A regular version of Smilansky model

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Talk outline

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Motivation: $\sigma_{\rm disc}$ for infinite phase-space volume

Among many ideas we owe to Hermann Weyl semiclassical method is of the most successful as more than hundred years of its use demonstrates.

Nevertheless, it is not universal: there are systems with discrete spectrum for which the classically allowed phase-space volume is infinite. A classical example due to [Simon'83] is a 2D Schrödinger operator with the potential

$$V(x,y) = x^2y^2$$

or more generally, $V(x,y) = |xy|^p$ with $p \geq 1$.

Similar behavior one can observe for Dirichlet Laplacians in regions with hyperbolic cusps – see [Geisinger-Weidl'11] for recent results and a survey. Moreover, using the dimensional-reduction technique of Laptev and Weidl one can prove spectral estimates for such operators.

A common feature of these models is that the particle motion is confined into channels narrowing towards infinity.

Motivation: potentials unbounded from below

A similar behavior may occur even for Schrödinger operators with potential unbounded from below in which a classical particle can escape to infinity with an increasing velocity.

The situation changes, however, if the attraction is strong enough

As an illustration, let us analyze the following class of operators:

$$L_p(\lambda)\,:\; L_p(\lambda)\psi = -\Delta\psi + \Big(|xy|^p - \lambda(x^2+y^2)^{p/(p+2)}\Big)\,\psi\,, \quad p \geq 1$$

on $L^2(\mathbb{R}^2)$, where (x,y) are the standard Cartesian coordinates in $\mathbb{R}^2$ and the parameter λ in the second term of the potential is non-negative; unless the value of λ is important we write it simply as L_p.

Note that $\frac{2p}{p+2} < 2$ so the operator is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ by Faris-Lavine theorem; the symbol L_p or $L_p(\lambda)$ will always mean its closure.

The subcritical case

The spectral properties of $L_p(\lambda)$ depend crucially on the value of λ and there is a transition between different regimes as λ changes.

Let us start with the subcritical case which occurs for small values of λ. To characterize the smallness quantitatively we need an auxiliary operator which will be an (an)harmonic oscillator Hamiltonian on line,

$$\tilde{H}_p \,:\, \tilde{H}_p u = -u'' + |t|^p u$$

on $L^2(\mathbb{R})$ with the standard domain. Let γ_p be the minimal eigenvalue of this operator; in view of the potential symmetry we have $\gamma_p = \inf \sigma(H_p)$, where

$$H_p \,:\, H_p u = -u'' + t^p u$$

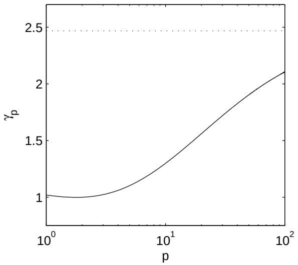
on $L^2(\mathbb{R}_+)$ with Neumann condition at $t = 0$.

The subcritical case – continued

The eigenvalue $\gamma_p = \inf \sigma(H_p)$ equals one for $p = 2$; for $p \to \infty$ it becomes $\gamma_\infty = \frac{1}{4}\pi^2$; it smoothly interpolates between the two values.

Since $x^p \geq 1 - \chi_{[0,1]}(x)$ we have $\gamma_p \geq \epsilon_0 \approx 0.546$, where ϵ_0 is the ground-state energy of the rectangular potential well of depth one.

In fact, a numerical solution gives true minimum $\gamma_p \approx 0.998995$ attained at $p \approx 1.788$; in the semilogarithmic scale the plot is as follows:



The subcritical case – continued

The spectrum is naturally bounded from below and discrete if $\lambda = 0$; our aim is to show that this remains to be the case provided λ is small enough.

Theorem (E-Barseghyan'12)

For any $\lambda \in [0, \lambda_{\mathrm{crit}}]$, where $\lambda_{\mathrm{crit}} := \gamma_p$, the operator $L_p(\lambda)$ is bounded from below for $p \geq 1$; if $\lambda < \gamma_p$ its spectrum is purely discrete.

