The intriguing δ'

Pavel Exner

Doppler Institute
for Mathematical Physics and Applied Mathematics
Prague

with grateful remembrance of a common work which inspired various further explorations

A talk at the conference Quantum Spectra and Transport

The Hebrew University of Jerusalem, June 30, 2013

Some conferences are particular

It is a good habit to celebrate jubilees by discussing results related to the honoree work. In the present case

this opens a rather wide field because Yosi made a footprint in many areas.

One is nevertheless tempted to choose a problem around which our paths crossed, especially if the subject proved inspirative and lead to various other investigations.

Where it started?

The root of the present story extend as far as to 1980 when A. Grossmann, R. Høegh-Krohn and M. Mebkhout investigated point interactions on line and introduced a counterpart to the usual δ interaction which the called δ'.

By this they meant operator $-\Delta$ on $L^2(\mathbb{R})$ defined on $\psi \in H^2(\mathbb{R} \setminus \{0\})$ satisfying the conditions

$$\psi'(0+) = \psi'(0-) =: \psi'(0) \quad \text{and} \quad \psi(0+) - \psi(0-) = \beta\psi'(0)$$

for some $\beta \in \mathbb{R}$, and its generalizations to the many-center case.

The name they choose was not particularly fortunate but it stuck. Recall that while while the δ interaction can be obtained as a limit of *scaled potentials*, the δ' is *not* the limit of scaled potentials of *zero mean* – cf. [Šeba'86, Zolotaryuk et al.'03].

Why we should be excited?

It all looks like a simple example to illustrate self-adjoint extensions of second-order differential equations.

Recall, however, a quote from George Elliott used, in particular, as a motto in Reed-Simon II: *Any blockhead can cite generalities, but the mastermind discerns the particular cases they represent*.

The intriguing features of the interaction came first to attention when Yosi proposed to look at the δ' version of the *Wannier-Stark model* combining singular periodic and linear potentials, i.e the system formally described by the Hamiltonian

$$H(\beta, F, a) = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n\in Z} \beta\delta'_{na} - eFx$$

δ' WS exhibits no transport

We have the following surprising result:

Theorem (Avron-E-Last'94)

$\sigma_{\mathrm{ac}}(H(\beta, F, a)) = \emptyset$ *holds any non-zero* F, λ

Proof sketch: (a) The periodic operator, $F = 0$, has a band spectrum, in contrast to more regular potentials, it is *dominated by gaps* as $k \to \infty$.

(b) A heuristic argument: the tilted gaps are *classically forbidden regions*; if their widths grow they may prevent indefinite propagation.

(c) Make it rigorous by a *Simon-Spencer-type* argument. Inserting a sequence of Neumann conditions we get an operator with pure point spectrum; if the 'chops' are placed in the middle of the tilted gaps, one can check that the perturbation is *trace class*. □

Robustness of the effect

The assumptions can be substantially weakened in two directions:

- the background potential need not be linear
- one may consider a lattice of non-identical δ' interactions

It is sufficient consider the halfline problem only assuming that

(a) the linear potential is replaced by a locally bounded V satisfying $V(x) = -U(x) + W(x)$ for $x > x_0$ with some $x_0 > 0$, where U, V are such that

(a1) U is nondecreasing, $\lim_{x\to\infty} U(x) = \infty$
(a2) U is C^2 smooth with $|U'(x)| \le c$ and $|U''(x)| \le \tilde{c}U(x)$ for some $c, \tilde{c} > 0$
(a3) W is piecewise continuous and bounded

Under these conditions the standard solutions between neighboring δ's are

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}(x) = \begin{pmatrix} \cos \\ \sin \end{pmatrix} k_n(x-n)\left(1 + \mathcal{O}(U(x)^{-1/2})\right)$$

