text{ess}}(H_{\alpha,\beta}) = \sigma_{\text{ac}}(H_{\alpha,\beta}) = [-\frac{1}{4}\alpha^2, \infty)$$

σ_{disc} given by *generalized Birman-Schwinger principle:*

$$\begin{aligned} \dim\ker\Gamma(z) &= \dim\ker R_{\alpha,\beta}(z)\,, \\ H_{\alpha,\beta}\phi_z = z\phi_z \;\Leftrightarrow\; \phi_z &= \sum_{i=0}^{1} \mathbf{R}_{L,i}(z)\eta_{i,z}\,, \end{aligned}$$

where $(\eta_{0,z}, \eta_{1,z}) \in \ker\Gamma(z)$. Moreover, it is clear that $0 \in \sigma_{\text{disc}}(\Gamma(z)) \Leftrightarrow 0 \in \sigma_{\text{disc}}(D(z))$; this reduces the task of finding the spectrum to an algebraic problem

Spectral properties of $H_{\alpha,\beta}$

Theorem [E.-Kondej, 2003]: (a) Let $n = 1$ and denote $\mathrm{dist}\,(\sigma, \Pi) =: a$, then $H_{\alpha,\beta}$ has one isolated eigenvalue $-\kappa_a^2$. The function $a \mapsto -\kappa_a^2$ is increasing in $(0, \infty)$,

$$\lim_{a\to\infty}(-\kappa_a^2) = \min\left\{\epsilon_\beta,\, -\frac{1}{4}\alpha^2\right\},$$

where $\epsilon_\beta := -4\mathrm{e}^{2(-2\pi\beta+\psi(1))}$, while $\lim_{a\to 0}(-\kappa_a^2)$ is finite.

Spectral properties of $H_{\alpha,\beta}$

Theorem [E.-Kondej, 2003]: (a) Let $n = 1$ and denote $\mathrm{dist}\,(\sigma, \Pi) =: a$, then $H_{\alpha,\beta}$ has one isolated eigenvalue $-\kappa_a^2$. The function $a \mapsto -\kappa_a^2$ is increasing in $(0, \infty)$,

$$\lim_{a\to\infty}(-\kappa_a^2) = \min\left\{\epsilon_\beta,\, -\frac{1}{4}\alpha^2\right\},$$

where $\epsilon_\beta := -4\mathrm{e}^{2(-2\pi\beta+\psi(1))}$, while $\lim_{a\to 0}(-\kappa_a^2)$ is finite.
(b) For any $\alpha > 0,\ \beta \in \mathbb{R}^n$, and $n \in \mathbb{N}_+$ the operator $H_{\alpha,\beta}$ has N isolated eigenvalues, $1 \le N \le n$. If all the point interactions are strong enough, we have $N = n$

Spectral properties of $H_{\alpha,\beta}$

Theorem [E.-Kondej, 2003]: (a) Let $n = 1$ and denote $\mathrm{dist}\,(\sigma, \Pi) =: a$, then $H_{\alpha,\beta}$ has one isolated eigenvalue $-\kappa_a^2$. The function $a \mapsto -\kappa_a^2$ is increasing in $(0, \infty)$,

$$\lim_{a\to\infty} (-\kappa_a^2) = \min\left\{\epsilon_\beta,\, -\frac{1}{4}\alpha^2\right\},$$

where $\epsilon_\beta := -4\mathrm{e}^{2(-2\pi\beta+\psi(1))}$, while $\lim_{a\to 0}(-\kappa_a^2)$ is finite.
(b) For any $\alpha > 0,\ \beta \in \mathbb{R}^n$, and $n \in \mathbb{N}_+$ the operator $H_{\alpha,\beta}$ has N isolated eigenvalues, $1 \le N \le n$. If all the point interactions are strong enough, we have $N = n$

Remark: Embedded eigenvalues due to mirror symmetry w.r.t. Σ possible if $n \ge 2$

Resonance for $n = 1$

Assume the point interaction eigenvalue *becomes embedded* as $a \to \infty$, i.e. that $\epsilon_\beta > -\frac{1}{4}\alpha^2$