Idea of the proof: Let $\lambda < \gamma_p$. By minimax we need to estimate L_p from below by a s-a operator with a purely discrete spectrum. To construct it we employ bracketing imposing additional Neumann conditions at concentric circles of radii $n = 1, 2, \dots$.

In the estimating operators the variables decouple asymptotically and the spectral behavior is determined by the angular part of the operators.

Subcritical behavior – the proof

Specifically, in polar coordinates we get direct sum of operators acting as

$$L^{(1)}_{n,p}\psi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{1}{n^2}\frac{\partial^2\psi}{\partial\varphi^2} + \left(\frac{r^{2p}}{2^p}|\sin 2\varphi|^p - \lambda r^{2p/(p+2)}\right)\psi$$

on the annuli $G_n := \{(r,\varphi)\,:\, n-1 \le r < n,\ 0 \le \varphi < 2\pi\}$, $n = 1, 2, \dots$ with Neumann conditions imposed on ∂G_n.

Obviously $\sigma(L^{(1)}_{n,p})$ is purely discrete for each $n = 1, 2, \dots$, hence it is sufficient to check that $\inf \sigma(L^{(1)}_{n,p}) \to \infty$ holds as $n \to \infty$.

We estimate $L^{(1)}_{n,p}$ from below by an operator with separating variables, note that the radial part does not contribute and use the symmetry of the problem; for $\varepsilon \in (0,1)$ the question is then to analyze

$$L^{(2)}_{n,p}\,:\; L^{(2)}_{n,p}u = -u'' + \left(\frac{n^{2p+2}}{2^p}\sin^p 2x - \frac{\lambda}{1-\varepsilon}\,n^{(4p+4)/(p+2)}\right)u$$

on $L^2(0,\pi/4)$ with Neumann conditions, $u'(0) = u'(\pi/4) = 0$.

Subcritical behavior – proof continued

We have $n^2 \inf\sigma(L^{(1)}_{n,p}) \geq \inf\sigma(L^{(2)}_{n-1,p})$ if n is large enough, specifically for $n > \left(1-(1-\varepsilon)^{(p+2)/(4p+4)}\right)^{-1}$, hence it is sufficient to investigate the spectral threshold $\mu_{n,p}$ of $L^{(2)}_{n,p}$ as $n\to\infty$.

The trigonometric potential can be estimated by a powerlike one with the similar behavior around the minimum introducing, e.g.

$$L^{(3)}_{n,p} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + n^{2p+2}x^p \left(\chi_{(0,\delta(\varepsilon)]}(x) + \left(\frac{2}{\pi}\right)^p \chi_{[\delta(\varepsilon),\pi/4)}(x)\right) - \lambda'_\varepsilon\, n^{(4p+4)/(p+2)}$$

for small enough $\delta(\varepsilon)$ with Neumann boundary conditions at $x=0, \frac14\pi$, where we have denoted $\lambda'_\varepsilon := \lambda(1-\varepsilon)^{-p-1}$.

We have $L^{(2)}_{n,p} \geq (1-\varepsilon)^p L^{(3)}_{n,p}$. To estimate the *rhs* by comparing the indicated potential contributions it is useful to pass to the rescaled variable $x = t\cdot n^{-(2p+2)/(p+2)}$.

Subcritical behavior – proof concluded

In this way we find that $\mu'_{n,p} := \inf \sigma(L^{(3)}_{n,p})$ satisfies

$$\frac{\mu'_{n,p}}{n^2} \to \infty \qquad \text{as} \qquad n \to \infty$$

Through the chain of inequalities we come to conclusion that $\inf \sigma(L^{(1)}_{n,p}) \to \infty$ holds as $n \to \infty$ which proves discreteness of the spectrum for $\lambda < \gamma_p$.

If $\lambda = \gamma_p$ the sequence of spectral thresholds no longer diverges but it remains bounded from below and the same is by minimax principle true for the operator $L_p(\lambda)$. $\square$

Remark

It is natural to conjecture that $\sigma(L_p(\gamma_p)) \supset \mathbb{R}_+$. It is less clear whether the critical operator can have also a negative discrete spectrum.