Robustness of the effect, continued

In addition, we suppose that the coupling constants β_n are such that

(b) $|\beta_n| \geq \beta > 0$ for all n

(c) there is a monotonic sequence $\{n_\ell\} \subset \mathbb{Z}_+$ such that $\mathrm{Re}\, k_{n_\ell} = \pi(n_\ell + \epsilon_\ell)$ with $\epsilon_\ell \in (\frac{1}{4}, \frac{3}{4})$, and $\beta_{n_\ell}\beta_{n_{\ell+1}}^{-1}$ remains bounded as $n_\ell \to \infty$

Using the same trace-class perturbation argument we can conclude:

Theorem (E'95)

Under the assumptions (a)–(c) the ac spectrum of

$$H(\{\beta_n\}, V, a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n \in Z_+} \beta_n \delta'_{na} + V(x)$$

is empty

How does the spectrum look like?

Character of the spectrum: using a KAM-type argument one can prove – cf. [Asch-Duclos-E'98] – that for all but a 'small set' of the parameters the spectrum of $H(\beta, F, a)$ is *pure point*, and extend this result to the result to a class of nonlinear background potentials.

Conjecture: The same conclusion can be made for *all* parameter values

Spectrum as a set depends on parameter values, in particular, one is able to *conjecture* that the following relation holds,

$$\sigma_{\text{ess}}(H(\beta, F, a)) = \left\{ \frac{4}{\beta a} + \left(\frac{m\pi}{a}\right)^2 - F\left(n + \frac{1}{2}\right) a \,:\, m, n \in \mathbb{Z} \right\}$$

This would imply a dichotomy:

- if $\gamma := \left(\frac{a}{\pi}\right)^2 Fa$ is rational, the spectrum is *nowhere dense*, and therefore automatically *pure point*.
- on the other hand, $\sigma((H(\beta, E, a)) = \sigma_{ess}((H(\beta, E, a)) = \mathbb{R}$ holds if γ is irrational.

A digression: the δ problem

- while the spectrum of a regular Wannier-Stark problem is purely *ac* and that of its δ' version is purely singularly continuous, the borderline case remains open
- I mean the corresponding δ problem, or the *Kronig-Penney model* amended with a linear potential, viz

$$H(\beta, F, a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n\in\mathbb{Z}} \alpha\delta_{na} - eFx$$

- the problem is open for a long time – *Yosi can wittness*
- a heuristic argument suggest that the spectrum might exhibit a transition from (singularly?) continuous at small F to a point one for larger field values
- such a behavior is observed in the random case with probability one [Delyon-Simon-Souillard'85], however, the deterministic problem remains open

What is then the δ'?

Note that between δ and δ' there is a *duality* with respect to the interchange of the energy/momentum values, $k \leftrightarrow k^{-1}$.

As a result, in contrast to the 'usual' potentials, regular or singular, the δ' becomes *opaque at high energies*, and need a mechanism which could produce this feature, at least in an *approximation*.

The first attempt we made in [Avron-E-Last'94] was *to replace points by 'onion' type graphs*,

Each 'onion' consists of N links of length L; we consider the limit $N \to \infty$ keeping the product $NL = \beta$ fixed.

'Onion' graph limits

- Assuming 'Kirchhoff' conditions at the graph vertices, one easily finds the reflection amplitude of a single 'onion' to be

$$r(kL;N) = \frac{-N^2+1}{N^2+2iN\cot(kL)+1}$$

- it oscillates as $k\to\infty$ but in the described limit $N\to\infty$ we get

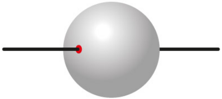
$$r(kL;N)\,\to\,-\frac{1}{1+2i/\beta k}$$

  which is nothing else than the δ' reflection amplitude

- hence the two halflines asymptotically decouple as as $k\to\infty$, even if the decoupling is Dirichlet instead of Neumann appropriate for the δ'
- for an 'onion' string we get similarly the *band-gap structure* of the δ' in the limit $k\to\infty$

