Spectral properties of spiral quantum waveguides

Pavel Exner

Doppler Institute
for Mathematical Physics and Applied Mathematics
Prague

in collaboration with Miloš Tater

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Geometrically induced bound states

This is a traditional topic, and some of the present made significant contributions to it, so there is no need for an extensive introduction

As a warm-up, just a brief reminder: let $-\Delta_{\mathrm{D}}^{\Omega}$ be the Dirichlet Laplacian in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a strip of the width $2a$ built over an infinite curve Γ without self-intersections

If Γ is straight line the spectrum is found by separation of variables, $\sigma(-\Delta_{\mathrm{D}}^{\Omega}) = \left[\left(\frac{\pi}{2a}\right)^2, \infty\right)$, and it is absolutely continuous

If, on the other hand, the curve Γ is *not* straight, but it is *asymptotically straight* – expressed in terms of suitable technical assumptions – then there are curvature-induced bound states, i.e. $\sigma_{\mathrm{disc}}(-\Delta_{\mathrm{D}}^{\Omega}) \neq \emptyset$

There is a huge number of related results involving systems in other dimensions and different geometric perturbations; for a survey and bibliography we refer to

P.E., H. Kovařík: *Quantum Waveguides*, Springer, Cham 2015

Spiral waveguides

The assumption of asymptotic straightness is often not satisfied. Here we are going to discuss such a class waveguide systems, *two-dimensional spiral-shaped strips with Dirichlet boundaries*.

Such waveguides appear often in physics. A few examples:

- guides for cold atoms with application to atomic gyroscopes

Jiang Xiao-Jun, Li Xiao-Lin, Xu Xin-Ping, Zhang Hai-Chao, Wang Yu-Zhu: Archimedean-spiral-based microchip ring waveguide for cold atoms, *Chinese Phys. Lett. 32* (2015), 020301.

Xiaojun Jiang, Xiaolin Li, Haichao Zhang, Yuzhu Wang: Smooth Archimedean-spiral ring waveguide for cold atomic gyroscope, *Chinese Opt. Lett.* **14** (2016), 070201.

- electromagnetic or optical systems

N. Bamiedakis, J. Beals, R.V. Penty, I.H. White, J.V. DeGroot, T.V. Clapp: Cost-effective multimode polymer waveguides for high-speed on-board optical interconnects, *IEEE J. Quant. Electronics* **45** (2009), 415–424.

Zhitian Chen *et al.*: Spiral Bragg grating waveguides for TM mode Silicon photonics, *Optics Express* **23** (2015), 25295–25307.

- with applications such as nanoparticle detection or spectrometry

Shui-Jing Tang *et al.*: On-chip spiral waveguides for ultrasensitive and rapid detection of nanoscale objects, *Advanced Materials* **30** (2018), 1800262.

B. Redding, Seng Fatt Liew, Y. Bromberg, Raktim Sarma, Hui Cao: Evanescently coupled multimode spiral spectrometer, *Optica* **3** (2016), 956–962.

Tong Chen, Hansuek Lee, K.J. Vahala: Design and characterization of whispering-gallery spiral waveguides, *Optics Express 22* (2014), 5196–5208.

- spiral shapes appear also in acoustic waveguides

S. Periyannan, P. Rajagopal, K. Balasubramaniam: Multiple temperature sensors embedded in an ultrasonic “spiral-like” waveguide, *AIP Advances* **7** (2017), 035201.

A mathematical motivation

Spirals involved in physical experiments are finite, of course, but we are going to discuss spiral regions of infinite length.

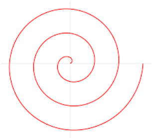
This is not only the usual theoretical license, but our aim is also to show that that many spectral properties of such systems have a truly global character.

Consider a quantum particle in the plane divided by Dirichlet conditions at concentric circles of radii $r_n = 2\pi an$, $n = 1, 2, \dots,$ into the family of annular domains with impenetrable boundaries. The spectrum of this system covers the halfline $\big((2a)^{-1}, \infty\big)$ being there *dense pure point*. In addition, there is a *discrete spectrum* below $(2a)^{-1}$ which is infinite and accumulates at $(2a)^{-1}$.

We will see that the spectral *character changes* profoundly if the Dirichlet boundary is instead imposed on an *Archimedean spiral* of the slope a, despite the fact that if we observe the two boundaries far from the center they look very similar.

Case study: Archimedean waveguide

Let Γ_a be the Archimedean spiral in the plane with the slope $a > 0$, expressed in terms of the polar coordinates, $\Gamma_a = \{r = a\theta :\ \theta \geq 0\}$, and denote by $\mathcal{C}_a$ its complement, $\mathcal{C}_a := \mathbb{R}^2 \setminus \Gamma_a$ which is an open set.



Source: Wikipedia

The object of our interest is the operator

$$H_a = -\Delta_{\mathrm{D}}^{\mathcal{C}_a},$$

the Dirichlet Laplacian in $L^2(\mathcal{C}_a)$. Since scaling transformations change Γ_a into Archimedean spiral again, with a different slope, the above introduced *H_a is unitarily equivalent to $\left(\frac{a'}{a}\right)^2 H_{a'}$* for any $a' > 0$. One could, e.g., consider $a = \frac{1}{2}$; the general case is restored easily multiplying the length scale quantities by $2a$, energy by $(2a)^{-2}$, etc.

Polar coordinates

Our Hilbert space is $L^2((0,\infty)\times[0,2\pi);\, r\mathrm{d}r\mathrm{d}\theta)$ with Dirichlet condition at $\{(\theta+2\pi n,\theta):\, n\in\mathbb{N}_0\}$. This can be equivalently written as $L^2(\Omega_a;\, r\mathrm{d}r\mathrm{d}\theta)$ where Ω_a is the *skewed strip* $\Omega_a := \{(r,\theta):\; r\in(r_{\min}(\theta), a\theta),\; \theta>0\}$ and $r_{\min}(\theta) := \max\{0, a(\theta-2\pi)\}$.

The Dirichlet condition is imposed at the boundary points of Ω_a with $r>0$. As for $r=0$ we note that the boundary of $\mathcal{C}_a$ *is not convex* there and the spiral end represents an angle 2π, hence the operator domain is

$$D(H_a) = \mathcal{H}^2(\Omega_a)\cap\mathcal{H}^1_0(\Omega_a)\oplus\mathbb{C}(\psi_{\mathrm{sing}}),$$

where

$$\psi_{\mathrm{sing}}(r,\theta) = \chi(r)\, r^{1/2}\sin\frac{1}{2}\theta$$

and χ is a smooth function with compact support not vanishing at $r=0$.