Resonance for $n = 1$

Assume the point interaction eigenvalue *becomes embedded* as $a \to \infty$, i.e. that $\epsilon_\beta > -\frac{1}{4}\alpha^2$

Observation: Birman-Schwinger works in the complex domain too *(recall P. Hislops's talk for regular potentials)*. Thus it is enough to look for analytical continuation of $D(\cdot)$, which acts for $z \in \mathbb{C} \setminus [-\frac{1}{4}\alpha^2, \infty)$ as a multiplication by

$$d_a(z) := s_\beta(z) - \phi_a(z) = s_\beta(z) - \int_0^\infty \frac{\mu(z,t)}{t - z - \frac{1}{4}\alpha^2}\,\mathrm{d}t\,,$$

$$\mu(z,t) := \frac{i\alpha}{16\pi}\,\frac{\left(\alpha - 2i(z-t)^{1/2}\right)\mathrm{e}^{2ia(z-t)^{1/2}}}{t^{1/2}(z-t)^{1/2}}$$

Thus we have a situation reminiscent of Friedrichs model, just the functions involved are more complicated

Analytic continuation

Take a region Ω_- of the other sheet with $(-\frac{1}{4}\alpha^2, 0)$ as a part of its boundary. Put $\mu^0(\lambda, t) := \lim_{\varepsilon\to 0} \mu(\lambda+i\varepsilon, t)$, define

$$I(\lambda) := \mathcal{P} \int_0^\infty \frac{\mu^0(\lambda, t)}{t - \lambda - \frac{1}{4}\alpha^2}\, \mathrm{d}t\,,$$

and furthermore, $g_{\alpha,a}(z) := \frac{i\alpha}{4}\, \frac{\mathrm{e}^{-\alpha a}}{(z+\frac{1}{4}\alpha^2)^{1/2}}$.

Analytic continuation

Take a region Ω_- of the other sheet with $(-\frac{1}{4}\alpha^2, 0)$ as a part of its boundary. Put $\mu^0(\lambda, t) := \lim_{\varepsilon\to 0} \mu(\lambda + i\varepsilon, t)$, define

$$I(\lambda) := \mathcal{P} \int_0^\infty \frac{\mu^0(\lambda, t)}{t - \lambda - \frac{1}{4}\alpha^2}\, \mathrm{d}t\,,$$

and furthermore, $g_{\alpha,a}(z) := \frac{i\alpha}{4} \frac{\mathrm{e}^{-\alpha a}}{(z + \frac{1}{4}\alpha^2)^{1/2}}$.

Lemma: $z \mapsto \phi_a(z)$ is continued analytically to Ω_- as

$$\phi_a^0(\lambda) = I(\lambda) + g_{\alpha,a}(\lambda) \quad \text{for} \quad \lambda \in \left(-\frac{1}{4}\alpha^2, 0\right),$$

$$\phi_a^-(z) = -\int_0^\infty \frac{\mu(z, t)}{t - z - \frac{1}{4}\alpha^2}\, \mathrm{d}t - 2g_{\alpha,a}(z)\,, \;\; z \in \Omega_-$$

Analytic continuation

Proof: By a direct computation one checks

$$\lim_{\varepsilon\to 0^+} \phi_a^{\pm}(\lambda \pm i\varepsilon) = \phi_a^0(\lambda)\,, \qquad -\frac{1}{4}\alpha^2 < \lambda < 0\,,$$

so the claim follows from edge-of-the-wedge theorem. $\square$

Analytic continuation

Proof: By a direct computation one checks

$$\lim_{\varepsilon\to 0^+} \phi_a^{\pm}(\lambda \pm i\varepsilon) = \phi_a^0(\lambda)\,, \qquad -\frac{1}{4}\alpha^2 < \lambda < 0\,,$$

so the claim follows from edge-of-the-wedge theorem. □

The continuation of d_a is thus the function $\eta_a : M \mapsto \mathbb{C}$, where $M = \{z : \mathrm{Im}\, z > 0\} \cup (-\frac{1}{4}\alpha^2, 0) \cup \Omega_-$, acting as

$$\eta_a(z) = s_\beta(z) - \phi_a^{l(z)}(z)\,,$$

and our problem reduces to solution if the implicit function problem $\eta_a(z) = 0$.