Other geometric scatterers

As an example, consider a sphere with two leads attached

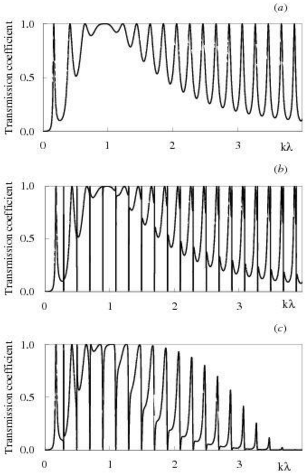


They are coupled through boundary conditions which involve boundary values on the halfline with the *generalized ones* on the sphere which are the coefficients in the expansion $\Phi(\vec{x}) = L_0(\Phi) \ln|\vec{x}| + L_1(\Phi_2) + \mathcal{O}(|\vec{x}|)$

The system was investigated in [Kiselev'97, E-Tater-Vaněk'01, Brüning et al'02] showing, in particular, the following properties

- scattering *en gross* is not very sensitive to the choice of the coupling, it is *sensitive to relative junction positions*
- there are *numerous resonances* in such systems
- the *background reflection dominates* at high energies, $k \to \infty$

Transmission through the sphere



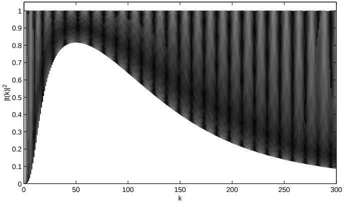
(a) Junctions at opposed poles, (b) tilt 2^{o}, (c) tilt 4^{o}

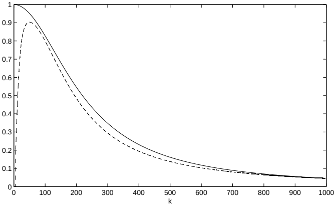
(reproduced from [Brüning-Geyler-Margulis-Pyataev'02])

Further properties of the polar case

By the results of [Kiselev'97, E-Tater-Vaněk'01] we can also claim that

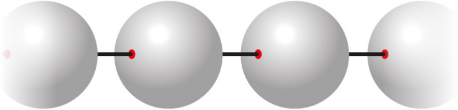
- transmission at resonance energies is *asymptotically perfect*
- the background scattering *asymptotically dominates* and the transmission probability there is $\sim (E \ln E)^{-1}$ as the energy $E \to \infty$
- *Conjecture:* The *coarse-grained* transmission probability decays as that of the δ', that is, $\sim E^{-1}$
- The conjecture is supported to numerical results, for instance

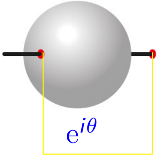




Arrays of geometric scatterers

One can also consider 'necklaces' of such geometric scatterers:





The band spectrum of such system can be found using standard Floquet analysis investigating the dependence of single cell eigenvalues on the *quasimomentum* θ

Large gaps again

Denote by B_n, G_n the nth band and gap of the sphere array, respectively; then we have the following claim:

Theorem (Brüning-E-Geyler'03)

There is a $c > 0$ such that $\frac{|B_n|}{|G_n|} \leq c\, n^{-1/2}$ holds as $n \to \infty$.

Remarks: (a) The result is proven for a subset of admissible couplings but one can *conjecture* its validity generally.

(b) Similar result holds also for 'carpets' of scatterers with $n^{-1/2}$ replaced by $n^{-1/4}$, or even for 'tight' systems where the spheres touch each other; there the gaps dominate *logarithmically* at high energies.

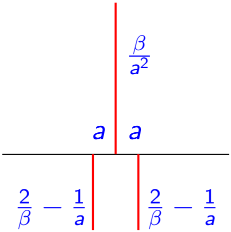
Question: Is the *ac* spectrum again absent if we add an electric field parallel to the array?

Approximation by Schrödinger operators

Moral of the following story: *Mathematicians are sometimes wrong!*

For a decade every involved mathematician new that such an approximation does not exist — that is, without introducing velocity-dependent potentials or similar things.