V.A. Kondratev: Boundary-value problems for elliptic equations in domains with conical or angular points, *Trans. Moscow Math. Soc.* **16** (1967), 227–313.

Equivalent formulations

As usual we pass to the unitarily equivalent operator $\tilde{H}_a$ on $L^2(\Omega_a)$ by $U : L^2(\Omega_a; r\mathrm{d}r\mathrm{d}\theta) \to L^2(\Omega_a),\ (U\psi)(r,\theta) = r^{1/2}\psi(r,\theta)$, which acts as

$$\tilde{H}_a f = -\frac{\partial^2 f}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{4r^2}.$$

Note that this differential expression is independent of the parameter a, the difference is in the curve at which Dirichlet condition is imposed.

We can also write the quadratic form associated with H_a which is

$$\begin{aligned} q_a :\ q_a[\psi] &= \int_0^\infty \int_{r_{\min}(\theta)}^{a\theta} \left[r\left|\frac{\partial\psi}{\partial r}\right|^2 + \frac{1}{r}\left|\frac{\partial\psi}{\partial\theta}\right|^2 \right] \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^\infty \int_{r/a}^{(r+2\pi a)/a} \left[r\left|\frac{\partial\psi}{\partial r}\right|^2 + \frac{1}{r}\left|\frac{\partial\psi}{\partial\theta}\right|^2 \right] \mathrm{d}\theta\mathrm{d}r \end{aligned}$$

defined for all $\psi \in H^1(\Omega_a)$ satisfying Dirichlet condition at points of $\partial\Omega_a$ with $r > 0$ and such that $\lim_{r\to 0+} \frac{\psi(r,\theta)}{\sin\frac{1}{2}\theta}$ exists being independent of θ.

Essential spectrum threshold

Proposition

We have $\inf \sigma_{\rm ess}(\tilde H_a) \ge (2a)^{-2}$.

Proof sketch: We put $a=\frac12$ and employ *Neumann bracketing* dividing the skewed strip into

$$\Omega^{\rm N} := \big\{(r,\theta) :\, r\in \big(\max\{0,\tfrac12\theta-\pi\},\tfrac12\theta\big),\, 0\le\theta<\theta_{\rm N}\big\}$$

and its infinite complement $\Omega_{\rm c}^{\rm N}$ imposing Neumann condition at $\theta=\theta_{\rm N}>2\pi$ which allows to us to estimate our operator from below.

The first component in $\tilde H^{\rm N}\oplus\tilde H_{\rm c}^{\rm N}$ does not contribute to $\sigma_{\rm ess}(\tilde H_a)$; estimating the quadratic form associated with $\tilde H_{\rm c}^{\rm N}$ we get

$$\tilde H_{\rm c}^{\rm N} \ge -\frac{\partial^2}{\partial r^2} - \frac{3}{16r^2}.$$

However, the 'vertical' width of $\Omega_0^{\rm c}$ is π and $r>\frac12\theta_{\rm N}-\pi$ where $\theta_{\rm N}$ can be chosen arbitrarily large, hence $\inf\sigma_{\rm ess}(\tilde H) = \inf\sigma_{\rm ess}(\tilde H_{\rm c}^{\rm N}) \ge 1$. $\square$

Discrete spectrum?

The question about the existence of discrete spectrum below $(2a)^{-2}$ is equivalent to the positivity violation of the shifted quadratic form,

$$\psi \mapsto q_a[\psi] - \frac{1}{(2a)^2}\|\psi\|^2.$$

Since $\psi(r, r/a) = \psi(r, (r+2\pi a)/a) = 0$, we find easily

$$q_a[\psi] - \frac{1}{(2a)^2}\|\psi\|^2 \geq p_{(0,\infty)}[\psi],$$

where

$$p_{(\alpha,\beta)}[\psi] := \int_\alpha^\beta \mathrm{d}\theta \int_{r_{\min}(\theta)}^{a\theta} \left[r\left|\frac{\partial\psi(r,\theta)}{\partial r}\right|^2 + \left(\frac{1}{4r} - \frac{r}{4a^2}\right)|\psi(r,\theta)|^2 \right] \mathrm{d}r$$

Using the Dirichlet conditions in the 'vertical' direction we can check that

$$p_{(\alpha,\beta)}[\psi] \geq 0 \qquad \text{for any } \; 2\pi \leq \alpha < \beta \leq \infty,$$

and similarly, $p_{(\alpha,\beta)}[\psi] \geq 0$ for $\beta \leq \pi$. Consequently, the only negative contribution can come from the interval $(\pi, 2\pi)$, in particular, that there can be *at most a finite number of bound states*. Later we will present a convincing numerical evidence that the discrete spectrum is in fact *empty*.

Another parametrization

From what we know about curved Dirichlet waveguides this conclusion may seem surprising. To understand the reason, we look at the problem from a different point of view introducing locally orthogonal coordinates, sometimes called *Fermi* or *parallel*.

The Cartesian coordinates of the spiral are $x_1 = a\theta\cos\theta,\ x_2 = a\theta\sin\theta$, and we introduce the transverse coordinate u measuring the distance from Γ_a along the inward pointing normal, writing the points of $\mathcal{C}_a$ as

$$\begin{aligned} x_1(\theta,u) &= a\theta\cos\theta - \frac{u}{\sqrt{1+\theta^2}}\big(\sin\theta + \theta\cos\theta\big), \\ x_2(\theta,u) &= a\theta\sin\theta + \frac{u}{\sqrt{1+\theta^2}}\big(\cos\theta - \theta\sin\theta\big) \end{aligned}$$

A natural counterpart to the variable u is the *arc length* of Γ_a given by

$$s(\theta) = a\int_0^\theta \sqrt{1+\xi^2}\,\mathrm{d}\xi = \tfrac{1}{2}a\big(\theta\sqrt{1+\theta^2} + \ln(\theta + \sqrt{1+\theta^2})\big)$$

which for large values of θ behaves as

$$s(\theta) = \tfrac{1}{2}a\theta^2 + \mathcal{O}(\ln\theta).$$

The curvature

Another quantity of interest is the *curvature* of the spiral given by

$$\kappa(\theta) = \frac{2+\theta^2}{a(1+\theta^2)^{3/2}} = \frac{1}{a\theta} + \mathcal{O}(\theta^{-2}) \quad \text{as } \theta \to \infty$$

which means that

$$\kappa(s) = \frac{1}{\sqrt{2as}} + \mathcal{O}(s^{-1}) \quad \text{as } s \to \infty.$$

With an abuse of notation we will denote the points of $\mathcal{C}_a$ as $x(s,u)$ keeping in mind that the described parametrization *cannot be used globally*, as it becomes non-unique for small θ when the normal to Γ fails to cross the previous coil of the spiral; this obviously happens for $\theta < \theta_0$ with some $\theta_0 \in (\frac{3}{2}\pi, 2\pi)$.

