Periodic quantum graphs and their local perturbations

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Talk overview

In this talk I am going to discuss several recent results on spectral properties of periodic quantum graphs:

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- *Gap structure,* in particular, the high-energy behaviour of the spectrum
- *Dispersion functions,* in particular, location of the band edges in the Brillouin zone
- *Local perturbations:* eigenvalues in gaps and resonances they produce, in a simple model framework

Introduction: the quantum graph concept

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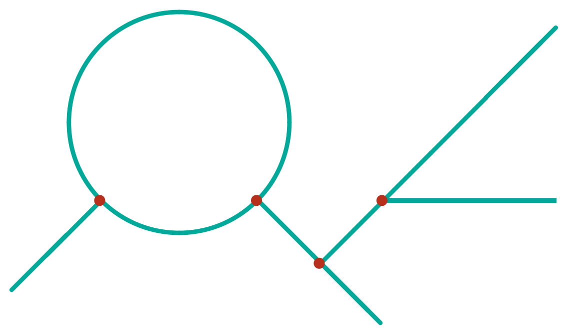
The concept extends, however, to graphs of *arbitrary shape*



Hamiltonian: $-\frac{\partial^2}{\partial x_j^2} + v(x_j)$
on graph edges,
boundary conditions at vertices

and what is important, it became *practically important* after experimentalists learned in the last two decades to fabricate tiny graph-like structure for which this is a good model

Remarks

- There are many graph-like systems based on *semiconductor* or *metallic* materials, *carbon nanotubes*, etc. The dynamics can be also simulated by *microwave network* built of optical cables – see [Hul et al.'04]

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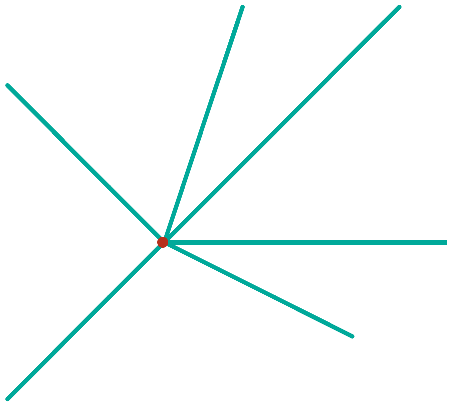
- Graphs can support also *Dirac operators*, see [Bulla-Trenckler'90], [Bolte-Harrison'03], and many recent applications to *graphene* and its derivates
- The graph literature is extensive; a good up-to-date reference are proceedings of the recent semester AGA Programme at INI Cambridge

Quantum graphs: vertex coupling



The most simple example is a *star graph* with the state Hilbert space $\mathcal{H} = \bigoplus_{j=1}^{n} L^2(\mathbb{R}_+)$ and the particle Hamiltonian acting on $\mathcal{H}$ as $\psi_j \mapsto -\psi_j''$

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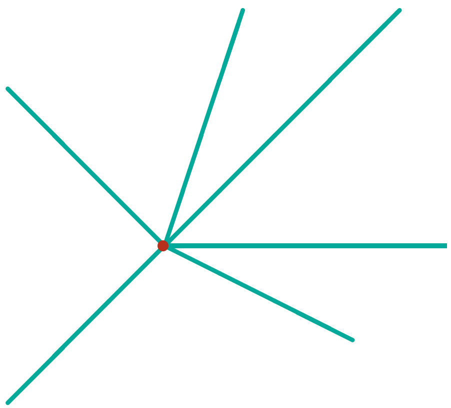

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Since it is second-order, the boundary condition involve $\Psi(0) := \{\psi_j(0)\}$ and $\Psi'(0) := \{\psi_j'(0)\}$ being of the form

$$A\Psi(0) + B\Psi'(0) = 0\,;$$

by [Kostrykin-Schrader'99] the $n \times n$ matrices A, B give rise to a self-adjoint operator if they satisfy the conditions

- $\operatorname{rank}(A, B) = n$
- AB^* is self-adjoint

Unique boundary conditions

The non-uniqueness of the above b.c. can be removed:

Proposition [Harmer'00, K-S'00]: Vertex couplings are uniquely characterized by unitary $n \times n$ matrices U such that

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One can derive them modifying the argument used in [Fülöp-Tsutsui'00] for generalized point interactions, $n = 2$

Self-adjointness requires vanishing of the boundary form,

$$\sum_{j=1}^{n} (\bar{\psi}_j \psi'_j - \bar{\psi}'_j \psi_j)(0) = 0\,,$$

which occurs *iff* the norms $\|\Psi(0) \pm i\ell\Psi'(0)\|_{\mathbb{C}^n}$ with a fixed $\ell \neq 0$ coincide, so the vectors must be related by an $n \times n$ unitary matrix; this gives $(U - I)\Psi(0) + i\ell(U + I)\Psi'(0) = 0$

Examples of vertex coupling

- Denote by $\mathcal{J}$ the $n \times n$ matrix whose all entries are equal to one; then $U = \frac{2}{n+i\alpha}\mathcal{J} - I$ corresponds to the standard δ *coupling*,

$$\psi_j(0) = \psi_k(0) =: \psi(0)\,, \; j,k = 1,\dots,n\,, \;\; \sum_{j=1}^{n} \psi_j'(0) = \alpha\psi(0)$$

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- Similarly, $U = I - \frac{2}{n-i\beta}\mathcal{J}$ describes the *δ_s' coupling*

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  with $\beta \in \mathbb{R}$; for $\beta = \infty$ we get *Neumann* decoupling, etc.

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Periodic quantum graphs

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- the spectrum has in general a *gap structure*
- it *need not be absolutely continuous* since the unique continuation principle may not hold, in particular, if graph edge lengths are rationally related
- local perturbations can produce *eigenvalues*, in the gaps or embedded, and *resonances*

Gap structure

The first question is about the *gap structure*:

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- A rectangular lattice with Kirchhoff coupling has *no gaps* above the spectral threshold
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Can a *more general vertex coupling* produce other types of gap behaviour at high energies?

Square-lattice graphs

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Theorem [E-Turek'10]: Let the coupling at each vertex be described by a fixed unitary matrix U. Then

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(a) The spectrum of H_U consists of a.c. bands and infinitely degenerate ev's. There are at most four bands in $\mathbb{R}_-$

(b) The high-energy asymptotics of bands and gaps w.r.t. the band index n includes the following classes:

- *flat bands*, i.e. infinitely degenerate point spectrum,
- *bands* behaving as $\mathcal{O}(n^j),\ j = 1, 0, -1, -2, -3$, $n \to \infty$,
- *gaps* behaving as $\mathcal{O}(n^j),\ j = 1, 0$, as $n \to \infty$.

Depending on U the high-energy asymptotics of the spectrum may be a combination of the above listed types.

An alternative coupling description

Theorem [Cheon-E.-Turek'10]: Consider a quantum graph vertex of degree n. If $m \leq n$, $S \in \mathbb{C}^{m,m}$ is a self-adjoint matrix and $T \in \mathbb{C}^{m,n-m}$, then the relation

$$\begin{pmatrix} I^{(m)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^* & I^{(n-m)} \end{pmatrix} \Psi$$

expresses self-adjoint boundary conditions of the KS-type. Conversely, for any self-adjoint vertex coupling there is an $m \leq n$ and a numbering of the edges such that the coupling is described by the KS boundary conditions with uniquely given matrices $T \in \mathbb{C}^{m,n-m}$ and self-adjoint $S \in \mathbb{C}^{m,m}$.