Then a scheme appeared in [Cheon-Shigehara'98] which gave formally the δ' conditions in the limit $a \to 0$, namely



Approximation by Schrödinger operators, contd

In fact, the convergence of those operators, in standard notation notation [AGHH'05] written as $-\Delta_{\mathcal{A}_a, Y_a}$ and $\Xi_{\beta,y}$ is by far not only formal:

- convergence of transfer (thus also scattering) matrices was proven in [Albeverio-Nizhnik'00]
- the norm-resolvent convergence, $\|(-\Delta_{\mathcal{A}_{a(\epsilon)}, Y_{a(\epsilon)}} + \kappa^2)^{-1} - (\Xi_{\beta,y} + \kappa^2)^{-1}\| \to 0$ as $a \to 0$ was proven in [E-Neidhardt-Zagrebnov'01]
- One should note how subtle the convergence is: both resolvents are strongly divergent as $a \to 0$, but in the difference *the first four orders cancel* and the fifth gives a convergent result

Moreover, since δ interaction is a limit of squeezed potentials, one can approximate the δ' by *regular potentials*.

Approximation by Schrödinger operators, contd

Consider operator $H^a_{\epsilon,y} := -\Delta + W^a_{\epsilon,y}$ with the potential

$$W^a_{\epsilon,0}(x) = \frac{\beta}{\epsilon a(\epsilon)^2}\, V_0\left(\frac{x}{\epsilon}\right) + \left(\frac{2}{\beta} - \frac{1}{a(\epsilon)}\right)\left\{\frac{1}{\epsilon}\, V_{-1}\left(\frac{x+a(\epsilon)}{\epsilon}\right) + \frac{1}{\epsilon}\, V_1\left(\frac{x-a(\epsilon)}{\epsilon}\right)\right\}$$

and let $\|V_j\|_{L^1} = 1$ and $\int_{-\infty}^{\infty} \mathrm{d}x\, |x|^{1/2}\, |V_0(x)| < \infty$ hold for $j = 0, \pm 1$.

Moreover, assume that $a(\epsilon) \to 0$ as $\epsilon \to 0$ and $\lim_{\epsilon\to 0} \frac{\epsilon}{a(\epsilon)^{12}} = 0$.

Theorem (E-Neidhardt-Zagrebnov'01)

Under the stated assumptions, one has

$$\lim_{\epsilon\to 0} \left\|(H^a_{\epsilon,y} + \kappa^2)^{-1} - (\Xi_{\beta,y} + \kappa^2)^{-1}\right\| = 0\,.$$

Remark: The power 12 is for sure not optimal.

Passing to higher dimensions

The δ' is an essentially one-dimensional thing, however, on can obtain interesting models considering singular Schrödinger operators with a δ' interaction supported by a manifold of *codimension one*.

Let us look a the radially periodic potentials in $\mathbb{R}^\nu$, $\nu \geq 2$. If they are regular, the spectrum mixes by [Hempel-Herbst-Hinz-Kalf'91] absolutely continuous and dense pure point components. The same is true by [E-Fraas'07] for concentric δ potentials.

One can treat similarly other point interactions, in particular, the δ'. As long as the system has a radial symmetry, we can employ the partial-wave decomposition, $H_\beta := \bigoplus_l U^{-1} H_{\beta,l} U \otimes I_l$, where

$$H_{\beta,l} := -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r^2}\left[\frac{(\nu-1)(\nu-3)}{4} + l(l+\nu-2)\right]$$

defined on functions which are locally H^2 and satisfy the δ' conditions *with the same coupling constant β* at the radii r_n, $n = 1, 2, \dots$ with $r_{n+1} - r_n = d > 0$ (an extra condition at the origin needed if $\nu \leq 3$).

Comparison to the δ' KP model

The radial motion can be naturally compared to the one described by

$$h_\beta := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n\in\mathbb{Z}} \beta\delta'_{x_n}$$

on $L^2(\mathbb{R})$ with the δ' interactions at the points $x_n = d(n+\frac{1}{2})$.