Nevertheless, we can use it elucidate the properties of H_a that depend on the behavior of Γ_a *at large values of s* using a 'perpendicular' DN bracketing with additional conditions imposed, say, at $s = s(2\pi)$.

Transverse width

We ask about $d(s)$, the range of the variable u, that is, the transverse width of $\mathcal{C}_a$ at a given s. The intersection of the normal to Γ with the previous coil occurs at the angle θ_-, for which we

$$a\theta\cos\theta - \frac{u}{\sqrt{1+\theta^2}}\big(\sin\theta + \theta\cos\theta\big) = a\theta_-\cos\theta_-,$$
$$a\theta\sin\theta + \frac{u}{\sqrt{1+\theta^2}}\big(\cos\theta - \theta\sin\theta\big) = a\theta_-\sin\theta_-.$$

from which we get equations for u and θ_-, namely

$$\theta^2\Big(a^2 - \frac{2au}{\sqrt{1+\theta^2}}\Big) + u^2 = a^2\theta_-^2$$

and

$$\theta\big(1 + \theta_-\sin(\theta-\theta_-)\big) = \theta_-\cos(\theta-\theta_-).$$

An estimate of the transverse width

Abusing again the notation we write $d(\theta)$ meaning $d(s(\theta))$. The radial dropped from the point $x(\theta, 0)$ towards the coordinate center crosses the previous coil of Γ_a at $x(\theta', 0)$ where $\theta' = \theta - 2\pi$. The *spiral slope* at this point, i.e. the angle $\beta(\theta')$ between the tangent to Γ_a and the tangent the the circle passing through this point is $\cos\beta(\theta') = \frac{\theta'}{\sqrt{1+\theta'^2}}$, and since the radial distance between the two coils is $2\pi a$, we get

$$d(\theta) > \frac{2\pi a\theta'}{\sqrt{1+\theta'^2}} = \frac{2\pi a\theta}{\sqrt{1+\theta^2}}\big(1+\mathcal{O}(\theta^{-1})\big).$$

To see that this bound to $u = d(\theta)$ is asymptotically exact, we put $\theta_- = \theta' + \delta$ and solve the second equation using the implicit function theorem; this shows that *inequality in the last relation is in fact equality*, and

$$\frac{\pi^2}{d(\theta)^2} = \frac{1}{(2a)^2}\big(1+(\theta-2\pi)^{-2}\big) = \frac{1}{(2a)^2}\Big(1+\frac{1}{\theta^2}+\mathcal{O}(\theta^{-3})\Big)$$

The straightened strip

The coordinates s, u allow us to pass from $H^{\mathrm{D}}_{a,\mathrm{nc}}$ and its Neumann counterpart to a unitarily equivalent operator on $L^2(\Sigma_{a,\mathrm{nc}})$, where

$$\Sigma_{a,\mathrm{nc}} = \big\{(s,u):\ s > s(2\pi), u \in (0, d(s))\big\}$$

is a semi-infinite strip with a varying, but asymptotically constant width, which acts as

$$\hat{H}^{\mathrm{D}}_{a,\mathrm{nc}}\psi = -\frac{\partial}{\partial s}(1-u\kappa(s))^{-2}\frac{\partial\psi}{\partial s}(s,u) - \frac{\partial^2\psi}{\partial u^2}(s,u) + V(s,u)\psi(s,u),$$

where

$$V(s,u) := -\frac{\kappa(s)^2}{4(1-u\kappa(s))^2} - \frac{u\ddot{\kappa}(s)}{2(1-u\kappa(s))^3} - \frac{5}{4}\,\frac{u^2\dot{\kappa}(s)^2}{(1-u\kappa(s))^4},$$

with Dirichlet condition at the boundary of $\mathcal{C}_{a,\mathrm{nc}}$, which in $\hat{H}^{\mathrm{N}}_{a,\mathrm{nc}}$ is replaced by Neumann one at the cut, $s = s(2\pi)$.

The essential spectrum

Proposition

We have $\sigma_{\rm ess}(H_a) = [(2a)^{-2}, \infty)$.

Proof sketch: We have to check the inclusion $\sigma_{\rm ess}(H_a) \supset [(2a)^{-2}, \infty)$; we do it first for $\hat{H}^{\rm D}_{a,\rm nc}$ putting $a = \frac{1}{2}$ and using Weyl's criterion with

$$\psi_{k,\lambda}(s,u) := \mu(\lambda s)\, \mathrm{e}^{iks} \sin \frac{\pi u}{d(s)},$$

where $k \in \mathbb{R}$ and $\mu \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\mu \subset (1,2)$. For small enough s the support of $\psi_{k,\lambda}$ lies in $\Sigma_{a,\rm nc}$, and since $d(s) = \pi\big(1 - \frac{1}{8s} + \mathcal{O}(s^{-3/2})\big)$, we have

$$\|\psi_{k,\lambda}\| = \lambda^{-1/2}\|\mu\| + \mathcal{O}(1)$$

as $\lambda \to 0$; this has to be compared with $\big\|(\hat{H}^{\rm D}_{a,\rm nc} - 1 - k^2)\psi_{k,\lambda}\big\|$.

Proof sketch, continued

The norm $\left\|(\hat{H}^{\mathrm{D}}_{a,\mathrm{nc}} - 1 - k^2)\psi_{k,\lambda}\right\|$ is a rather complicated expression, however, it is sufficient to single out the terms which dominate in the limit $\lambda \to 0$; a tedious but straightforward computation gives

$$\frac{\left\|(\hat{H}^{\mathrm{D}}_{a,\mathrm{nc}} - 1 - k^2)\psi_{k,\lambda}\right\|}{\|\psi_{k,\lambda}\|} = \mathcal{O}(\lambda) \quad \text{as} \;\; \lambda \to 0.$$

Hence $1 + k^2 \in \sigma(\hat{H}^{\mathrm{D}}_{a,\mathrm{nc}})$ for any $k \in \mathbb{R}$, and the same holds for $\hat{H}^{\mathrm{N}}_{a,\mathrm{nc}}$ because the supports of the functions $\psi_{k,\lambda}$ are separated from the boundary of $\Sigma_{a,\mathrm{nc}}$ at $s = s(2\pi)$.