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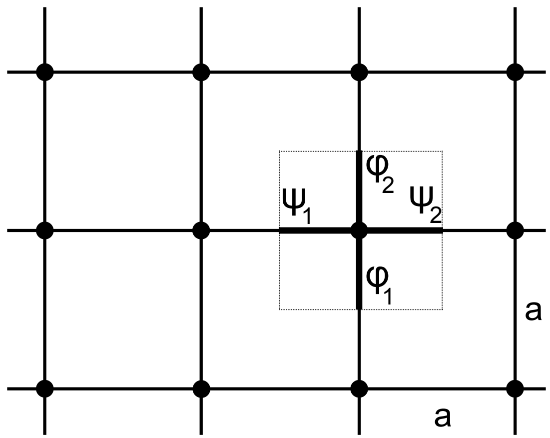
Remark: [Kuchment'04] writes b.c. in terms of eigenspaces of U. Here we single out the one corresponding to ev -1; there is also a symmetrical form referring to ev's ± 1

Proof outline

One has to perform Floquet analysis



for solutions with energy $E = k^2,\ k > 0$,

$$
\begin{aligned}
\psi_1(x) &= C_1^+ \mathrm{e}^{ikx} + C_1^- \mathrm{e}^{-ikx}, \quad x \in [-a/2, 0] \\
\psi_2(x) &= C_2^+ \mathrm{e}^{ikx} + C_2^- \mathrm{e}^{-ikx}, \quad x \in [0, a/2] \\
\varphi_1(x) &= D_1^+ \mathrm{e}^{ikx} + D_1^- \mathrm{e}^{-ikx}, \quad x \in [-a/2, 0] \\
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Proof outline, continued

We introduce the following matrices,

$$\left\{ \begin{array}{c} M \\ N \end{array} \right\} := \begin{pmatrix} \pm e^{-\frac{i}{2}(\theta_1 - ak)} & e^{-\frac{i}{2}(\theta_1 + ak)} & 0 & 0 \\ e^{\frac{i}{2}(\theta_1 - ak)} & \pm e^{\frac{i}{2}(\theta_1 + ak)} & 0 & 0 \\ 0 & 0 & \pm e^{-\frac{i}{2}(\theta_2 - ak)} & e^{-\frac{i}{2}(\theta_2 + ak)} \\ 0 & 0 & e^{\frac{i}{2}(\theta_2 - ak)} & \pm e^{\frac{i}{2}(\theta_2 + ak)} \end{pmatrix}$$

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then the spectral condition becomes $\det(AM + ikBN) = 0$ where the KS matrices are

$$-A = \begin{pmatrix} S & 0 \\ -T^* & I^{(4-m)} \end{pmatrix}, \quad B = \begin{pmatrix} I^{(m)} & T \\ 0 & 0 \end{pmatrix},$$

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The claim comes from a straightforward but rather tedious analysis of the particular cases $m = 0, 1, 2, 3, 4$ $\square$

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- Each case with a fixed m covers a situations with flat bands corresponding to lattice decoupling into separated edges, or pairs of edges
- The lattice can separate into ‘one-dimensional’ subsets describing generalized Kronig-Penney models on lines or zigzag curves, or to ‘combs’
- From the spectral point of view the case $m = 3$ is the richest, including situations with a power-like shrinking bands that occur for the graph decomposed into ‘combs’

More remarks

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Recall the example of [E-Gawlista'96] concerning a *rectangular lattice* graph with basic cell sides a and $a\theta^{-1}$ where $\theta = \frac{1}{2}(1+\sqrt{5})$ is the *golden mean*:

Let H_U have *δ-coupling* with parameter α, then

- $\sigma(H_U)$ has *no open gaps* (above threshold) if $|\alpha| \le \frac{\pi^2}{a\sqrt{5}}$
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Question: Are there ‘Bohr-Sommerfeld’ (periodic, connected) graphs with a *finite number of open gaps* above the spectral threshold?

Spectral edges positions

Next we address another ‘dimension-related’ question. It is known that looking for band edges of one-dimensional periodic Schrödinger operator it is enough to check *endpoints of the Brillouin zone*

The same is often done in higher dimensions even if numerical counterexamples hint that caution is needed

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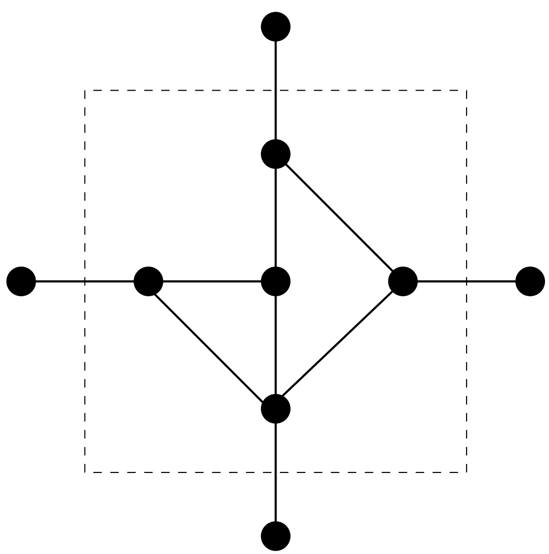
[Harrison-Kuchment-Sobolev-Winn’07] provided example of a periodic graph with the following basic cell



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They demonstrated that in this and some other examples spectral edges correspond to quasimomentum values *inside the Brillouin zone*

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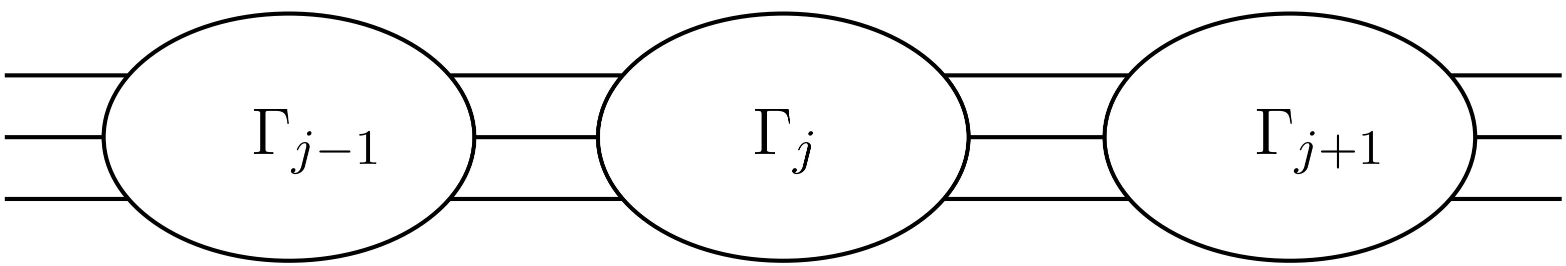
In order to see for which periodic systems such a claim can be made, let us look at periodic *chain graphs*