In particular, one has $\sigma_{\mathrm{ess}}(H_{\beta,l}) = \sigma_{\mathrm{ess}}(h_\beta)$ which yields

$$\sigma_{\mathrm{ess}}(H_\beta) = [\inf \sigma_{\mathrm{ess}}(h_\beta), \infty)$$

through a minimax estimate and construction of suitable Weyl sequences; the idea is that choosing an interval far from the origin, we get an almost constant background from the centrifugal term.

Remark: If $\nu = 2$ the operator H_β has infinitely many eigenvalues below $\inf \sigma_{\mathrm{ess}}(H_\beta)$ – they are analogous to the *'Welsh eigenvalues'* discussed in [Brown et al'98].

Character of the spectrum

Theorem (E-Fraas'08)

(a) For any gap (E_{2k-1}, E_{2k}) in the essential spectrum of h_β we have
(i) H_β has no continuous spectrum in (E_{2k-1}, E_{2k}),
(ii) eigenvalues of H_β are dense in (E_{2k-1}, E_{2k}).

(b) On every compact K with the support in the interior of the band of h_β the spectrum of H_β is purely absolutely continuous.

Proof sketch: By the partial-wave decomposition there is no continuous spectrum in the gaps of h_β, at the same time they belong to $\sigma_{\mathrm{ess}}(H_\beta)$. The absolute continuity on K is obtained by a subordinacy argument. □

Remark: While the spectrum consists of interlaced intervals of *ac* and *dense p.p.* spectrum, in contrast to more regular potentials including δ, for δ' the *dense point component dominates at high energies*.

Strong coupling behavior

So far we have seen that a sub-manifold supported δ' potentials can lead to spectral properties different from its more regular counterparts.

Now we are going to discuss an effect where the difference does not show — the reason is that from the δ' point of view we will deal with the lower part of the spectrum.

For simplicity we consider operators in $L^2(\mathbb{R}^2)$ with the interaction support being a smooth closed curve Γ, being graph of a function $\Gamma:\ [0, L] \to \mathbb{R}^2$.

The operator acts as Laplacian outside the interaction support,

$$(H_{\beta,\Gamma}\psi)(x) = -(\Delta\psi)(x)$$

for $x \in \mathbb{R}^2 \setminus \Gamma$, with the domain $\mathcal{D}(H_{\beta,\Gamma}) = \{\psi \in H^2(\mathbb{R}^2 \setminus \Gamma) \mid \partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) =: \psi'(x)|_\Gamma,\ \beta\psi'(x)|_\Gamma = \psi(x)|_{\partial_+\Gamma} - \psi(x)|_{\partial_-\Gamma}\}$, where n_Γ is the outer normal to Γ and $\psi(x)|_{\partial_\pm\Gamma}$ are the appropriate traces.

Strong coupling behavior, continued

Alternatively, the singular Schrödinger operator $H_{\beta,\Gamma}$ can be defined through its quadratic form. We introduce locally orthogonal coordinates (s,u) in the vicinity of Γ — s is the arc length of Γ and u the distance from the curve — and set

$$h_{\beta,\Gamma}[\psi] = \|\nabla\psi\|^2 + \beta^{-1}\int_\Gamma |\psi(s,0_+) - \psi(s,0_-)|^2\,\mathrm{d}s$$

defined on functions $\psi \in C(\mathbb{R}^2) \cap H^1(\mathbb{R}^2 \setminus \Gamma)$ written, with an abuse of notation, also as $\psi(s,u)$.

We are interest in the asymptotic behavior of $\sigma(H_{\beta,\Gamma})$ for *strong δ' interaction*; recall that it corresponds to the limit $\beta \to 0-$.

For this purpose we introduce a comparison operator

$$S := -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$$

on $L^2(0,L)$ with periodic boundary conditions, where $\gamma(s)$ denotes the *signed curvature* of Γ at the point s.