Resonance for $n = 1$

Theorem [E.-Kondej, 2003]: Assume $\epsilon_\beta > -\frac{1}{4}\alpha^2$. For any a large enough the equation $\eta_a(z) = 0$ has a unique solution $z(a) = \mu(b) + i\nu(b) \in \Omega_-$, i.e. $\nu(a) < 0$, with the following asymptotic behaviour as $a \to \infty$,

$$\mu(a) = \epsilon_\beta + \mathcal{O}(\mathrm{e}^{-a\sqrt{-\epsilon_\beta}})\,, \quad \nu(a) = \mathcal{O}(\mathrm{e}^{-a\sqrt{-\epsilon_\beta}})$$

Resonance for $n = 1$

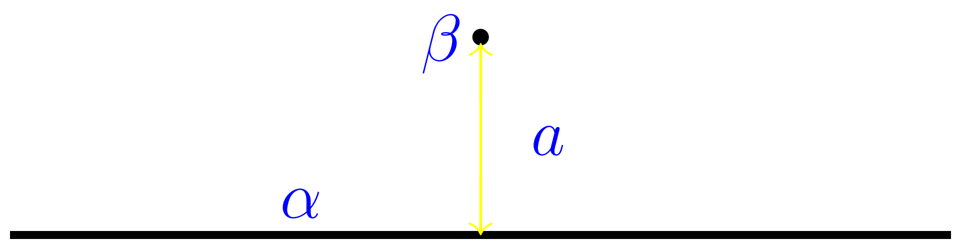
Theorem [E.-Kondej, 2003]: Assume $\epsilon_\beta > -\frac{1}{4}\alpha^2$. For any a large enough the equation $\eta_a(z) = 0$ has a unique solution $z(a) = \mu(b) + i\nu(b) \in \Omega_-$, i.e. $\nu(a) < 0$, with the following asymptotic behaviour as $a \to \infty$,

$$\mu(a) = \epsilon_\beta + \mathcal{O}(\mathrm{e}^{-a\sqrt{-\epsilon_\beta}}), \quad \nu(a) = \mathcal{O}(\mathrm{e}^{-a\sqrt{-\epsilon_\beta}})$$

Remark: We have $|\phi_a^-(z)| \to 0$ uniformly in a and $|s_\beta(z)| \to \infty$ as $\operatorname{Im} z \to -\infty$. Hence the imaginary part $z(a)$ is bounded as a function of a, in particular, *the resonance pole survives* as $a \to 0$.

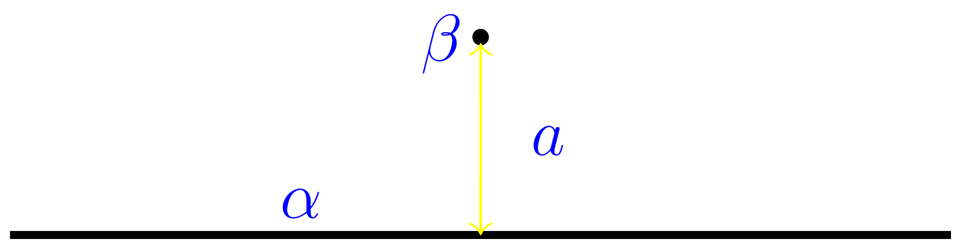
Scattering for $n = 1$

The same as scattering problem for $(H_{\alpha,\beta}, H_\alpha)$



Scattering for $n = 1$

The same as scattering problem for $(H_{\alpha,\beta}, H_\alpha)$



Existence and completeness by Birman-Kuroda theorem; we seek on-shell S-matrix in $(-\frac{1}{4}\alpha^2, 0)$. By Krein formula, resolvent for $\operatorname{Im} z > 0$ expresses as