Furthermore, by the unitary equivalence their preimages in the original coordinates constitute a *Weyl sequence of the operator H_a*, the 'full' one as the cut as $s = s(2\pi)$ is again irrelevant from the viewpoint of the essential spectrum. Finally, one can choose as sequence $\{\lambda_n\}_{n=1}^{\infty}$ in such a way that the supports of different ψ_{k,λ_n} do nor overlap; then $\psi_{k,\lambda_n} \to 0$ *weakly* as $n \to \infty$ which means that $1 + k^2 \in \sigma_{\mathrm{ess}}(H_a)$. $\square$

Absolute continuity

As usual in waveguides, we exclude from the consideration the family of transverse thresholds at which the spectral multiplicity changes, that is, $\mathcal{T} = \left\{ \left(\frac{n}{2a}\right)^2 : \; n = 1, 2, \dots \right\}$.

Proposition

Let I be an open interval, $I \subset [(2a)^{-2}, \infty) \setminus \mathcal{T}$, then the spectrum of H_a in I is purely absolutely continuous.

Proof sketch: In the spirit of *Mourre's method*, we have to find a suitable conjugate operator A to $\tilde{H}_a$ on Ω_a. We choose

$$A = -\frac{i}{2}\Big(r\frac{\partial}{\partial r} + \frac{\partial}{\partial r} r \Big)$$

with the domain consisting of functions from $\mathcal{H}^1(\Omega_a)$ satisfying Dirichlet condition at the boundary of Ω_a except its part corresponding to $r = 0$. This is the *generator of the group $\{\mathrm{e}^{itA} : \; t \in \mathbb{R}\}$ of dilations* of the skew strip Ω_a in the direction parallel to the line $r = a\theta$.

Proof sketch, continued

It is obvious that the scaling e^{itA} preserves the domain of $\tilde{H}_a$ referring to that of $D(H_a)$, and from the self-similarity it follows that *the map $t \mapsto \mathrm{e}^{itA}(\tilde{H}_a - i)^{-1}\mathrm{e}^{-itA}$ has the needed regularity*. The commutator $[\tilde{H}_a, iA]$ is easily evaluated,

$$[\tilde{H}_a, iA]f = -2\,\frac{\partial^2 f}{\partial r^2} - \frac{2}{r^2}\,\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{2r^2}f.$$

The corresponding quadratic form can be estimated similarly to $p_{(\alpha,\beta)}$ above; this shows that the contribution of the of the last two terms in nonnegative and

$$E_{\tilde{H}_a}(I)[\tilde{H}_a, iA]E_{\tilde{H}_a}(I) \geq -2\,\frac{\partial^2}{\partial r^2}\,E_{\tilde{H}_a}(I) \geq \frac{1}{8}\,E_{\tilde{H}_a}(I).$$

Since this bound contains no compact part, there are *no embedded eigenvalues* and the spectrum of H_a in I is purely absolutely continuous. □

Why a rich $\sigma_{\rm disc}(H_a)$ is absent?

As I said, one might expect a rich discrete spectrum below $(2a)^{-2}$ because the effective potential $V(s,u)$ is attractive and long-range. It is not the case, and the reason is that *$d(s)$ equals $(2a)^{-2}$ only asymptotically*. In the sense of quadratic forms we have

$$\hat{H}^{\rm D}_{a,\rm nc} \geq -\frac{\partial}{\partial s}(1-u\kappa(s))^{-2}\frac{\partial}{\partial s} + W(s,u),$$

where $W(s,u) := \big(\frac{\pi}{d(s)}\big)^2 + V(s,u)$. Taking the implicit-function theorem evaluation of $d(s)$ further, in terms of the angular variable θ we get

$$\Big(\frac{\pi}{d(s)}\Big)^2 = \frac{1}{4a^2} + \frac{1}{4a^2\theta^2} + \frac{\pi}{2a^2\theta^3} + \frac{\pi^2}{a^2\theta^4} + \frac{\pi(4\pi^2-1)}{4a^2\theta^5} + \mathcal{O}(\theta^{-6}).$$

One the other hand, $V(\theta,u)$ can be expressed from $V(s,u)$ with the help of $\dot\kappa(s) = a^{-1}(1+\theta^2)^{-1/2}\dot\kappa(\theta)$ and $\ddot\kappa(s) = a^{-2}(1+\theta^2)^{-1}\ddot\kappa(\theta)$ in the form

$$V(s,u) = -\frac{1}{4a^2\theta^2} - \frac{u}{2a^2\theta^3} - \frac{a^2+3u^2}{4a^2\theta^4} + \frac{u(7a^2+4u^2)}{4a^5\theta^5} + \mathcal{O}(\theta^{-6}).$$

Why a rich $\sigma_{\mathrm{disc}}(H_a)$ is absent?

Thus the contributions of the strip width and the effective potential are competing: from the above expansions we get

$$W(s,u) = \frac{1}{4a^2} + \frac{\pi a - u}{2a^3\theta^3} + \frac{a^2(4\pi^2-1) - 3u^2}{4a^4\theta^4} + \mathcal{O}(\theta^{-5}).$$

Apart from the constant corresponding the continuum threshold, *the leading terms have canceled* mutually, and the transverse mean of the following one vanishes, modulo a small correction coming from the difference between $d(\theta)$ and $2\pi a$.

These heuristic considerations correspond well to the observation made above about the sign of the contribution to the form p_a from the region with $\theta > 2\pi$, and next bring a numerical evidence that *the discrete spectrum is in fact empty*.

A variation: spiral waveguide with a cavity

Let us *'erase' a part of the Dirichlet boundary*, that is, we impose the condition on the 'cut' Archimedean spiral $\Gamma_{a,\beta}$ for some $\beta > 0$, where $\Gamma_{a,\beta} = \{r = a\theta :\, \theta \geq \beta\}$. The particle thus 'lives' in $\mathcal{C}_{a,\beta} := \mathbb{R}^2 \setminus \Gamma_{a,\beta}$ and its Hamiltonian, modulo unimportant physical constants, is

$$H_{a,\beta} = -\Delta_{\mathrm{D}}^{\mathcal{C}_{a,\beta}},$$

the Dirichlet Laplacian in $L^2(\mathcal{C}_{a,\beta})$. Obviously, we have

$$\sigma_{\mathrm{ess}}(H_{a,\beta}) = [(2a)^{-2}, \infty).$$

By bracketing, the discrete spectrum is nonempty for β large enough:

Proposition

There is a critical $\beta_1 = 2j_{0,1} \approx 4.805 \approx 1.531\pi$ such that $\sigma_{\mathrm{disc}}(H_{a,\beta}) \neq \emptyset$ holds for all $\beta > \beta_1$. Furthermore, let $\mathcal{B} = \{\beta_j\}_{j=1}^{\infty}$ be the sequence $\mathcal{B} = \big\{2j_{0,1}, 2j_{1,1}, 2j_{1,1}, 2j_{2,1}, 2j_{2,1}, 2j_{0,2}, 2j_{1,2}, 2j_{1,2}, \dots\big\}$ composed of zeros of Bessel functions J_n, $n = 0, 1, \dots,$ then for any $\beta > \beta_j$ the operator $H_{a,\beta}$ has at least j eigenvalues, the multiplicity taken into account.