Strong coupling on a δ' loop

Theorem (E-Jex'13)

Let Γ be a C^4-smooth closed curve without self-intersections. Then $\sigma_{\mathrm{ess}}(H_{\beta,\Gamma}) = [0,\infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that $\#\sigma_{\mathrm{disc}}(H_{\beta,\Gamma}) \geq n$ holds for $\beta \in (0, \beta_n)$. Denoting for such a β by $\lambda_j(\beta)$ the j-th eigenvalue of $H_{\beta,\Gamma}$, again counted with its multiplicity, we have the asymptotic expansion

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}\big(\beta \ln|\beta|\big)\,, \quad j = 1, \dots, n\,,$$

valid as $\beta \to 0-$, where μ_j is the j-th eigenvalue of the comparison operator S introduced above. Moreover, for the counting function $\beta \mapsto \#\sigma_d(H_{\beta,\Gamma})$ we have

$$\#\sigma_{\mathrm{disc}}(H_{\beta,\Gamma}) = \frac{2L}{\pi|\beta|} + \mathcal{O}(-\ln|\beta|) \quad \text{as} \quad \beta \to 0-\,.$$

Strong coupling behavior, continued

Remark: Compare the above with the asymptotics for δ interaction,

$$\lambda_j(\beta) = -\frac{\alpha^2}{4} + \mu_j + \mathcal{O}\big(-\alpha^{-1}\ln|\alpha|\big)\,, \quad j = 1,\dots,n\,,$$

as $\alpha \to -\infty$. The divergent term is different, but the second term in the asymptotics refers to *the same comparison operator S*.

Sketch of the proof: Choose $\Omega_a := \{x \in \mathbb{R}^2 :\, \mathrm{dist}\,(x,\Gamma) < a\}$; for a small enough passing to curvilinear coordinates (s,u) is a diffeomorphism on Ω_a.

We employ a bracketing argument: imposing Dirichlet and Neumann conditions at $\partial\Omega_a$, we get $H_N(\beta) \le H_\beta \le H_D(\beta)$. Furthermore, the 'outer' parts of the estimating operators are positive, hence for our purpose it is only the strip part which matters. The corresponding quadratic forms are

$$h_{N/D,\beta}[f] = \|\nabla f\|^2 + \beta^{-1}\int_\Gamma |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s$$

defined on $H^1(\Omega_a \setminus \Gamma)$ and $H^1_0(\Omega_a \setminus \Gamma)$, respectively.

Proof sketch

Next we 'straighten' Ω_a passing to the coordinates (s,u); in this way the estimating operators are equivalent to those associated with the forms

$$q_D[f] = \left\| \frac{1}{g}\partial_s f \right\|^2 + \|\partial_u f\|^2 + (f, Vf) + \beta^{-1}\int_0^L |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^L \gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,\mathrm{d}s$$

$$q_N[g] = q_D[g] - \int_0^L \frac{\gamma(s)}{2(1+a\gamma(s))}|f(s,a)|^2\,\mathrm{d}s + \int_0^L \frac{\gamma(s)}{2(1-a\gamma(s))}|f(s,-a)|^2\,\mathrm{d}s$$

on $H_0^1((0,L)\times((-a,0)\cup(0,a)))$ and $H^1((0,L)\times((-a,0)\cup(0,a)))$, respectively, with periodic boundary conditions in the variable s. The geometrically induced potential in these formulæ is given by $V = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$ with $g(s) := 1 + u\gamma(s)$.

This is still not easy to handle, therefore we pass to slightly cruder estimates by the operators $Q^{\pm}_{a,\beta} = U^{\pm}_a \otimes I + \int^{\oplus}_{[0,L)} T^{\pm}_{a,\beta}(s)\,\mathrm{d}s$, where $U^{\pm}_a$ refers to a u-independent estimate of the first and the third terms.