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \eta_a(z)^{-1}(\cdot, v_z)v_z,$$

where $v_z := R_{\alpha;L,1}(z)$

Scattering for $n = 1$

Apply this operator to vector

$$\omega_{\lambda+i\varepsilon}(x) := \mathrm{e}^{i(\lambda+i\varepsilon+\alpha^2/4)^{1/2}x_1}\,\mathrm{e}^{-\alpha|x_2|/2}$$

and take limit $\varepsilon \to 0+$ in the sense of distributions; then a straightforward calculation give generalized eigenfunction of $H_{\alpha,\beta}$. In particular, we have

Scattering for $n = 1$

Apply this operator to vector

$$\omega_{\lambda+i\varepsilon}(x) := \mathrm{e}^{i(\lambda+i\varepsilon+\alpha^2/4)^{1/2}x_1}\,\mathrm{e}^{-\alpha|x_2|/2}$$

and take limit $\varepsilon \to 0+$ in the sense of distributions; then a straightforward calculation give generalized eigenfunction of $H_{\alpha,\beta}$. In particular, we have

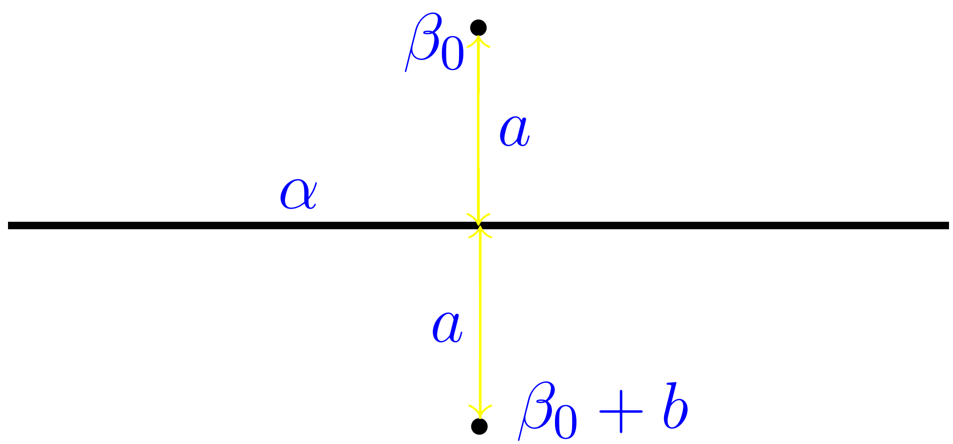
Proposition: For any $\lambda \in (-\frac{1}{4}\alpha^2, 0)$ the reflection and transmission amplitudes are

$$\mathcal{R}(\lambda) = \mathcal{T}(\lambda) - 1 = \frac{i}{4}\,\alpha\eta_a(\lambda)^{-1}\,\frac{\mathrm{e}^{-\alpha a}}{(\lambda+\frac{1}{4}\alpha^2)^{1/2}}\,;$$

they have the same pole in the analytical continuation to Ω_- as the continued resolvent