The supercritical case

The situation is different for large values of λ:

Theorem (E-Barseghyan'12)

The spectrum of $L_p(\lambda)$, $p \geq 1$, is unbounded below from if $\lambda > \lambda_{\text{crit}}$.

Idea of the proof: Similar as above with a few differences:

- now we seek an upper bound to $L_p(\lambda)$ by a below unbounded operator, hence we impose Dirichlet conditions on concentric circles
- the estimating operators have now a nonzero contribution from the radial part, however, it is bounded by π^2 independently of n
- the estimate of the angular part is simpler; the negative λ-dependent term now outweights the anharmonic oscillator part so that $\inf \sigma(L_{n,p}^{(1,D)}) \to -\infty$ holds as $n \to \infty$ $\square$

Also: a lower bounds to eigenvalue sums

To state the result we introduce the following quantity:

$$\alpha := \frac{1}{2}\left(1+\sqrt{5}\right)^2 \approx 5.236 > \gamma_p^{-1}$$

We denote by $\{\lambda_{j,p}\}_{j=1}^{\infty}$ the eigenvalues of $L_p(\lambda)$ arranged in the ascending order; then we can make the following claim.

Theorem (E-Barseghyan'12)

To any nonnegative $\lambda < \alpha^{-1} \approx 0.19$ there exists a positive constant C_p depending on p only such that the following estimate is valid,

$$\sum_{j=1}^{N} \lambda_{j,p} \geq C_p(1-\alpha\lambda)\frac{N^{(2p+1)/(p+1)}}{(\ln^p N+1)^{1/(p+1)}} - c\lambda N, \quad N = 1, 2, \dots,$$

where $c = 2\left(\frac{\alpha^2}{4}+1\right) \approx 15.7$.

Proof outline

Proof is technically demanding, we just review the main steps. We denote by $\{\psi_{j,p}\}_{j=1}^{\infty}$ normalized eigenfunctions corresponding to $\{\lambda_{j,p}\}_{j=1}^{\infty}$, i.e.

$$-\Delta\psi_{j,p} + (|xy|^p - \lambda(x^2+y^2)^{p)/(p+2)})\psi_{j,p} = \lambda_{j,p}\psi_{j,p}, \quad j = 1, 2, \dots;$$

without loss of generality we may suppose that they are real-valued.

Our potential forms hyperbolic-shaped "valleys" and we have to estimate eigenfunction integrals in them. Specifically, we check that for any natural j and $\delta > 0$ one has

$$\int_1^\infty \int_0^{(1+\delta)y^{-p/(p+2)}} y^{2p/(p+2)}\psi_{j,p}^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \le 2(1+\delta)^2 \int_1^\infty \int_0^\infty \left(\frac{\partial\psi_{j,p}}{\partial x}\right)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$+2\frac{1+\delta}{\delta}\int_1^\infty \int_0^{(1+\delta)y^{-p/(p+2)}} x^p y^p \psi_{j,p}^2(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Proof outline – continued

and that for any $\varepsilon > 0$ there is a number $1 \le \theta(\varepsilon) \le 1+\delta$ such that

$$\int_1^\infty y^{p/(p+2)} \psi_{j,p}^2 \left(\frac{\theta(\varepsilon)}{y^{p/(p+2)}}, y \right) \mathrm{d}y < \frac{1}{\delta} \int_1^\infty \int_{y^{-p/(p+2)}}^{(1+\delta)y^{-p/(p+2)}} x^p y^p \psi_{j,p}^2(x,y)\, \mathrm{d}x\, \mathrm{d}y + \varepsilon\,,$$

together with the symmetry counterparts of these relations.