Formulation of the problem

Graph G consists of a chain of identical copies Γ_j of some graph Γ, consecutive copies being connected by m edges. The internal structure of Γ is not important

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The standard Floquet theory applies. For quasimomentum $k \in [-\pi, \pi]$ we consider the fiber operator $H(k)$ defined by means of the condition $f(\tau_n x) = \mathrm{e}^{ikn} f(x)$ for $(n, x) \in \mathbb{Z} \times \Gamma$. Note that values $k = 0, \pm\pi$ refer to periodic and antiperiodic solutions, respectively

Spectrum of $H(k)$ is discrete consisting of eigenvalues $\lambda_j(k)$, $j = 1, 2, \ldots$. We look for values of k where extrema of the *band functions* $\lambda_j(\cdot)$ are attained

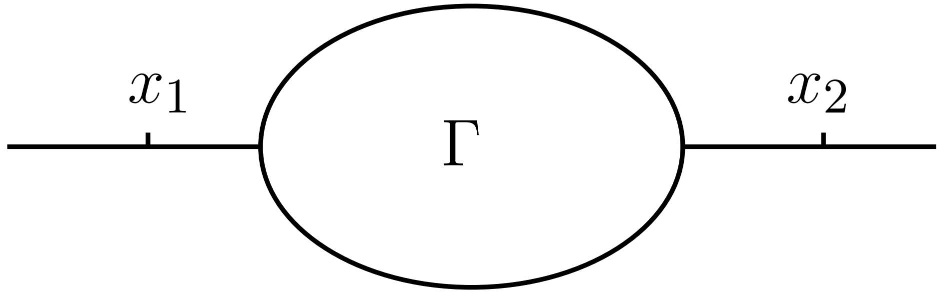
The result

Theorem [E-Kuchment-Winn'10]: Let G be a $\mathbb{Z}$-periodic "chain" graph G with m connecting edges and H be the corresponding Hamiltonian operator acting on $L^2(G)$. Then

(a) If $m = 1$, the endpoints of the bands $I_j = \lambda_j([-\pi, \pi])$, i.e. the extrema of the band functions, are *attained at the points* $k = 0, k = \pm\pi$ (although, they might be attained at some other points as well). In other words, the spectra of the periodic and anti-periodic problems provide the ends of the bands of the spectrum

(b) If $m > 1$, this is *not always true*

The case $m = 1$



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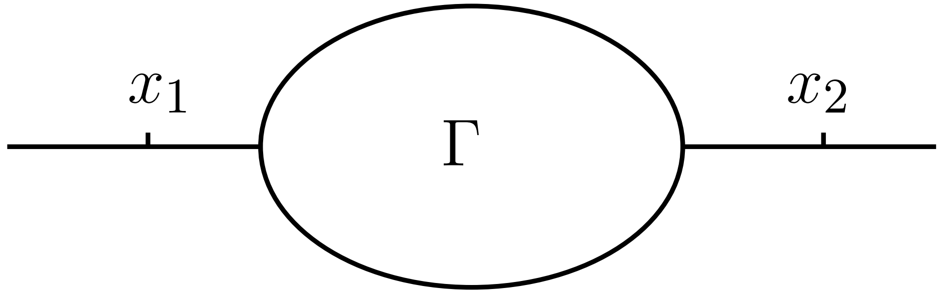

Lemma: Assume that $m = 1$.

(i) If some value λ is attained by the band functions $\lambda_j(k)$ at more than two points k in the segment $[-\pi, \pi]$, then there is a constant branch $\lambda(k) \equiv \lambda$ for all k, and thus this value is attained at all points of the segment

(ii) The set D of all such values λ is discrete (possibly empty)

(iii) If $\lambda \notin D$, then in a neighborhood of this value all band functions are strictly monotonous on $[0, \pi]$

The case $m = 1$, continued

Proof of the Lemma, outline: Suppose that a value λ is taken by $\lambda_j(k)$ at more than two points $k \in [-\pi, \pi]$. Then the solution space of $(H - \lambda)u = 0$ on G is more than two-dimensional, so there is a non-trivial solution u vanishing with its first derivative at a point x_1 on the connecting edge, and consequently, u vanishes on the whole edge containing x_1.

Using general results from Floquet theory we infer from here that $\sigma(H)$ contains the flat branch $\lambda_j(k) \equiv \lambda$ and that this can occur only at a discrete set D of values of λ

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Consider now $\lambda \notin D$. Due to the invariance of H w.r.t. complex conjugation, the band functions $\lambda_j(\cdot)$ are even, $\lambda_j(-k) = \lambda_j(k)$. This, together with $\lambda \notin D$ implies that all values near λ are attained by the (continuous) function $\lambda_j(\cdot)$ only once on $[0, \pi]$, so the function is monotonous there

Proof of the theorem

Suppose that λ is an extremum of a band function $\lambda_j(\cdot)$. If $\lambda \in D$, then the statement (i) of the Lemma claims that this value λ is attained at all values of k, in particular for $k = \pi, 0$ that correspond to the (anti)periodic problems

Let next $\lambda \notin D$. Then the statement (iii) of the Lemma implies that the corresponding value of k cannot be in the interior $(0, \pi)$ of the segment $[0, \pi]$. Thus, either $k = 0$ and λ belongs to the spectrum of the periodic problem, or $k = \pi$ and λ belongs to the spectrum of the anti-periodic problem

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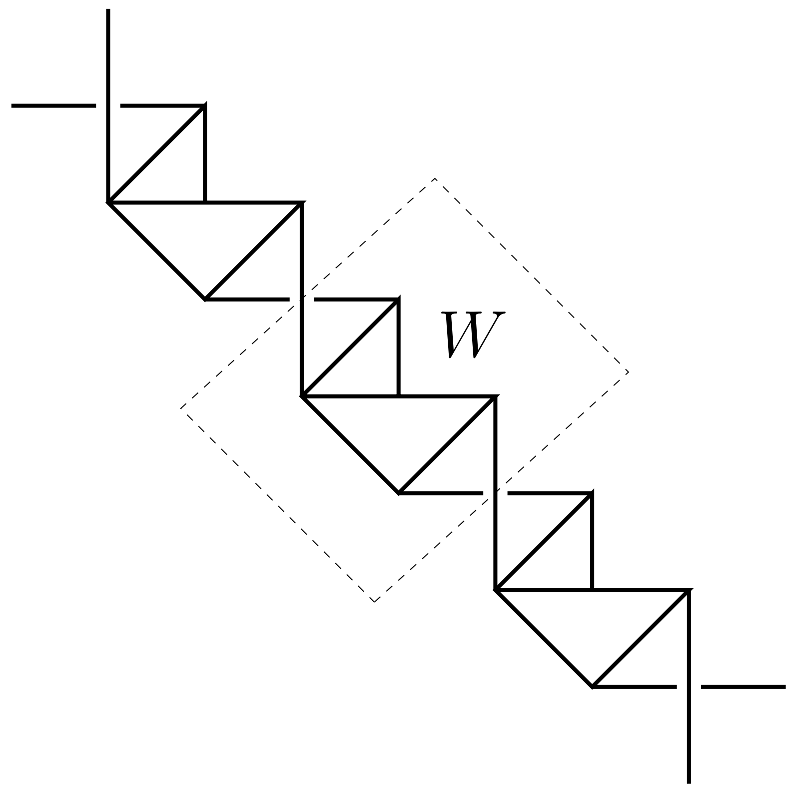
To prove the other part we construct an *example* of a chain with $m = 2$ by a suitable *folding of the* $\mathbb{Z}^2$ *periodic graph* from the paper [Harrison-Kuchment-Sobolev-Winn'07] we have mentioned above