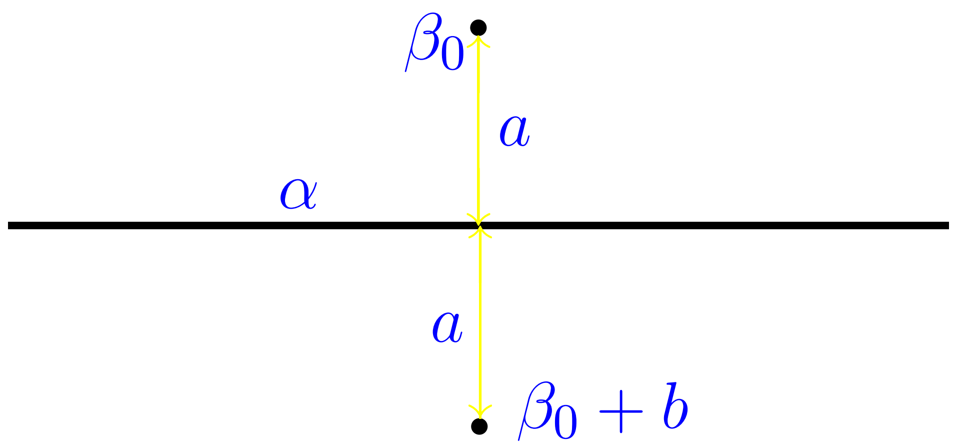
Resonances from perturbed symmetry

Take the simplest situation, $n = 2$



Resonances from perturbed symmetry

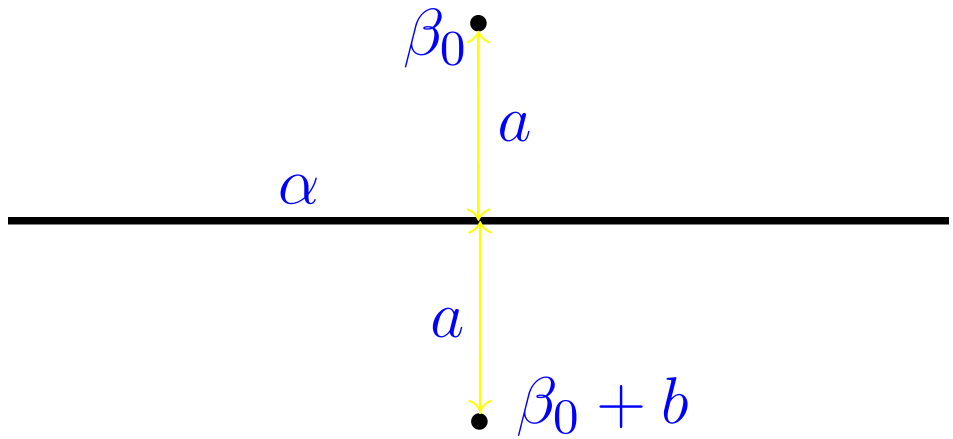
Take the simplest situation, $n = 2$



Let $\sigma_{\mathrm{disc}}(H_{0,\beta_0}) \cap \left(-\frac{1}{4}\alpha^2, 0\right) \neq \emptyset$, so that Hamiltonian H_{0,β_0} has two eigenvalues, the larger of which, ϵ_2, exceeds $-\frac{1}{4}\alpha^2$. Then H_{α,β_0} has the same eigenvalue ϵ_2 embedded in the negative part of continuous spectrum

Resonances from perturbed symmetry

Take the simplest situation, $n = 2$



Let $\sigma_{\text{disc}}(H_{0,\beta_0}) \cap \left(-\frac{1}{4}\alpha^2, 0\right) \neq \emptyset$, so that Hamiltonian H_{0,β_0} has two eigenvalues, the larger of which, ϵ_2, exceeds $-\frac{1}{4}\alpha^2$. Then H_{α,β_0} has the same eigenvalue ϵ_2 embedded in the negative part of continuous spectrum

One has now to continue analytically the 2×2 matrix function $D(\cdot)$. Put $\kappa_2 := \sqrt{-\epsilon_2}$ and $\breve{s}_\beta(\kappa) := s_\beta(-\kappa^2)$

Resonances from perturbed symmetry

Proposition: Assume $\epsilon_2 \in (-\frac{1}{4}\alpha^2, 0)$ and denote $\tilde{g}(\lambda) := -i g_{\alpha,a}(\lambda)$. Then for all b small enough the continued function has a unique zero $z_2(b) = \mu_2(b) + i\nu_2(b) \in \Omega_-$ with the asymptotic expansion

$$\begin{aligned}
\mu_2(b) &= \epsilon_2 + \frac{\kappa_2 b}{\breve{s}'_\beta(\kappa_2) + K'_0(2a\kappa_2)} + \mathcal{O}(b^2)\,, \\
\nu_2(b) &= -\frac{\kappa_2 \tilde{g}(\epsilon_2) b^2}{2(\breve{s}'_\beta(\kappa_2) + K'_0(2a\kappa_2))|\breve{s}'_\beta(\kappa_2) - \phi^0_a(\epsilon_2)|} + \mathcal{O}(b^3)
\end{aligned}$$