In combination with $\|\psi_{j,p}\| = 1$ this allows us to estimate the modulus of the attractive term by a combination of the kinetic and repulsive ones:

$$\begin{aligned} &\int_{\mathbb{R}^2} (x^2+y^2)^{\frac{p}{p+2}} \psi_{j,p}^2(x,y)\, \mathrm{d}x\, \mathrm{d}y \\ &\qquad \le 2(1+\delta) \max\left\{ (1+\delta), \frac{1}{\delta} \right\} \left(\int_{\mathbb{R}^2} \left|\nabla \psi_{j,p}\right|^2(x,y)\, \mathrm{d}x\, \mathrm{d}y \right. \\ &\qquad\qquad \left. + \int_{\mathbb{R}^2} |xy|^p \psi_{j,p}^2(x,y)\, \mathrm{d}x\, \mathrm{d}y + (1+\delta)^2 \right) + 2 \end{aligned}$$

Proof outline – continued

We choose $\delta = \frac{-1+\sqrt{5}}{2}$ and put $c := \alpha(1+\delta)^2 + 2 = 2\left(\frac{\alpha^2}{4}+1\right)$; using then the fact that $\lambda_{j,p}$ is the eigenvalue corresponding to $\psi_{j,p}$ we get

$$\int_{\mathbb{R}^2} |\nabla\psi_{j,p}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^2} |xy|^p \psi_{j,p}^2 \,\mathrm{d}x\,\mathrm{d}y \le \frac{1}{1-\alpha\lambda}(\lambda_{j,p} + c\lambda), \quad j = 1, 2, \dots .$$

Subtracting a number ϱ from both sides and rewriting the first integral using Fourier-Plancherel image of $\psi_{j,p}$ we get

$$-\sum_{j=1}^{N} \int_{R^2} [\varrho - x^2 - y^2]_+ |\hat{\psi}_{j,p}|^2 \,\mathrm{d}x\,\mathrm{d}y + \sum_{j=1}^{N} \int_{R^2} |xy|^p \psi_{j,p}^2 \,\mathrm{d}x\,\mathrm{d}y \le \frac{1}{1-\alpha\lambda} \sum_{j=1}^{N} (\lambda_{j,p} + c\lambda) - N\varrho \,.$$

Proof outline – continued

Lemma (Barseghyan'09)

There is a constant C_p' such that for any orthonormal system of real-valued functions, $\Phi = \{\varphi_j\}_{j=1}^N \subset L^2(\mathbb{R}^2)$, $N = 1, 2, \dots,$ the inequality

$$\int_{\mathbb{R}^2} \rho_\Phi^{p+1}\,\mathrm{d}x\,\mathrm{d}y \le C_p'(\ln^p N + 1) \sum_{j=1}^N \int_{\mathbb{R}^2} |\xi\eta|^p |\hat{\varphi}_j|^2\,\mathrm{d}\xi\,\mathrm{d}\eta\,,$$

holds true, where $\rho_\Phi := \sum_{j=1}^N \varphi_j^2$.

We use it to estimate the second integral on the *lhs*. To the first one we apply Hölder inequality and find the minimum of the obtained expression with respect to the variable $z = \left(\int_{\mathbb{R}^2} \rho_\Phi^{p+1}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/(p+1)}$

Proof outline – concluded

After a short computation we get in this way

$$C_p''(1+\ln^p N)^{1/p}\varrho^{(2p+1)/p} \ge N\varrho - \frac{1}{1-\alpha\lambda}\sum_{j=1}^{N}(\lambda_{j,p}+c\lambda)$$

with the new constant being an explicit function of C_p'.

Hence we have to find $\widetilde{g}(N) = \sup_{\varrho\ge 0}\left(N\varrho - C_p''\,\varrho^{(2p+1)/p}(1+\ln^p N)^{1/p}\right)$, in other words, the Legendre transformation of the *lhs* of the last inequality. It is straightforward to check that

$$\widetilde{g}(N) = C_p \frac{N^{(2p+1)/(p+1)}}{(1+\ln^p N)^{1/(p+1)}}$$

with the constant given by $C_p := \left(\frac{p}{(2p+1)C_p''}\right)^{p/(p+1)} \frac{p+1}{2p+1}$. This is equivalent to the sought bound concluding thus the proof. $\square$