Proof of the theorem, continued

By folding we get the following $\mathbb{Z}$ periodic graph with $m = 2$



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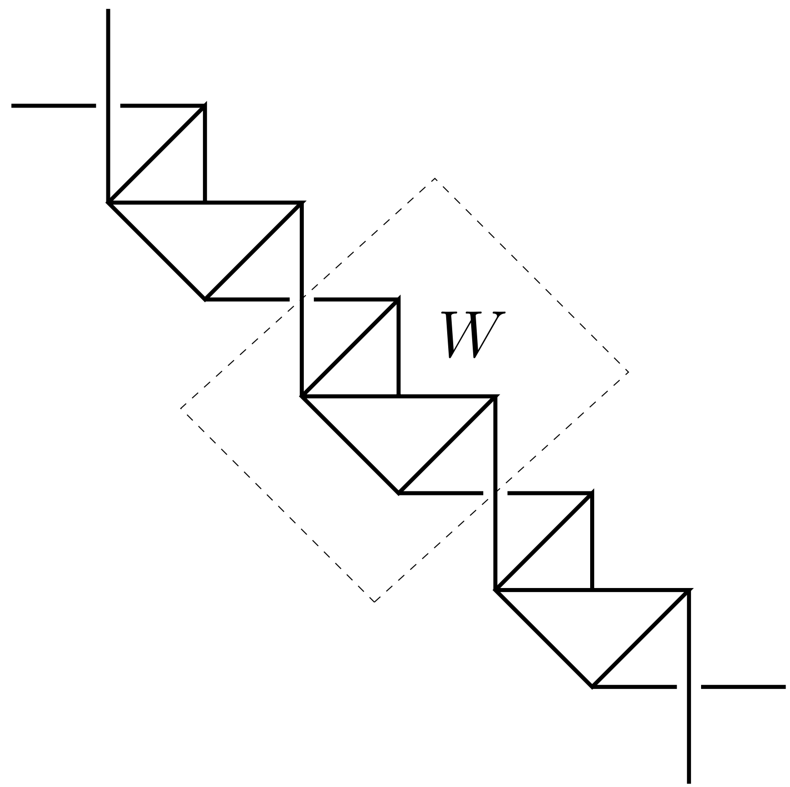

The second element of the proof is the *duality* between quantum (metric) graphs and combinatorial graphs [E'97]

Proof of the theorem, continued

The easiest thing is to assume that the above graph is *equilateral* then folding its spectrum (by $k \mapsto \cos k$) we get the spectrum of the discrete Laplace-Beltrami operator Δ on functions defined on the vertices of G, given by

$$(\Delta f)(v) := \frac{1}{\sqrt{d_v}} \sum_{u \sim v} \frac{1}{\sqrt{d_u}} f(u)\,,$$

where u and v are vertices, d_u is the degree of u, and the sum is taken over all vertices u adjacent to v

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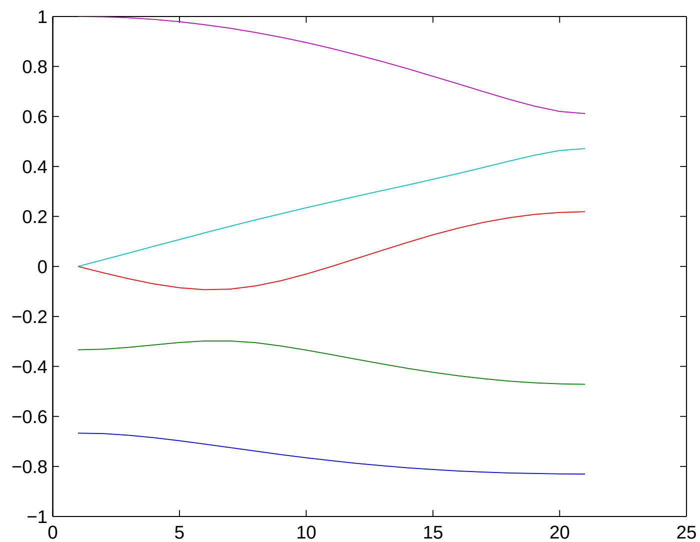
We arrive thus at the spectral problem for the matrix

$$\Delta(k) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 3^{-1/2} & \mathrm{e}^{ik}/2 & 0 & 3^{-1/2} \\ 3^{-1/2} & 0 & 1/2 & \mathrm{e}^{ik}/3^{1/2} & 0 \\ \mathrm{e}^{-ik}/2 & 1/2 & 0 & 1/2 & 1/2 \\ 0 & \mathrm{e}^{-ik}/3^{1/2} & 1/2 & 0 & 3^{-1/2} \\ 3^{-1/2} & 0 & 1/2 & 3^{-1/2} & 0 \end{pmatrix}$$

Proof of the theorem, completed

It is straightforward to see that two spectral branches out of five have extremum *inside the Brillouin zone*



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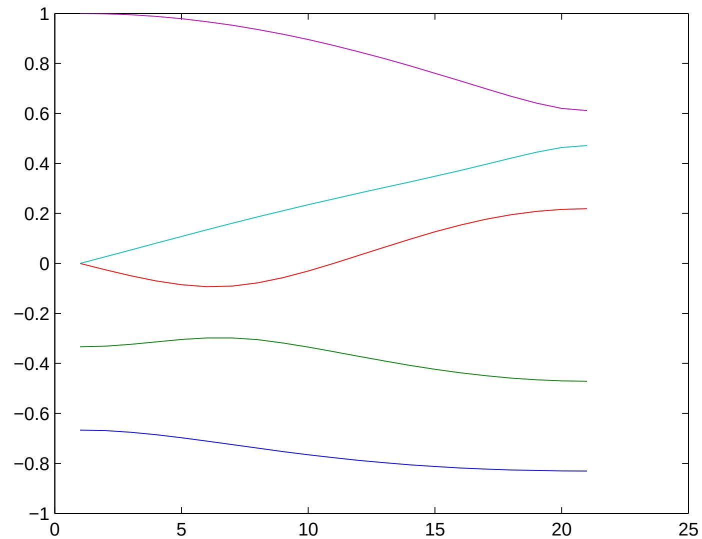