Unstable state decay, $n = 1$

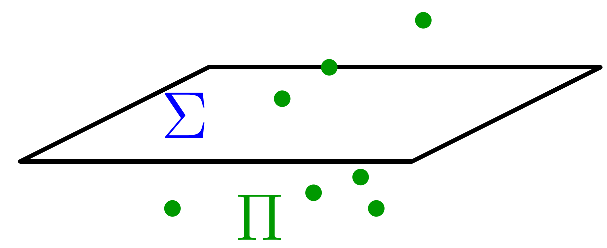
Complementary point of view: investigate decay of unstable state associated with the resonance; assume again $n = 1$. We found that if the "unperturbed" ev ϵ_β of H_β is embedded in $(-\frac{1}{4}\alpha^2, 0)$ and a is large, the corresponding resonance has a long halflife. In analogy with *Friedrichs model* [Demuth, 1976] one conjectures that in weak coupling case, the resonance state would be similar up to normalization to the eigenvector $\xi_0 := K_0(\sqrt{-\epsilon_\beta}\,\cdot)$ of H_β, with the *decay law being dominated by the exponential term*

Unstable state decay, $n = 1$

Complementary point of view: investigate decay of unstable state associated with the resonance; assume again $n = 1$. We found that if the "unperturbed" ev ϵ_β of H_β is embedded in $(-\frac{1}{4}\alpha^2, 0)$ and a is large, the corresponding resonance has a long halflife. In analogy with *Friedrichs model* [Demuth, 1976] one conjectures that in weak coupling case, the resonance state would be similar up to normalization to the eigenvector $\xi_0 := K_0(\sqrt{-\epsilon_\beta}\,\cdot)$ of H_β, with the *decay law being dominated by the exponential term*

At the same time, $H_{\alpha,\beta}$ has always an isolated ev with ef which is *not* orthogonal to ξ_0 for any a (recall that both functions are positive). Consequently, the decay law $|(\xi_0, U(t)\xi_0)|^2 \|\xi_0\|^{-2}$ *has always a nonzero limit* as $t \to \infty$

Extension: plane and points



In a similar way one can treat a 3D model with interaction supported by a *plane* and a family of *points*, formally

$$-\Delta - \alpha\delta(x-\Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)})$$

in $L^2(\mathbb{R}^3)$ with $\alpha > 0$. The point interactions at $\Pi = \{y^{(i)}\}$ with couplings $\beta = \{\beta_1, \dots, \beta_n\}$ are properly introduced through appropriate b.c., giving Hamiltonian $H_{\alpha,\beta}$

Extension: plane and points

- The resolvent is given by a similar *Krein-type formula* using the resolvent of $-\Delta$ in $L^2(\mathbb{R}^3)$

Extension: plane and points

- The resolvent is given by a similar *Krein-type formula* using the resolvent of $-\Delta$ in $L^2(\mathbb{R}^3)$
- $\sigma_{\text{ess}}(H_{\alpha,\beta}) = \sigma_{\text{ac}}(H_{\alpha,\beta}) = [-\frac{1}{4}\alpha^2, \infty)$

Extension: plane and points

- The resolvent is given by a similar *Krein-type formula* using the resolvent of $-\Delta$ in $L^2(\mathbb{R}^3)$
- $\sigma_{\mathrm{ess}}(H_{\alpha,\beta}) = \sigma_{\mathrm{ac}}(H_{\alpha,\beta}) = [-\frac{1}{4}\alpha^2, \infty)$
- If $n = 1$ there is one isolated ev $-\kappa_a^2 < -\frac{1}{4}\alpha^2$. If $\beta > 0$ or $\tilde{\epsilon}_\beta \in [-\frac{1}{4}\alpha^2, \infty)$, where $\tilde{\epsilon}_\beta := -(4\pi\beta)^2$, then

$$-\lim_{a\to\infty} \kappa_a^2 = \tilde{\epsilon}_\beta \,,$$

otherwise we have

$$-\lim_{a\to\infty} \kappa_a^2 = -\frac{1}{4}\alpha^2$$

Recall that $\sigma_{\mathrm{disc}}(H_{0,\beta}) = \emptyset$ for $\beta > 0$

Extension: plane and points

- In distinction to 2D situation we have $-\lim_{a\to 0}\kappa_a^2 = -\infty$

Extension: plane and points

- In distinction to 2D situation we have $-\lim_{a\to 0}\kappa_a^2 = -\infty$
- In general there are $1 \le N \le n$ ev's, in particular, exactly n if all the $-\beta_i$ are large enough