Other eigenvalue moments

We can use the result also to derive – by far not optimal – bounds to other eigenvalue moments. Assuming $\lambda_{1,p} \le \lambda_{2,p} \le \dots$ of operator $L_p(\lambda)$ we have the inequality

$$\sum_{j=K}^{K+N} \lambda_{j,p} \ge \frac{1}{2}\, C_p(1-\alpha\lambda)\, \frac{N^{(2p+1)/(p+1)}}{(1+\ln^p N)^{1/(p+1)}}, \quad K = 1, 2, \dots .$$

Using it for $K = N$ we get a lower bound for $\lambda_{2N,p}$ which further implies

Corollary

$$\sum_{j=1}^{N} \lambda_{j,p}^{\sigma} \ge \widetilde{C}_{p,\sigma}(1-\alpha\lambda)^{\sigma}\, \frac{N^{(p(\sigma+1)+1)/(p+1)}}{(1+\ln^p N)^{\sigma/(p+1)}}$$

holds any $\sigma > 0$ with some positive constant $\widetilde{C}_{p,\sigma}$.

Smilansky model

The model was originally proposed in [Smilansky'04] to describe a one-dimensional system interacting with a caricature *heat bath* represented by a harmonic oscillator.

Due to a particular choice of the coupling the model exhibited a *spectral transition* with respect to the coupling parameter.

Mathematical properties of the model were analyzed in [Solomyak'04], [Evans-Solomyak'05], [Naboko-Solomyak'06]. Recently in [Guarneri'11] time evolution in such a (slightly modified) model was analyzed.

In PDE terms, the model is described through a 2D Schrödinger operator

$$H_{\mathrm{Sm}} = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \lambda y \delta(x)$$

on $L^2(\mathbb{R})$ with various modifications to be mentioned later.

What is known about the model

- *Spectral transition:* if $|\lambda| > \sqrt{2}$ the particle can escape to infinity along the singular 'channel' in the *y* direction. In spectral terms, it corresponds to switch from a below bounded to below unbounded spectrum at $|\lambda| = \sqrt{2}$.
- *Eigenvalue absence:* for any $\lambda \geq 0$ there are *no eigenvalues* $\geq \frac{1}{2}$. If $|\lambda| > \sqrt{2}$, the point spectrum of H_{Sm} is *empty*.
- *Existence of eigenvalues:* for $0 < |\lambda| < \sqrt{2}$ we have $H_{\mathrm{Sm}} \geq 0$. The point spectrum is nonempty and finite, and

$$N(\tfrac{1}{2}, H_{\mathrm{Sm}}) \sim \frac{1}{4\sqrt{2(\mu(\lambda)-1)}}$$

holds as $\lambda \to \sqrt{2}-$, where $\mu(\lambda) := \sqrt{2}/\lambda$.

What is further known

- *Absolute continuity:* in the supercritical case $|\lambda| > \sqrt{2}$ we have $\sigma_{\mathrm{ac}}(H_{\mathrm{Sm}}) = \mathbb{R}$

- *Extension* of the result to a two 'channel' case with different oscillator frequencies [Evans-Solomyak'05]

- *Extension* to multiple 'channels' on a system *periodic in x* [Guarneri'11]. In this paper the time evolution generated by H_{Sm} is investigated and proposed as a model of *wavepacket collapse*.

These results have been obtained by a combination of different methods: a reduction to an infinite system of ODE's, facts from Jacobi matrices theory, variational estimates, etc.

Our aim is to look whether the model can be modified *replacing the singular coupling of the two degrees of freedom by a regular one*.

A regular version of Smilansky model

A regular version requires a modification, in particular, the coupling cannot be linear in y and the profile of the channel has to change with y.

The effect leans on effective variable decoupling far from the x-axis, where the oscillator potential competes with the principal eigenvalue of the 'transverse' part of the operator equal to $\frac{1}{4}\lambda^2 y^2$.

We replacing the δ by a family of shrinking potentials whose mean matches the δ coupling constant, $\int U(x,y)\,\mathrm{d}x \sim y$. This can be achieved, e.g., by choosing $U(x,y) = \lambda y^2 V(xy)$ for a fixed function V.