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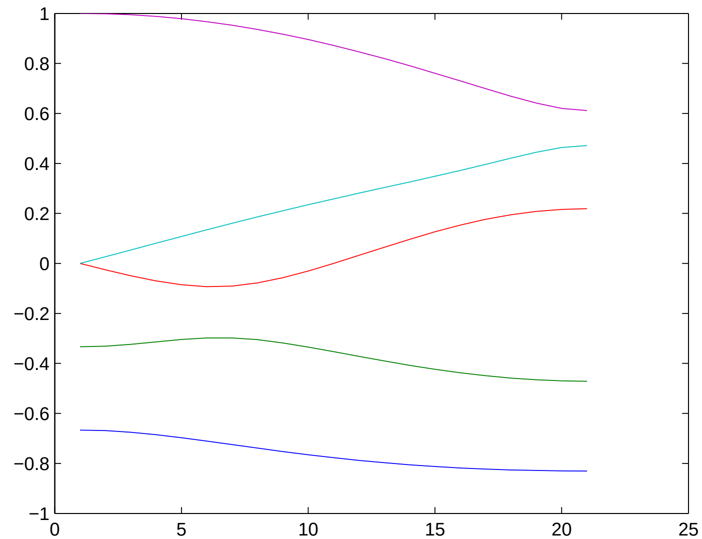

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Remark: Conclusions can be extended to more general Schrödinger operators on $\mathbb{Z}$-periodic graphs

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Instead of attempting general claims we will try to analyze the effect in a simple *model setting* in which the effect mentioned in the introduction will be seen

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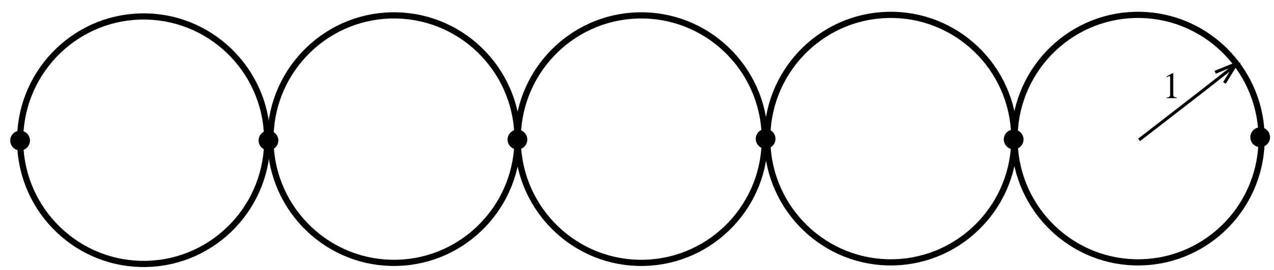
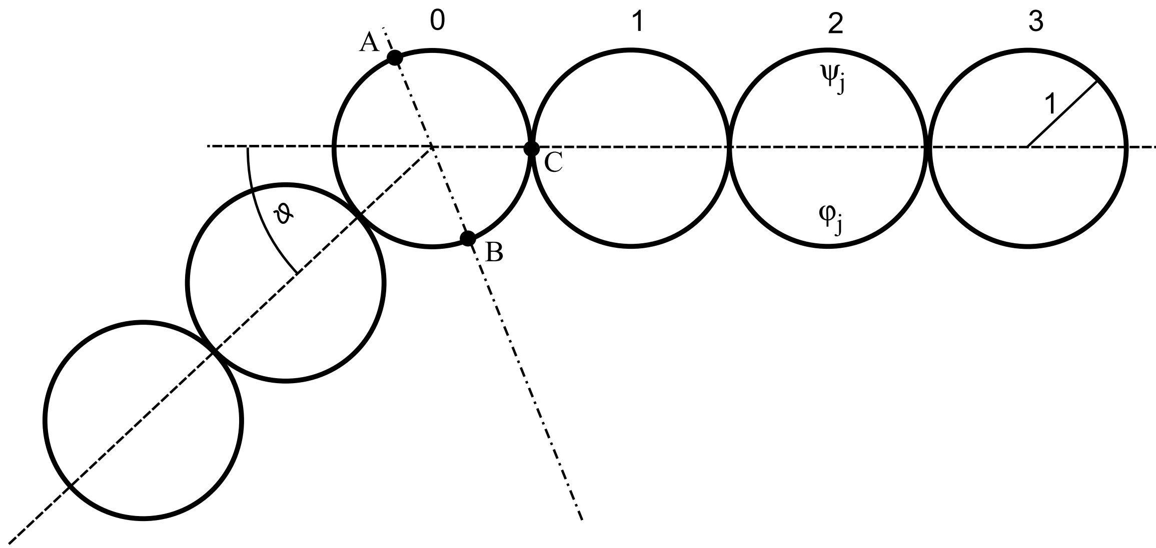
The model: we will analyze the *influence of a "bending" deformation* on a a "chain graph" which exhibits a one-dimensional periodicity



Without loss of generality we assume unit radii; the rings are connected by the δ*-coupling* of a strength $\alpha \neq 0$

Bending the chain

We will suppose that the chain is deformed as follows



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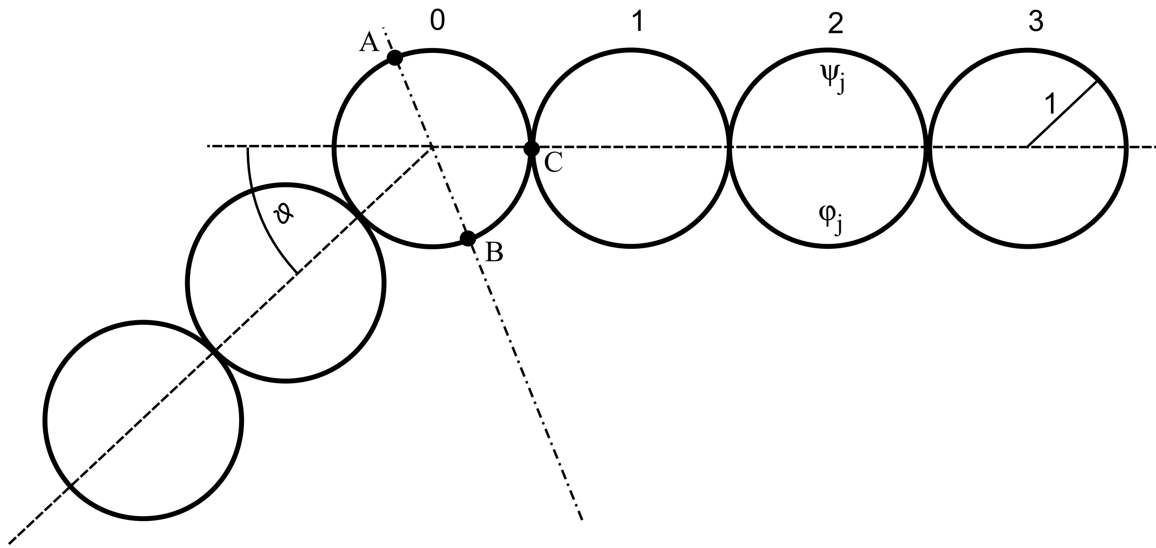
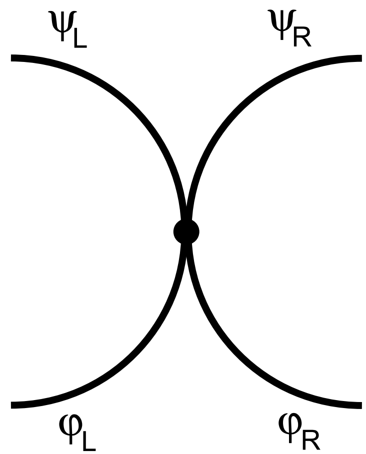