Extension: plane and points

- In distinction to 2D situation we have $-\lim_{a\to 0}\kappa_a^2 = -\infty$
- In general there are $1 \le N \le n$ ev's, in particular, exactly n if all the $-\beta_i$ are large enough
- Let $n=1$ and $\tilde\epsilon_\beta > -\frac{1}{4}\alpha^2$. For large enough a the resolvent $R_{\alpha,\beta}$ has 2nd sheet pole $z(a)=\mu(a)+i\nu(a)$, $\nu(a)<0$, which for $a\to\infty$ behaves as

$$\mu(a) = \tilde\epsilon_\beta + \mathcal{O}(\mathrm{e}^{-a\tilde\varsigma_\beta})\,,\ \nu(a) = \mathcal{O}(\mathrm{e}^{-a\tilde\varsigma_\beta})$$

Extension: plane and points

- In distinction to 2D situation we have $-\lim_{a\to 0}\kappa_a^2 = -\infty$
- In general there are $1 \le N \le n$ ev's, in particular, exactly n if all the $-\beta_i$ are large enough
- Let $n = 1$ and $\tilde{\epsilon}_\beta > -\frac{1}{4}\alpha^2$. For large enough a the resolvent $R_{\alpha,\beta}$ has 2nd sheet pole $z(a) = \mu(a) + i\nu(a)$, $\nu(a) < 0$, which for $a \to \infty$ behaves as

$$\mu(a) = \tilde{\epsilon}_\beta + \mathcal{O}(\mathrm{e}^{-a\tilde{\varsigma}_\beta})\,,\ \nu(a) = \mathcal{O}(\mathrm{e}^{-a\tilde{\varsigma}_\beta})$$

- The resonance pole exists even if the distance is not large. In contrast to the two dimensional case, however, the imaginary part of the pole position $\nu(a)$ diverges to $-\infty$ in the limit $a \to 0$

Open questions

- *Strong coupling asymptotics of* $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma})$ is not known for curves with *open ends* (manifolds with boundaries). For *smooth* Γ, one conjectures similar asymptotics, where S_Γ has *Dirichlet b.c.* For *non-smooth* Γ the leading term is expected to be different

Open questions

- *Strong coupling asymptotics of* $\sigma_{\rm disc}(H_{\alpha,\Gamma})$ is not known for curves with *open ends* (manifolds with boundaries). For *smooth* Γ, one conjectures similar asymptotics, where S_Γ has *Dirichlet b.c.* For *non-smooth* Γ the leading term is expected to be different
- *Scattering on leaky graphs:* existence and completess

Open questions

- *Strong coupling asymptotics of* $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma})$ is not known for curves with *open ends* (manifolds with boundaries). For *smooth* Γ, one conjectures similar asymptotics, where S_Γ has *Dirichlet b.c.* For *non-smooth* Γ the leading term is expected to be different
- *Scattering on leaky graphs:* existence and completess
- *Scattering on leaky curves:* strong coupling asymptotics

Open questions

- *Strong coupling asymptotics of* $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma})$ is not known for curves with *open ends* (manifolds with boundaries). For *smooth* Γ, one conjectures similar asymptotics, where S_Γ has *Dirichlet b.c.* For *non-smooth* Γ the leading term is expected to be different
- *Scattering on leaky graphs:* existence and completess
- *Scattering on leaky curves:* strong coupling asymptotics
- *Resonances:* existence, properties in less trivial models