This motivates us to investigate the following operator on $L^2(\mathbb{R}^2)$,

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \lambda y^2 V(xy)\chi_{\{|x|\le a\}}(x),$$

where ω, a are positive constants, $\chi_{\{|y|\le a\}}$ is the indicator function of the interval $(-a,a)$, and the potential V with $\operatorname{supp} V \subset [-a,a]$ is a nonnegative function with bounded first derivative.

A regular version of Smilansky model, continued

By Faris-Lavine theorem the operator is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ and the same is true for its generalization,

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \sum_{j=1}^{N} \lambda_j y^2 V_j(xy) \chi_{\{|x-b_j| \le a_j\}}(x)$$

with a finite number of channels, where functions V_j are positive with bounded first derivative, with the supports contained in $(b_j - a_j, b_j + a_j)$ and such that $\mathrm{supp}\, V_j \cap \mathrm{supp}\, V_k = \emptyset$ holds for $j \neq k$.

Remark

We note that the properties discussed below depend on the asymptotic behavior of the potential channels and would not change if the potential is modified in the vicinity of the x-axis, for instance, by replacing the above cut-off functions with $\chi_{|y| \ge a}(y)$ and $\chi_{|y| \ge a_j}(y)$, respectively.

Subcritical case

To state the result we employ a 1D comparison operator $L = L_V$,

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2 - \lambda V(x)$$

on $L^2(\mathbb{R})$ with the domain $H^2(\mathbb{R})$. What matters is the sign of its spectral threshold; since V is supposed to be nonnegative, the latter is a monotonous function of λ and there is a $\lambda_{\mathrm{crit}} > 0$ at which the sign changes.

Theorem (Barseghyan-E'13)

Under the stated assumption, the spectrum of the operator H is bounded from below provided the operator L is positive.

Proof outline

It is sufficient to prove the claim for $\lambda = 1$. We employ Neumann bracketing, similarly as for the previous model.

Let h_n and $\widetilde{h}_n$ be respectively the restrictions of operator H to the strips $G_n = \mathbb{R} \times \{y :\, \ln n < y \le \ln(n+1)\}$, $n = 1, 2, \ldots$, and $\widetilde{G}_n$, their mirror images w.r.t. y, with Neumann boundary conditions; then

$$H \ge \bigoplus_{n=1}^{\infty} (h_n \oplus \widetilde{h}_n)\,;$$

We find a uniform lower bound $\sigma(h_n)$ and $\sigma(\widetilde{h}_n)$ as $n \to \infty$.

Using the assumptions about V we find

$$V(xy) - V(x \ln n) = \mathcal{O}\left(\frac{1}{n \ln n}\right), \quad y^2 - \ln^2 n = \mathcal{O}\left(\frac{\ln n}{n}\right)$$

for any $(x, y) \in G_n$, and analogous relations for $\widetilde{G}_n$.

Proof outline – continued

This yields

$$y^2 V(xy) - \ln^2 n\, V(x \ln n) = \mathcal{O}\left(\frac{\ln n}{n}\right)$$

for for any $(x, y) \in \widetilde{G}_n$ which allows us to check that

$$\inf \sigma(h_n) \geq \inf \sigma(l_n) + \mathcal{O}\left(\frac{\ln n}{n}\right),$$

where $l_n := -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 \ln^2 n - \ln^2 n\, V(x \ln n)$ on $L^2(G_n)$.

The analogous relation holds for $\widetilde{l}_n$ on $L^2(\widetilde{G}_n)$. It is important that all these operators have separated variables.

Since the minimal eigenvalue of Neumann Laplacian $-\frac{\mathrm{d}^2}{\mathrm{d}y^2}$ on the strips $\ln n < y \leq \ln(n+1)$, $n = 1, 2, \dots,$ is zero, we have $\inf \sigma(l_n) = \inf \sigma\big(l_n^{(1)}\big)$, where the last operator on $L^2(\mathbb{R})$ acts as

$$l_n^{(1)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2\, \ln^2 n - \ln^2 n\, V(x \ln n)$$

Proof outline – concluded

Note that the cut-off function $\chi_{\{|x|\le a\}}$ plays no role in the asymptotic estimate as it affects a finite number of terms only.