Our aim is to show that

- the band spectrum of the straight Γ is preserved
- there are *bend-induced eigenvalues*, we analyze their behavior with respect to model parameters
- the bent chain exhibits also *resonances*

An infinite periodic chain

The "straight" chain Γ_0 can be treated as a periodic system analyzing the spectrum of the elementary cell



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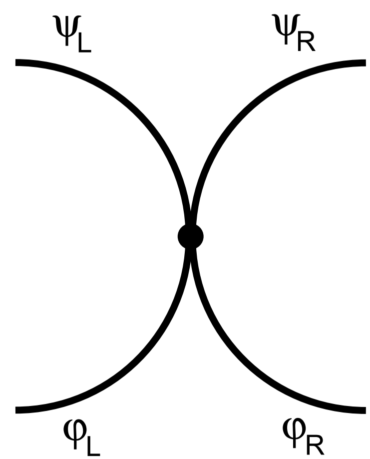

with Floquet-Bloch boundary conditions with the phase $e^{2i\theta}$

This yields the condition

$$e^{2i\theta} - e^{i\theta}\left(2\cos k\pi + \frac{\alpha}{2k}\sin k\pi\right) + 1 = 0$$

Straight chain spectrum

A straightforward analysis leads to the following conclusion:

Proposition: $\sigma(H_0)$ consists of *infinitely degenerate eigenvalues* equal to n^2 with $n \in \mathbb{N}$, and *absolutely continuous spectral bands* such that

If $\alpha > 0$, then every spectral band is contained in $(n^2, (n+1)^2]$ with $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and its upper edge coincides with the value $(n+1)^2$.

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If $\alpha < 0$, then in each interval $[n^2, (n+1)^2)$ with $n \in \mathbb{N}$ there is exactly one band with the lower edge n^2. In addition, there is a band with the lower edge (the overall threshold) $-\kappa^2$, where κ is the largest solution of

$$\left| \cosh \kappa\pi + \frac{\alpha}{4} \cdot \frac{\sinh \kappa\pi}{\kappa} \right| = 1$$

Straight chain spectrum

Proposition, cont'd: The upper edge of this band depends on α. If $-8/\pi < \alpha < 0$, it is k^2 where k solves

$$\cos k\pi + \frac{\alpha}{4} \cdot \frac{\sin k\pi}{k} = -1$$

in $(0,1)$. On the other hand, for $\alpha < -8/\pi$ the upper edge is negative, $-\kappa^2$ with κ being the smallest solution of the condition, and for $\alpha = -8/\pi$ it equals zero.

Finally, $\sigma(H_0) = [0, +\infty)$ holds if $\alpha = 0$.

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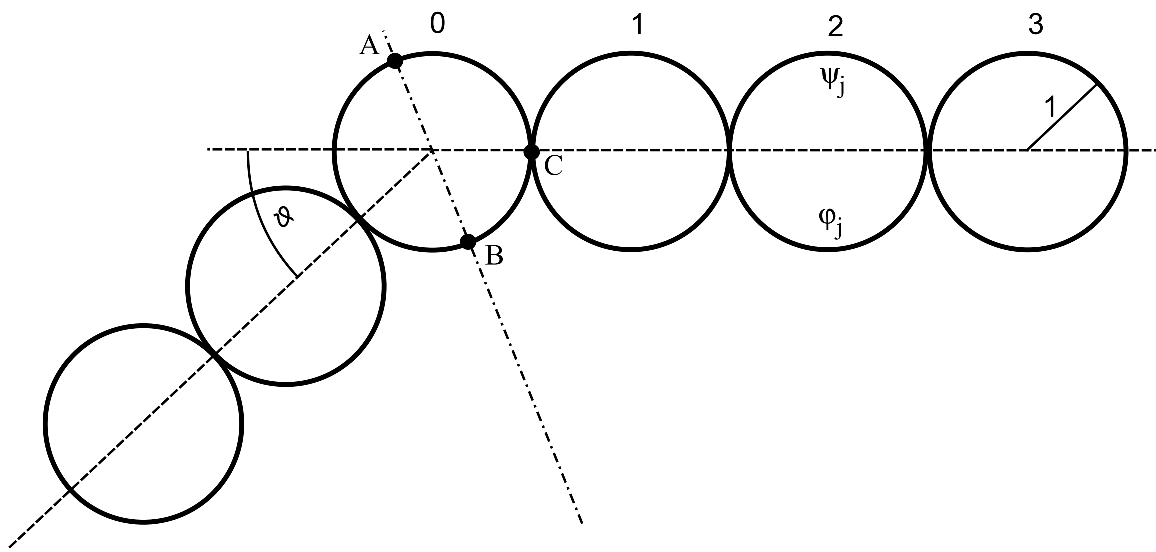
Let us add a couple of *remarks*:

- The bands correspond to *Kronig-Penney model* with the coupling $\frac{1}{2}\alpha$ instead of α, in addition one has here the *infinitely degenerate point spectrum*
- It is also an example of *gaps coming from decoration*

The bent chain spectrum

Now we pass to the bent chain denoted as Γ_ϑ:



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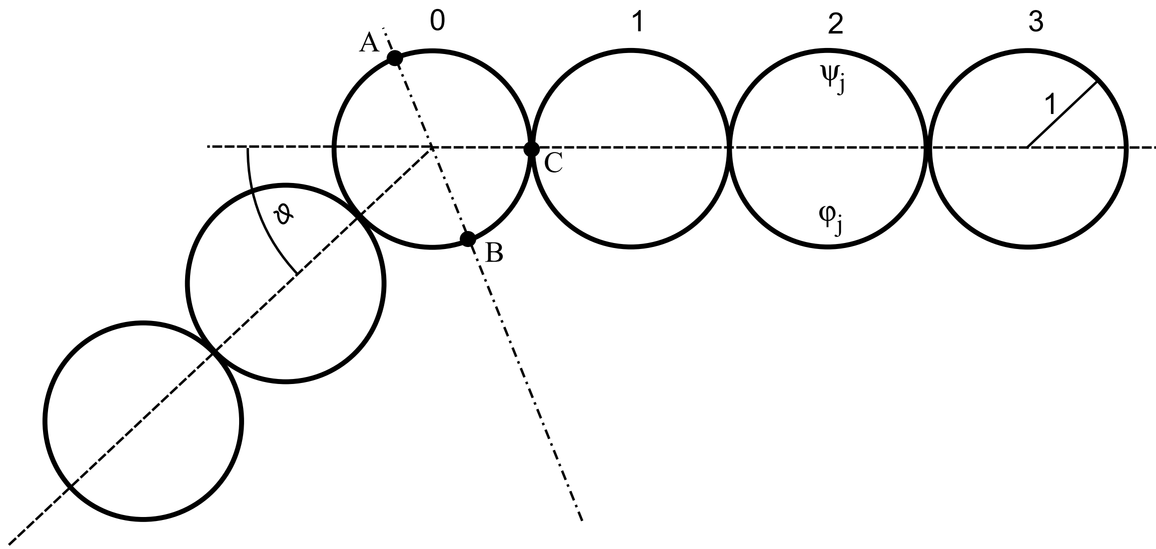