Proof sketch, continued

The transverse part of the upper bound corresponds to the quadratic form

$$t^+_{a,\beta}(s)[f] := \|f'\|^2 - \frac{1}{\beta}\,|f(0_+) - f(0_-)|^2 + \frac{1}{2}\gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,,$$

and has the following spectral properties:

Lemma

The operators $T^+_{a,\beta}(s)$ has exactly one negative eigenvalue $t_+ = -\kappa_+^2$ provided $\frac{a}{\beta} > 2$ which is independent of s and such that

$$\kappa_+ = \frac{2}{\beta} - \frac{4}{\beta}\,\mathrm{e}^{-4a/\beta} + \mathcal{O}(\beta^{-1}\mathrm{e}^{-8a/\beta}) \quad \textit{holds as} \quad \beta \to 0\,.$$

Proof sketch, concluded

In a similar way we estimate $T^{+}_{a,\beta}(s)$ finding that the influence of the boundary is again *exponentially small*.

As for the longitudinal part, we have

Lemma

There is a positive C independent of a and j such that

$$|\mu_j^{\pm}(a) - \mu_j| \le Caj^2$$

holds for $j \in \mathbb{N}$ and $0 < a < \frac{1}{2\gamma_+}$, where $\mu_j^{\pm}(a)$ are the eigenvalues of $U_a^{\pm}$, respectively, with the multiplicity taken into account.

Choosing finally $a(\beta) = \frac{3}{4}\beta \ln |\beta|$ and putting the estimates together, we get the first claim; the second one is demonstrated in a similar way. □

Geometrically induced bound states

Let us finally show that the δ' *need not be strong* to produce spectral effects related to the geometry of its support.

Consider a singular Schrödinger operator $H_{\beta,\Gamma}$ in $L^2(\mathbb{R}^2)$ with an attractive δ' interaction supported by an infinite curve Γ.

Theorem (Behrndt-E-Lotoreichik'13)

Suppose that Γ is piecewise C^1 smooth and obtained by a nontrivial local deformation of a straight line, then $\sigma_{\rm disc}(H_{\beta,\Gamma}) \neq \emptyset$ for any $\beta < 0$.

Proof idea: Choosing $\alpha = \frac{4}{\beta}$ we find by a comparison of the corresponding quadratic form and minimax principle that $\lambda_n(H^{\delta'}_{\beta,\Gamma}) \le \lambda_n(H^{\delta}_{\alpha,\Gamma})$ holds for any $n \in \mathbb{N}$; combining this inequality with the existence result obtained in [E-Ichinose'01] for the δ interaction we get the sought claim. □

The talk was based, in particular, on

[AEL94] J.E. Avron, P.E., Y. Last: Periodic Schrödinger operators with large gaps and Wannier–Stark ladders, *Phys. Rev. Lett.* **72** (1994), 896–899.

[E95] P.E.: The absence of the absolutely continuous spectrum for δ' Wannier–Stark ladders, *J. Math. Phys.* **36** (1995), 4561–4570.

[ADE98] J. Asch, P. Duclos, P.E.: Stability of driven systems with growing gaps. Quantum rings and Wannier ladders, *J. Stat. Phys.* **92** (1998), 1053–1069.

[ETV01] P.E., M. Tater, D. Vaněk: A single-mode quantum transport in serial-structure geometric scatterers, *J. Math. Phys.* **42** (2001), 4050–4078.

[BEG03] J. Brüning, P.E., V.A. Geyler: Large gaps in point-coupled periodic systems of manifolds, *J. Phys. A: Math. Gen.* **36** (2003), 4875–4890.

[EF08] P.E., M. Fraas: Interlaced dense point and absolutely continuous spectra for Hamiltonians with concentric-shell singular interactions, in *Proceedings of the QMath10 Conference* (Moeciu 2007; I. Beltita, G. Nenciu, R. Purice, eds.), World Scientific, Singapore 2008; pp. 48–65.

[EJ13] P.E., M. Jex: Spectral asymptotics of a strong δ' interaction on a planar loop, `arXiv:1304.7696`

[BEL13] J. Behrndt, P.E., V. Lotoreichik: Schrödinger operators with δ and δ'-interactions on Lipschitz surfaces and chromatic numbers of associated partitions, `arXiv:1307.????`

It remains to say

Happy birthday, Yosi!