By the change of variable $x = \frac{t}{\ln n}$ the last operator is unitarily equivalent to $\ln^2 n\, L$ which is non-negative as long as L is non-negative. In the same way one proves that $\widetilde{l_n}$ is non-negative ; this concludes the proof. □

In the same way one can treat systems restricted in the x direction:

Corollary

Let H be 'our' operator on $(-c,c)\times\mathbb{R}$ for some $c \ge a$ with Dirichlet (Neumann, periodic) boundary conditions in the variable x. The spectrum of H is bounded from below if $L \ge 0$ holds, where L is the comparison operator on $L^2(-c,c)$ with Dirichlet (respectively, Neumann or periodic) boundary conditions.

Supercritical case

Once the transverse channel principal eigenvalue dominates over the harmonic oscillator contribution, the spectral behavior changes:

Theorem (Barseghyan-E'13)

Under our hypotheses, $\sigma(H) = \mathbb{R}$ *holds if* $\inf \sigma(L) < 0$.

Proof relies on construction of an appropriate Weyl sequence: we have to find $\{\psi_k\}_{k=1}^{\infty} \subset D(H)$ such that $\|\psi_k\| = 1$ which contains no convergent subsequence, and at same time

$$\|H\psi_k - \mu\psi_k\| \to 0 \quad \text{as} \quad k \to \infty.$$

The construction is rather technical and we sketch just the main steps.

Proof outline

The claim is invariant under scaling transformations, hence we may suppose $\inf \sigma(L) = -1$. The spectral threshold is a simple isolated eigenvalue; we denote the corresponding normalized eigenfunction by h.

We want to show first that $0 \in \sigma_{\mathrm{ess}}(H)$. In fact, it would be enough for the proof to show that zero belongs to $\sigma(H)$ but we get the stronger claim at no extra expense.

We fix an $\varepsilon > 0$ and choose a natural $k = k(\varepsilon)$ with which we associate a function $\chi_k \subset C_0^2(1,k)$ satisfying the following conditions

$$\int_1^k \frac{1}{z}\chi_k^2(z)\,\mathrm{d}z = 1 \quad \text{and} \quad \int_1^k z(\chi_k'(z))^2\,\mathrm{d}z < \varepsilon.$$

Proof outline – continued

Such functions exist: as an example consider

$$\tilde{\chi}_k(z) = \frac{8\ln^3 z}{\ln^3 k}\chi_{\{1\le z\le\sqrt{k}\}}(z) + \frac{2\ln k - 2\ln z}{\ln k}\chi_{\{\sqrt{k}+1\le z\le k-1\}}(z)$$
$$+g_k(z)\chi_{\{\sqrt{k}<z<\sqrt{k}+1\}}(z) + q_k(z)\chi_{\{k-1<z\le k\}}(z),$$

where g_k and q_k are interpolating functions chosen in such a way that $\tilde{\chi}_k \in C_0^2(1,k)$, and define

$$\chi_k(z) = \left(\int_1^k \frac{1}{z}\tilde{\chi}_k^2(z)\,\mathrm{d}z\right)^{-1/2}\tilde{\chi}_k(z).$$

Given such functions χ_k, pu

$$\psi_k(x,y) := h(xy)\,\mathrm{e}^{iy^2/2}\chi_k\left(\frac{y}{n_k}\right) + \frac{f(xy)}{y^2}\,\mathrm{e}^{iy^2/2}\chi_k\left(\frac{y}{n_k}\right),$$

where $f(t) := -\frac{i}{2}\,t^2 h(t),\ t\in\mathbb{R}$, and $n_k \in \mathbb{N}$ is a positive integer, which we choose using the following auxiliary result.

Proof outline – continued

Lemma

Let ψ_k, $k = 1, 2, \ldots$, as defined above; then for any given k one can achieve that $\|\psi_k\|_{L^2(\mathbb{R}^2)} \geq \frac{1}{2}$ holds by choosing n_k large enough.