Since Γ_ϑ has mirror symmetry, the operator H_ϑ can be reduced by parity subspaces into a direct sum of an even part, H^+, and odd one, H^-; we drop mostly the subscript ϑ

Equivalently, we analyze the half-chain with *Neumann* and *Dirichlet* conditions at the points A, B, respectively

Eigenfunction components

At the energy k^2 they are are linear combinations of $\mathrm{e}^{\pm \mathrm{i}kx}$,

$$\begin{aligned} \psi_j(x) &= C_j^+ \mathrm{e}^{\mathrm{i}kx} + C_j^- \mathrm{e}^{-\mathrm{i}kx}, \quad x \in [0,\pi]\,, \\ \varphi_j(x) &= D_j^+ \mathrm{e}^{\mathrm{i}kx} + D_j^- \mathrm{e}^{-\mathrm{i}kx}, \quad x \in [0,\pi] \end{aligned}$$

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There are δ-couplings in the points of contact, i.e.

$$\psi_j(0) = \varphi_j(0)\,, \quad \psi_j(\pi) = \varphi_j(\pi)\,, \quad \text{and}$$

$$\psi_j(0) = \psi_{j-1}(\pi)\,; \quad \psi_j'(0) + \varphi_j'(0) - \psi_{j-1}'(\pi) - \varphi_{j-1}'(\pi) = \alpha \cdot \psi_j(0)$$

Transfer matrix

Using the above relations we get for all $j \geq 2$

$$\begin{pmatrix} C_j^+ \\ C_j^- \end{pmatrix} = \underbrace{\begin{pmatrix} \left(1 + \frac{\alpha}{4\mathrm{i}k}\right) \mathrm{e}^{\mathrm{i}k\pi} & \frac{\alpha}{4\mathrm{i}k}\mathrm{e}^{-\mathrm{i}k\pi} \\ -\frac{\alpha}{4\mathrm{i}k}\mathrm{e}^{\mathrm{i}k\pi} & \left(1 - \frac{\alpha}{4\mathrm{i}k}\right) \mathrm{e}^{-\mathrm{i}k\pi} \end{pmatrix}}_{M} \cdot \begin{pmatrix} C_{j-1}^+ \\ C_{j-1}^- \end{pmatrix},$$

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To have eigenvalues, one eigenvalue of M has to be *less than one* (they satisfy $\lambda_1\lambda_2 = 1$); this happens *iff*

$$\left|\cos k\pi + \frac{\alpha}{4k}\sin k\pi\right| > 1\,;$$

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Remark: By general arguments, σ_{ess} is preserved, and there are at most two eigenvalues in each gap

Spectrum of H^+

Combining the above with the Neumann condition at the mirror axis we get the spectral condition in this case,

$$\cos k\vartheta = -\cos k\pi + \frac{\sin^2 k\pi}{\frac{\alpha}{4k}\sin k\pi \pm \sqrt{\left(\cos k\pi + \frac{\alpha}{4k}\sin k\pi\right)^2 - 1}}$$

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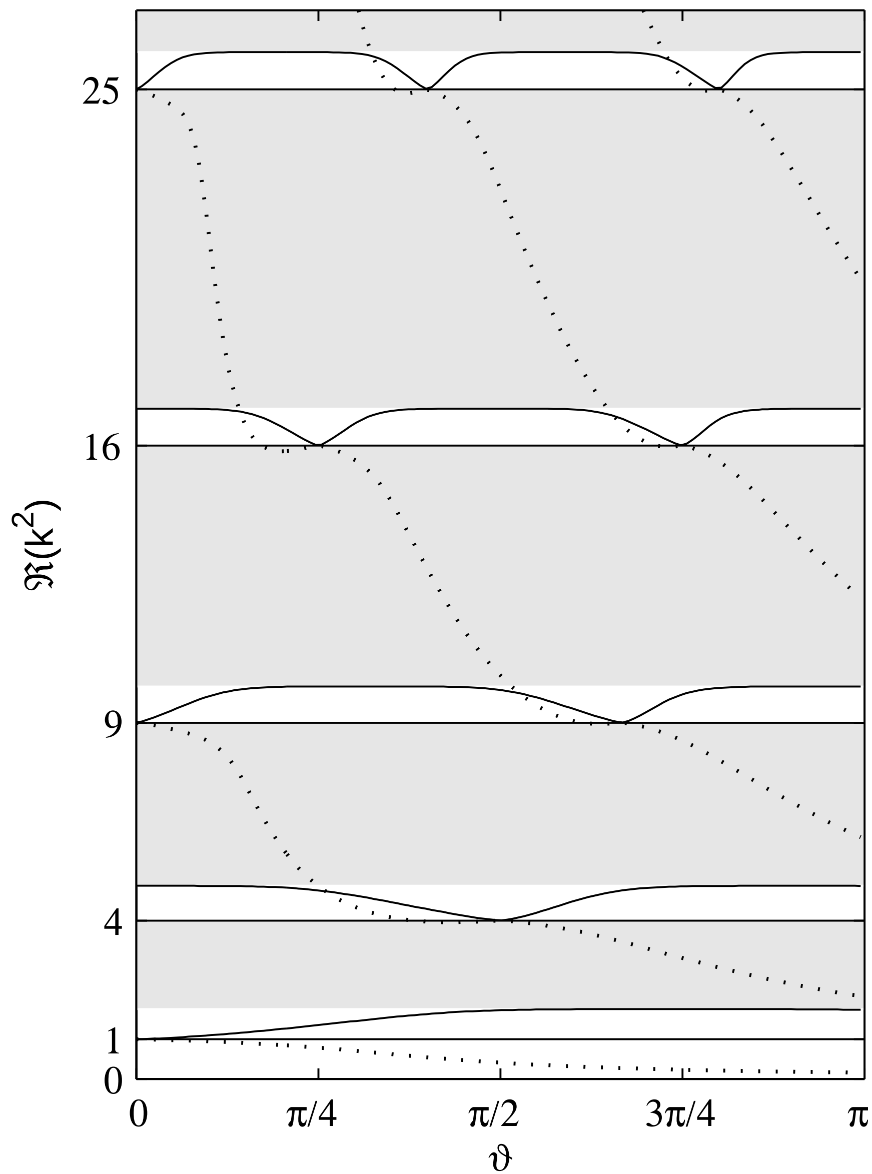
and an analogous expression for negative energies

After a tiresome but straightforward analysis one arrives then at the following conclusion:

Proposition: If $\alpha \geq 0$, then H^+ has no negative eigenvalues. On the other hand, for $\alpha < 0$ the operator H^+ has at least one negative eigenvalue which lies under the lowest spectral band and above the number $-\kappa_0^2$, where κ_0 is the (unique) solution of $\kappa \cdot \tanh \kappa\pi = -\alpha/2$

Spectrum of H^+ for $\alpha = 3$



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Replacing Neumann condition by Dirichlet at the mirror axis we get the spectral condition in this case,