Open questions

- *Strong coupling asymptotics of* $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma})$ is not known for curves with *open ends* (manifolds with boundaries). For *smooth* Γ, one conjectures similar asymptotics, where S_Γ has *Dirichlet b.c.* For *non-smooth* Γ the leading term is expected to be different
- *Scattering on leaky graphs:* existence and completess
- *Scattering on leaky curves:* strong coupling asymptotics
- *Resonances:* existence, properties in less trivial models
- *Periodic* Γ*, in one direction:* absolute continuity (proved so far only at the bottom of the spectrum)

Open questions

- *Strong coupling asymptotics of* $\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma})$ is not known for curves with *open ends* (manifolds with boundaries). For *smooth* Γ, one conjectures similar asymptotics, where S_Γ has *Dirichlet b.c.* For *non-smooth* Γ the leading term is expected to be different
- *Scattering on leaky graphs:* existence and completess
- *Scattering on leaky curves:* strong coupling asymptotics
- *Resonances:* existence, properties in less trivial models
- *Periodic* Γ*, in one direction:* absolute continuity (proved so far only at the bottom of the spectrum)
- *More questions:* random leaky graphs, adding magnetic fields, justification of the L^2 approach for leaky-graph resonances, etc.

The talk was based on

[EI01] P.E., T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys.* **A34** (2001), 1439-1450.

[EK02] P.E., S. Kondej: Curvature-induced bound states for a δ interaction supported by a curve in $\mathbb{R}^3$, *Ann. H. Poincaré* **3** (2002), 967-981.

[EK03a] P.E., S. Kondej: Bound states due to a strong δ interaction supported by a curved surface, *J. Phys.* **A36** (2003), 443-457.

[EK03c] P.E., S. Kondej: Schrödinger operators with singular interactions: a model of tunneling resonances, `math-ph/0312055`

[EN03] P.E., K. Němcová: Leaky quantum graphs: approximations by point interaction Hamiltonians, *J. Phys.* **A36** (2003), 10173-10193.

[EY01] P.E., K. Yoshitomi: Band gap of the Schrödinger operator with a strong δ-interaction on a periodic curve, *Ann. H. Poincaré* **2** (2001), 1139-1158.

[EY02a] P.E., K. Yoshitomi: Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop, *J. Geom. Phys.* **41** (2002), 344-358.

[EY02b] P.E., K. Yoshitomi: Persistent currents for 2D Schrödinger operator with a strong δ-interaction on a loop, *J. Phys.* **A35** (2002), 3479-3487.

[EY03] P.E., K. Yoshitomi: Eigenvalue asymptotics for the Schrödinger operator with a δ-interaction on a punctured surface, *Lett. Math. Phys.* **65** (2003), 19-26.

The talk was based on

[EI01] P.E., T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys.* **A34** (2001), 1439-1450.

[EK02] P.E., S. Kondej: Curvature-induced bound states for a δ interaction supported by a curve in $\mathbb{R}^3$, *Ann. H. Poincaré* **3** (2002), 967-981.

[EK03a] P.E., S. Kondej: Bound states due to a strong δ interaction supported by a curved surface, *J. Phys.* **A36** (2003), 443-457.

[EK03c] P.E., S. Kondej: Schrödinger operators with singular interactions: a model of tunneling resonances, `math-ph/0312055`

[EN03] P.E., K. Němcová: Leaky quantum graphs: approximations by point interaction Hamiltonians, *J. Phys.* **A36** (2003), 10173-10193.

[EY01] P.E., K. Yoshitomi: Band gap of the Schrödinger operator with a strong δ-interaction on a periodic curve, *Ann. H. Poincaré* **2** (2001), 1139-1158.

[EY02a] P.E., K. Yoshitomi: Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop, *J. Geom. Phys.* **41** (2002), 344-358.

[EY02b] P.E., K. Yoshitomi: Persistent currents for 2D Schrödinger operator with a strong δ-interaction on a loop, *J. Phys.* **A35** (2002), 3479-3487.

[EY03] P.E., K. Yoshitomi: Eigenvalue asymptotics for the Schrödinger operator with a δ-interaction on a punctured surface, *Lett. Math. Phys.* **65** (2003), 19-26.

for more information see *http://www.ujf.cas.cz/˜exner*