Numerical results

To analyze the problem numerically we use *FEM techniques* on finite spiral regions with θ large enough to ensure numerical stability, checked with the help of bracketing.

There is *more than one way* to address our problem numerically. One can apply the FEM technique to the Laplacian in the spiral region directly, alternatively one can analyze operator $\tilde{H}_a$ in the (truncated) skewed strip; comparison of the results provides another reliability check.

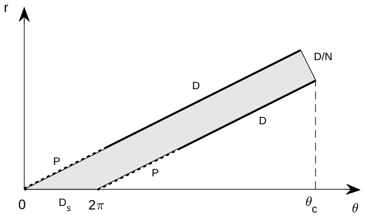


Figure: A truncated skewed strip for the spiral region with a cavity. For $0 < \theta < \beta$ the Dirichlet condition *is replaced by the periodic one*.

Eigenvalues

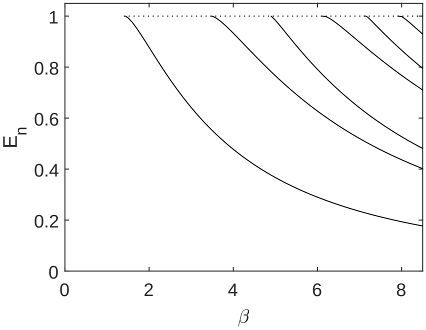


Figure: Eigenvalues of $H_{1/2,\beta}$ as functions of β.

As expected, they are monotonously decreasing functions. We also can identify the critical angle at which the first eigenvalue appears to be $\beta_1 \approx 1.43 \approx 0.455\pi$, a much smaller value than the above sufficient condition; what is more important, it provides the *indication that the discrete spectrum of the 'full' Archimedean spiral region is void*.

Eigenfunctions

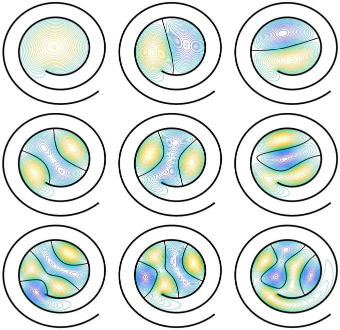


Figure: The first nine eigenfunctions of $H_{1/2,21/2}$ shown through their horizontal levels. The corresponding energies are 0.1280, 0.2969, 0.3456, 0.5312, 05811, 0.6825, 0.8266, 0.8852, and 0.9768, respectively.

The results agree with the Courant nodal domain theorem; the nodal lines are situated in the cavity only which, as well the finiteness of the spectrum, corresponds nicely to the observation that the part of $\mathcal{C}_{a,\beta}$ referring to the angles $\theta > \max\{2\pi, \beta_1\}$ is a *classically forbidden zone*.

A variation: multi-arm Archimedean waveguide

Let Γ_a^m be the union of m Archimedean spirals with slope $a > 0$ and an angular shift, $\Gamma_a^m = \{r = a\big(\theta - \frac{2\pi j}{m}\big) :\, \theta \ge \frac{2\pi j}{m},\, j = 0, \dots, m-1\}$. As before we consider its complement $\mathcal{C}_a^m := \mathbb{R}^2 \setminus \Gamma_a^m$ and the operator

$$H_a = -\Delta_{\mathrm{D}}^{\mathcal{C}_a^m}.$$

The analysis is similar, but there is a difference coming from regularity of the boundary. For $m = 2$ the set $\mathcal{C}_a^2$ consists of two connected components and has a *smooth boundary*, for $m \ge 3$ it consists of m connected components separated by the branches of Γ_a^m, each of them them has an angle at the origin of coordinates which is $\frac{2\pi}{m}$, that is, *convex*; this means that for $m \ge 2$ the *singular component is missing*.

It is sufficient to consider one connected component of $\mathcal{C}_a^m$ only, i.e. the operator $\tilde{H}_a^m$ referring to the skewed strip

$$\Omega_a^m := \big\{(r, \theta) :\, r \in (r_{\min}^m(\theta), a\theta), \theta > 0\big\},$$

where $r_{\min}^m(\theta) := \max\big\{0, a\big(\theta - \frac{2\pi}{m}\big)\big\}$ with $D(H_a^m) = \mathcal{H}^2(\Omega_a^m) \cap \mathcal{H}_0^1(\Omega_a^m)$.

Spectrum of multi-arm spiral region

Proposition

$\sigma(H_a^m) = \left[\left(\frac{m}{2a}\right)^2, \infty\right)$ for any natural $m \geq 2$. The spectrum is absolutely continuous outside $\mathcal{T}_m = \left\{\left(\frac{mn}{2a}\right)^2 : n = 1, 2, \dots\right\}$ and its multiplicity is divisible by m.

Proof sketch: The multiplicity claim is obvious. The above arguments used to determine the essential spectrum and to prove its absolute continuity outside the thresholds modify easily.

Furthermore, *the discrete spectrum is void*. Indeed, since the domain is now 'pure Sobolev', the bottom part, $r = 0$, of the skewed strip supports Dirichlet condition. This means that

$$p^m_{(\alpha,\beta)}[\psi] \geq 0 \qquad \text{now for any } \; 0 \leq \alpha < \beta \leq \infty$$

so that $q_a^m[\psi] - \left(\frac{m}{2a}\right)^2 \|\psi\|^2 \geq p^m_{(0,\infty)}[\psi] \geq 0$ for any $\psi \in \mathrm{dom}[q_a^m]$. $\square$

Eigenfuctions

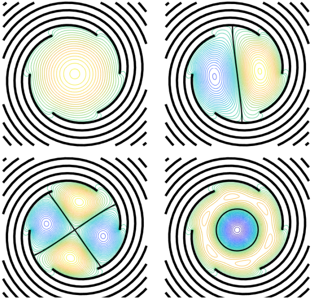
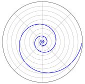


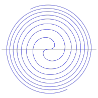
Figure: The jth eigenfunction, $j = 1, 2, 4, 6$, of $H^6_{3,2\pi}$, the corresponding energies are 0.1296, 0.3282, 0.5871, and 0.6783, respectively.