We need one more auxiliary result:

Lemma

Let ψ_k, $k = 1, 2, \ldots$, be again functions defined above; then the inequality $\|H\psi_k\|^2_{L^2(\mathbb{R}^2)} < c\varepsilon$ with a fixed constant c holds for $k = k(\varepsilon)$.

Proofs are in both cases straightforward but rather tedious.

Proof outline – concluded

Using the lemmata, we are able to complete the proof. We fix a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that $\varepsilon_j \searrow 0$ holds as $j \to \infty$ and to any j we construct a function $\psi_{k(\varepsilon_j)}$ in such a way that $n_{k(\varepsilon_j)} > k(\varepsilon_{j-1}) n_{k(\varepsilon_{j-1})}$.

The norms of $H\psi_{k(\varepsilon_j)}$ are bounded from above with $9\varepsilon_j$ on the right-hand side, and since the supports of $\psi_{k(\varepsilon_j)}$, $j = 1, 2, \dots$, do not intersect each other by construction, their sequence converges weakly to zero.

This yields the sought Weyl sequence for zero energy; for any nonzero real number μ we use the same procedure replacing the above ψ_k with

$$\psi_k(x,y) = h(xy)\, \mathrm{e}^{i\epsilon_\mu(y)} \chi_k\left(\frac{y}{n_k}\right) + \frac{f(xy)}{y^2}\, \mathrm{e}^{i\epsilon_\mu(y)} \chi_k\left(\frac{y}{n_k}\right),$$

where $\epsilon_\mu(y) := \int_{\sqrt{|\mu|}}^{y} \sqrt{t^2+\mu}\,\mathrm{d}t$, and furthermore, the functions f, χ_k are defined in the same way as above. $\square$

Restricted motion

In the supercritical case, too, the result extends to systems restricted in the x direction:

Theorem

Let H be the 'our' operator on $L^2(-c, c) \otimes L^2(\mathbb{R})$ for some $c > 0$ with Dirichlet condition at $x = \pm c$ and denote by L the corresponding Dirichlet operator on $L^2(-c, c)$. If the spectral threshold of L is negative, the spectrum of H covers the whole real axis.

Observing the domains of the quadratic form associated with such operators we get

Corollary

The claim of the above theorem remains valid if the Dirichlet boundary conditions at $x = \pm c$ are replaced by any other self-adjoint boundary conditions.

The multichannel case

The above results are interesting not only *per se* or to deal with the Guarneri-type periodic modification of the model.

Using a simple bracketing argument we can show how the spectral-regime transition looks like in the multichannel case:

Theorem (Barseghyan-E'13)

Let H be 'our' operator with the potentials satisfying the stated assumptions, namely the functions V_j are positive with bounded first derivative and $\mathrm{supp}\, V_j \cap \mathrm{supp}\, V_k = \emptyset$ holds for $j \neq k$. Denote by L_j the comparison operator on $L^2(\mathbb{R})$ with the potential V_j and set $t_V := \min_j \inf \sigma(L_j)$. Then H is bounded from below if and only if $t_V \geq 0$ and in the opposite case its spectrum covers the whole real axis.

Some open questions

- Are there eigenvalue-sum estimates for λ between $\alpha^{-1} \approx 0.19$ and $\gamma_p \gtrsim 0.999$?
- Find better estimates for other eigenvalue moments
- One expects that $\mathbb{R}_+ \subset \sigma(L_p(\lambda_{\text{crit}})$. Are there negative eigenvalues?
- Prove that $\sigma(L_p(\lambda_{\text{crit}})) = \mathbb{R}$ for $\lambda > \lambda_{\text{crit}}$
- Analyze the negative discrete spectrum for subcritical regular Smilansky model
- Find the time evolution in this model, in particular, its change when we pass from subcritical to supercritical regime

The talk was based on

[EB12] P.E., D. Barseghyan: Spectral estimates for a class of Schrödinger operators with infinite phase space and potential unbounded from below, *J. Phys. A: Math. Theor.* **A45** (2012), 075204.

[EB13] D. Barseghyan, P.E.: A regular version of Smilansky model, submitted; `arXiv: 1308.4249 [math-ph]`.

It remains to say

Thank you for your attention!