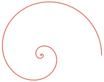
Here we plot result for a six-arm spiral region with the central cavity. As expected, with the growing m the eigenfunctions – with the possible exception of states close to the threshold – become similar to those of the Dirichlet Laplacian in a disc; it is instructive to compare the nodal lines to those of the single arm region shown above.

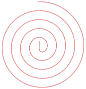
General spirals

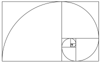
There are many spirals beyond the Archimedean case, for instance

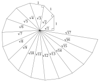

logarithmic


Fermat


Poinsot


Atzema


Fibonacci


Theodorus

Source: Wikipedia

A spiral curve Γ can be described in polar coordinates as the family of points $(r(\theta), \theta)$, where $r(\cdot)$ is a given increasing function. We assume that $r(\cdot)$ is a *C^2-smooth function* excluding thus well-known curves such as *Fibonacci spiral*, *spiral of Theodorus*, etc.

Unless specified otherwise, the spirals considered are semi-infinite $r : \mathbb{R}_+ \to \mathbb{R}_+$. In some cases we also consider *'fully' infinite spirals* for which $r : \mathbb{R} \to \mathbb{R}_+$.

General spirals

The monotonicity of r means that Γ does not intersect itself, in other words, the *width function* a : $a(\theta) = \frac{1}{2\pi}\big(r(\theta) - r(\theta - 2\pi)\big)$ is positive for any $\theta \geq 2\pi$, or for all $\theta \in \mathbb{R}$ in the fully infinite case (the 'inward' coil width is $2\pi a(\theta)$; we make this choice with the correspondence to the Archimedean case in mind).

As before we denote $\mathcal{C} := \mathbb{R}^2 \setminus \Gamma$ and ask about spectral properties of

$$H_r = -\Delta_{\mathrm{D}}^{\mathcal{C}},$$

the Dirichlet Laplacian in $L^2(\mathcal{C})$. Another modification concerns *multiarm-arm spirals*: given $0 = \theta_0 < \theta_1 < \cdots < \theta_{m-1} < 2\pi$ and increasing functions $r_j : [\theta_j, \infty) \to \mathbb{R}_+$, $j = 0, 1, \dots, m-1$, satisfying

$$a_j(\theta) := \frac{1}{2\pi}\big(r_j(\theta) - r_{j+1}(\theta)\big) > 0,$$

for all relevant values of θ. Note that a two-arm spiral can also be alternatively described by means of a function $r : \mathbb{R} \to \mathbb{R}$ such that $\pm r(\theta) > 0$ for $\pm\theta > 0$ interpreting *negative radii* as describing vectors rotated by π.

Types of general spirals

Asymptotic properties of the width function are decisive. We call a spiral-shaped region $\mathcal{C}$ *simple* if the function $a(\cdot)$ is monotonous (or monotonous on both the halflines $\mathbb{R}_\pm$). A simple $\mathcal{C}$ is called *expanding* and *shrinking* if a is respectively increasing and decreasing for $\theta \geq 0$; these qualifications are labeled as *strict* if $\lim_{\theta\to\infty} a(\theta) = \infty$ and $\lim_{\theta\to\infty} a(\theta) = 0$, respectively.

A spiral-shaped region is called *asymptotically Archimedean* if there is an $a_0 \in \mathbb{R}$ such that $\lim_{\theta\to\infty} a(\theta) = a_0$, for multi-arm spirals this means finite limits of all the a_j.

A region $\mathcal{C}$ is obviously unbounded *iff* $\lim_{\theta\to\infty} r(\theta) = \infty$. If the limit is finite, $\lim_{\theta\to\infty} r(\theta) = R$, the closure $\overline{\mathcal{C}}$ is contained in the circle of radius R, it may or may not be simply connected as the example of *Simon's jelly roll*, $r(\theta) = \frac{3}{4} + \frac{1}{2\pi} \arctan\theta$, shows (recall that the *Neumann* Laplacian in this region has a *continuous spectrum*).

Description of general spiral regions

The Hamiltonian domain is $D(H_r) = \mathcal{H}^2(\Omega_r) \cap \mathcal{H}_0^1(\Omega_r) \oplus \mathbb{C}(\psi_{\mathrm{sing}})$, where the singular element may be absent in the multi-arm case. We can again investigated H_r as an opeator on a *skewed strip*, now of a generally *nonconstant width*. The quadratic form associated with H_r is

$$\begin{aligned} q_r :\ q_r[\psi] &= \int_0^\infty \int_{r_{\min}(\theta)}^{r(\theta)} \left[r\left|\frac{\partial\psi}{\partial r}\right|^2 + \frac{1}{r}\left|\frac{\partial\psi}{\partial\theta}\right|^2 \right] \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^\infty \int_{\theta^{-1}(r)}^{\theta^{-1}(r)+2\pi} \left[r\left|\frac{\partial\psi}{\partial r}\right|^2 + \frac{1}{r}\left|\frac{\partial\psi}{\partial\theta}\right|^2 \right] \mathrm{d}\theta\mathrm{d}r \end{aligned}$$

where $\theta^{-1}(\cdot)$ is the pull-back of the function $r(\cdot)$; its domain consists of function $\psi \in H^1(\Omega_r)$ satisfying appropriate conditions at $\partial\Omega_a$.

Using function r and it derivatives, we also can express the *arc length* and the *curvature*; they are

$$s(\theta) = \int_0^\theta \sqrt{\dot{r}(\xi)^2 + r(\xi)^2}\,\mathrm{d}\xi \quad \text{and} \quad \kappa(\theta) = \frac{r(\theta)^2 + 2\dot{r}(\theta)^2 - r(\theta)\ddot{r}(\theta)}{(r(\theta)^2 + \dot{r}(\theta)^2)^{3/2}}.$$

Strictly expanding spiral regions

In contrast to the Archimedean case, it may not be possible to amend the arclength with the orthogonal coordinate u to parametrize $\mathcal{C}_r$ by

$$x_1(\theta,u)= r(\theta)\cos\theta - \frac{u}{\sqrt{\dot r(\theta)^2+r(\theta)^2}}\left(\dot r(\theta)\sin\theta + r(\theta)\cos\theta\right),$$

$$x_2(\theta,u) = r(\theta)\sin\theta + \frac{u}{\sqrt{\dot r(\theta)^2+r(\theta)^2}}\left(\dot r(\theta)\cos\theta - r(\theta)\sin\theta\right).$$

The reason is that for strictly expanding spirals the inward normal at a point may not intersect the previous spiral coil; it is easy to check that in the examples of a *logarithmic spiral*, $r(\theta)=a\,\mathrm{e}^{k\theta}$ with $a,k>0$, or *hyperbolic spiral*, $r(\theta)=a\theta^{-1}$.

Fortunately, some properties of H_r can be derived without the use of the locally orthogonal system. Using suitable Weyl sequences one can prove the following claim:

Proposition

$\sigma(H_r)=\sigma_{\mathrm{ess}}(H_r)=[0,\infty)$ *holds if* $\mathcal{C}$ *is simple and strictly expanding.*

Strictly shrinking spiral regions

On the other hand, parallel coordinates *can be used*, possibly outside a compact region, if $\mathcal{C}$ is generated by a shrinking or an asymptotically Archimedean spirals.

We combine bracketing with the unitarily equivalent form of the operator in parallel coordinates,

$$\hat{H}_{\mathrm{nc}}^{\mathrm{D}} \geq -\frac{\partial}{\partial s}(1-u\kappa(s))^{-2}\frac{\partial}{\partial s} + \frac{\pi^2}{d(s)^2} + V(s,u)$$

and similarly for $\hat{H}_{\mathrm{nc}}^{\mathrm{N}}$. Since $d(s) \to 0$ as $s \to \infty$ holds is a strictly shrinking region, the sum of the two last term explodes in the limit, and in the standard way we can check the following claim:

Proposition

If $\mathcal{C}$ is simple and strictly shrinking, the spectrum of H_r is purely discrete.

Fermat spiral eigenfuctions

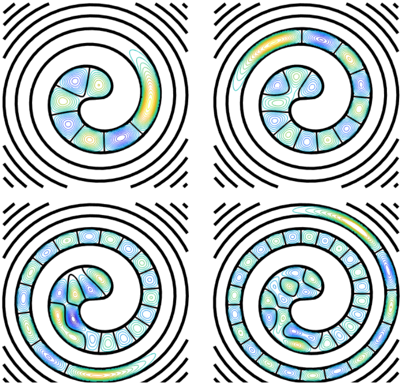


Figure: Eigenfunctions of the Fermat spiral region, $b = 1$, corresponding to the eigenvalues, $E_7 = 19.5462$, $E_{15} = 28.3118$, $E_{27} = 38.8062$, and $E_{42} = 48.8367$.

For *Fermat spiral*, $r(\theta)^2 = b^2\theta$, we have $a(\theta) = \frac{1}{2}b\theta^{-1/2} + \mathcal{O}(\theta^{-3/2})$ so the spectrum is discrete; note that apart from the central region the eigenfunctions have a quasi-one-dimensional character.

Fermat spiral region: number of eigenvalues

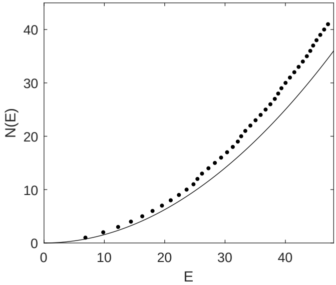


Figure: The number of eigenvalues *vs.* energy compared to the asymptotics taking into account the strip width only.

The dominant contribution comes from the transverse confinement potential $v(\theta) = \left(\frac{\pi}{d(\theta)}\right)^2$. For Fermat spiral region this leads to the asymptotics $N(E) \approx \frac{1}{64}\, b^4\, E^2$ as $E \to \infty$. Looking at the true number of eigenvalues, we see a significant excess which is naturally attributed to the geometry-related effects.

Asymptically Archimedean regions

Between the above discussed extremes the situation is much more interesting. Modifying the argument in the Archimedean case we get

Proposition

If the spiral Γ is asymptotically Archimedean with $\lim_{\theta\to\infty} a(\theta) = a_0$, we have $\sigma_{\mathrm{ess}}(H_r) = [(2a_0)^{-2}, \infty)$. In the case of a multi-arm region with$\lim_{\theta\to\infty} a_j(\theta) = a_{0,j}$, the essential spectrum is $[(2\overline{a})^{-2}, \infty)$, where $\overline{a} := \max_{0\le j\le m-1} a_{0,j}$.

The question about the discrete spectrum is more subtle and the type of asymptotics is decisive. Let us consider the spiral

$$r(\theta) = a_0\theta + b_0 - \rho(\theta),$$

where $\rho(\cdot)$ is a positive function such that, $\lim_{\theta\to\infty} \rho(\theta) = 0$; for the sake of definiteness we restrict our attention to functions satisfying

$$\dot\rho(\theta) = -\frac{c}{\theta^\gamma} + \mathcal{O}(\theta^{-\gamma-1}) \quad \text{as} \quad \theta\to\infty \quad \text{with} \quad 1<\gamma<3.$$

Infinite discrete spectrum

Proposition

For the described $r(\cdot)$, $\#\sigma_{\mathrm{disc}}(H_r) = \infty$ holds for any $c > 0$.

Proof sketch: By a variational argument using the function $\psi_{0,\lambda}$ from the Archimedean case. After a straightforward computation we get for the shifted quadratic form

$$p[\psi_{0,\lambda}] < \lambda \frac{4\pi}{a_0} \|\dot{\mu}\|^2 - \Big(\frac{4\pi^2 c}{a_0^4} \Big(\frac{a_0}{4} \Big)^{\gamma/2} \lambda^{(\gamma-2)/2} + \mathcal{O}\big(\lambda^{(\gamma'-2)/2}\big) \Big) \|\mu\|^2,$$

where the right-hand side is negative for all λ small enough. Moreover, since the support of μ is compact, one can choose a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ as $n \to \infty$ and the *supports of ψ_{0,λ_n} are mutually disjoint* which means that the discrete spectrum of H_r is infinite, accumulating at the threshold $(2a_0)^{-2}$. $\square$

Fermat meets Archimedes

As an example, consider an interpolation between Fermat and Archimedean spirals, in the simplest case described parametrically as

$$r(\theta) = a\sqrt{\theta\big(\theta + \tfrac{b^2}{a^2}\big)}, \quad a, b > 0,$$

with the asymptotic behavior

$$r(\theta) = b\sqrt{\theta} + \frac{a^2}{2b}\theta^{3/2} + \mathcal{O}(\theta^{5/2}),$$
$$r(\theta) = a\theta + \frac{b^2}{2a} + \mathcal{O}(\theta^{-1})$$

for $\theta \to 0+$ and $\theta \to \infty$, respectively.

The Fermat spiral is conventionally considered as a *two-arm one* dividing the plane into a pair of mutually homothetic regions, hence we interpolate with the two-arm Archimedean spiral; the essential spectrum is $[a^{-2}, \infty)$.

Fermat meets Archimedes, continued

As for the discrete spectrum, taking the expansion of $r(\theta)$ two terms further, we get $b_0 = \frac{b^2}{2a}$ and

$$\rho(\theta) = \frac{b^4}{8a^3\theta} - \frac{3b^6}{16a^5\theta^2} + \mathcal{O}(\theta^{-3}).$$

This means that the assumptions of the last proposition hold with with $c = \frac{b^4}{8a^3} > 0$ and $\gamma = 2$, and the the operator H_r has an *infinite discrete spectrum* in $(0, a^{-2})$ accumulating at the threshold.

One can also specify the accumulation rate: the one-dimensional effective potential is in this case $\frac{\pi b^4}{16a^5}\, s^{-1} + \mathcal{O}(s^{-3/2})$, with the leading term of Coulomb type, which shows that the number of eigenvalues *below* $a^{-2} - E$ behaves as

$$\mathcal{N}_{a^{-2}-E}(H_r) = \frac{\pi b^4}{32a^5}\,\frac{1}{\sqrt{E}} + o(E^{-1/2}) \quad \text{if} \quad E \to 0+$$

Remarks

- Experimentalists often label their spirals as Archimedean, but in fact *they are not*. The reason is that they are produced by coiling fibers of a *fixed cross section*, hence their transverse width is constant instead of changing with the angle as it would be the case for true Archimedean spiral. Such waveguides behave asymptotically rather as the current interpolation with $\frac{b}{a} = (2\pi)^{-1/4} \approx 0.632$.
- Some asymptotically Archimedean regions behave differently, for instance, *involute of a circle*, $r(\theta) = a\sqrt{1+\theta^2}$, for which $b_0 = 0$ and $\rho(\theta) < 0$; thus the polar width of the region is larger than that of its Archimedean asymptote acting against the existence of bound states.
- A similar conclusion holds for *Atzema spiral*, $r(t) = a(t + t^{-1})$, where $t > t_0$ for a suitable t_0 to avoid self-intersections. The parameter is *not* the polar angle, but $\theta = t - \frac{\pi}{2} + \mathcal{O}(t^{-1})$, hence we have $b_0 = 0$ and $\rho(\theta) < 0$ again.
- On the other hand, modifying the last example, $r(\theta) = a(\theta - \theta^{-1})$ with $\theta > 1$, we have $b_0 = 0$ and $\gamma = 2$ so there is an infinite discrete spectrum below $(2a)^{-1}$ accumulating at the threshold.

Eigenvalues

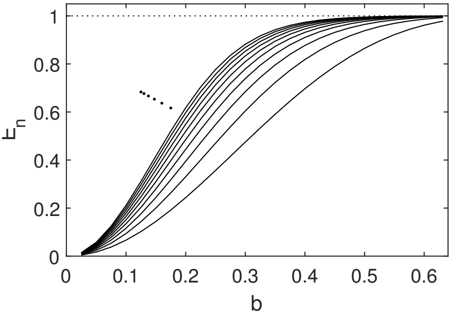


Figure: The lowest eigenvalues of interpolating region as functions of b.

As expected the ground state is close to the continuum threshold for (sufficiently) large values of b and the *whole discrete spectrum disappears in the limit $b \to \infty$*, while for small b the region has *a large bulge* in the center and the spectral bottom drops to appropriately low values. We also see in the picture how the eigenvalues accumulate towards the continuum.

An eigenfunction

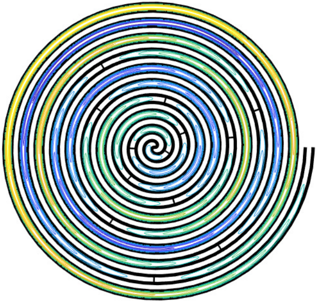


Figure: The eigenfunction with $E_{14} = 0.999952$ corresponding to $b = (2\pi)^{-1/4}$.

The difference from the two-arm Archimedean region is *hardly perceptible by a naked eye*, however, the discrete spectrum is now not only non-void but it is rich with the eigenfunctions the tails of which have a distinctively *quasi-one-dimensional character*.

Open questions

- There are other spiral-shaped regions of interest. We discussed situations in which $\overline{\mathcal{C}}$ *was simply connected*. This is often not the case in physical applications, sometimes the guide looks like a two-arm Archimedean region – in the experimentalist reckoning mentioned above – however, with an opening in the center where the two 'loose ends' meet each other in an S-shape way.
- A discrete spectrum can also be created by erasing a part of the Dirichlet boundary *away from the center*. There is no doubt that a large enough 'window' would give rise to bound states, the question is *how small it can be to have that effect*.
- Having in mind that (single-arm) spiral-shaped regions violate the rotational symmetry, one is naturally interested whether *the spectrum is simple*.
- Another question concerns the spectral statistics for strictly shrinking spirals, one would like to know whether they give rise to *quantum chaotic systems*.

More open questions

- We mentioned that properties of Neumann Laplacian may be completely different. On the other hand, one expects that a *Robin boundary* (with a nonzero parameter) would behave similarly to the Dirichlet one, with some natural differences: in the asymptotically Archimedean case *the essential spectrum threshold would be not* $(2a)^{-1}$ but the principal eigenvalue of the Robin Laplacian on an interval of length 2π, and in the case of an attractive boundary the operator *will no longer be positive*.
- In some physical applications a magnetic field is applied, hence it would be useful to investigate spectral properties of the *magnetic Dirichlet Laplacian* in spiral-shaped regions.
- In real physical system the separation of the spiral coils is never complete, which motivates one to look into the *'leaky' version of the present problem*, that is, singular Schrödinger operators of the type $-\Delta + \alpha\delta_\Gamma$.

P.E., M. Tater: Spectral properties of spiral-shaped quantum waveguides, *J. Phys. A: Math. Theor.*, to appear; `arXiv: 2009.02730 [math-ph]`

It remains to say

Thank you for your attention!