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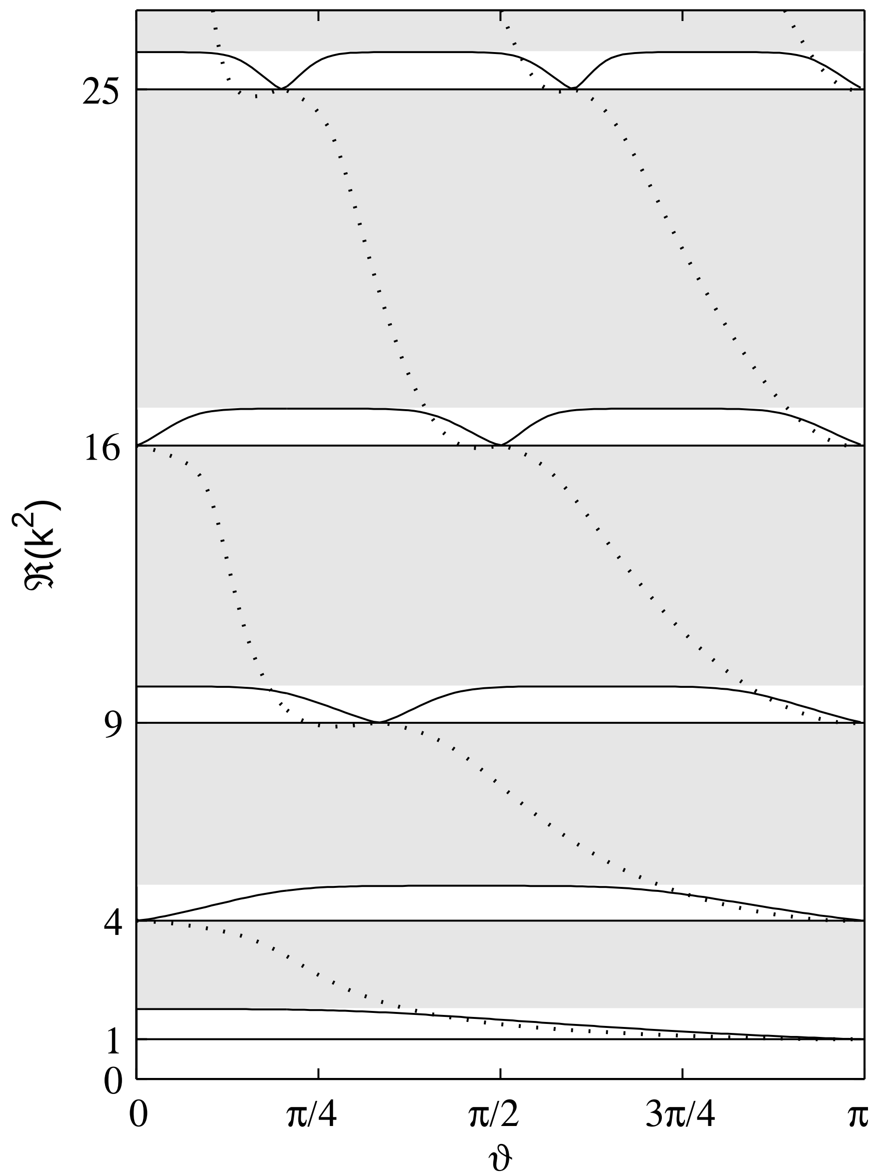
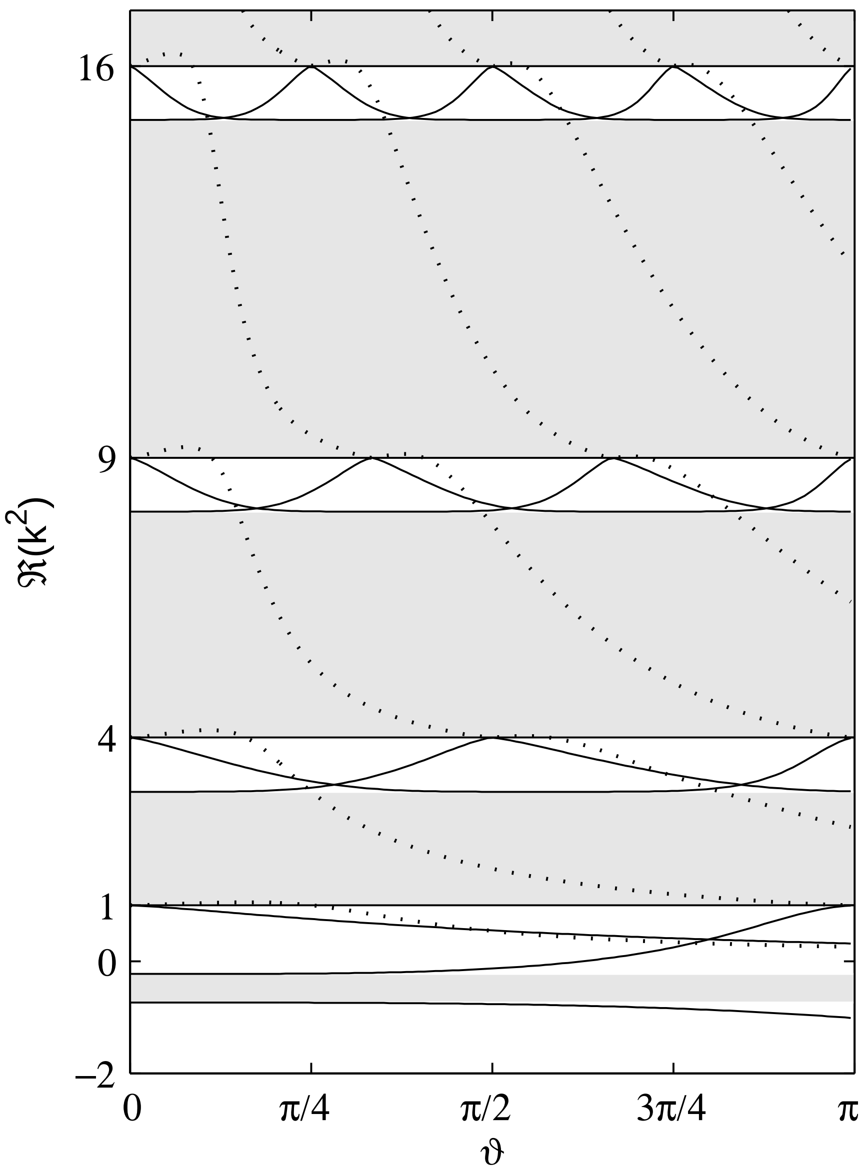
Summarizing, for each of the operators $H^\pm$ there is at least one eigenvalue in every spectral gap closure. It can lapse into a band edge n^2, $n \in \mathbb{N}$, and thus be in fact absent. The ev's of H^+ and H^- may coincide, becoming a single ev of multiplicity two; this happens only if

$$k \cdot \tan k\pi = \frac{\alpha}{2}$$

Spectrum of H^- for $\alpha = 3$



$\sigma(H)$ for attractive coupling, $\alpha = -3$



Resonances, analyticity

The above eigenvalue curves are not the only solutions of the spectral condition. There are also *complex solutions* representing *resonances* of the bent-chain system

In the above pictures their real parts are drawn as functions of ϑ by dashed lines.

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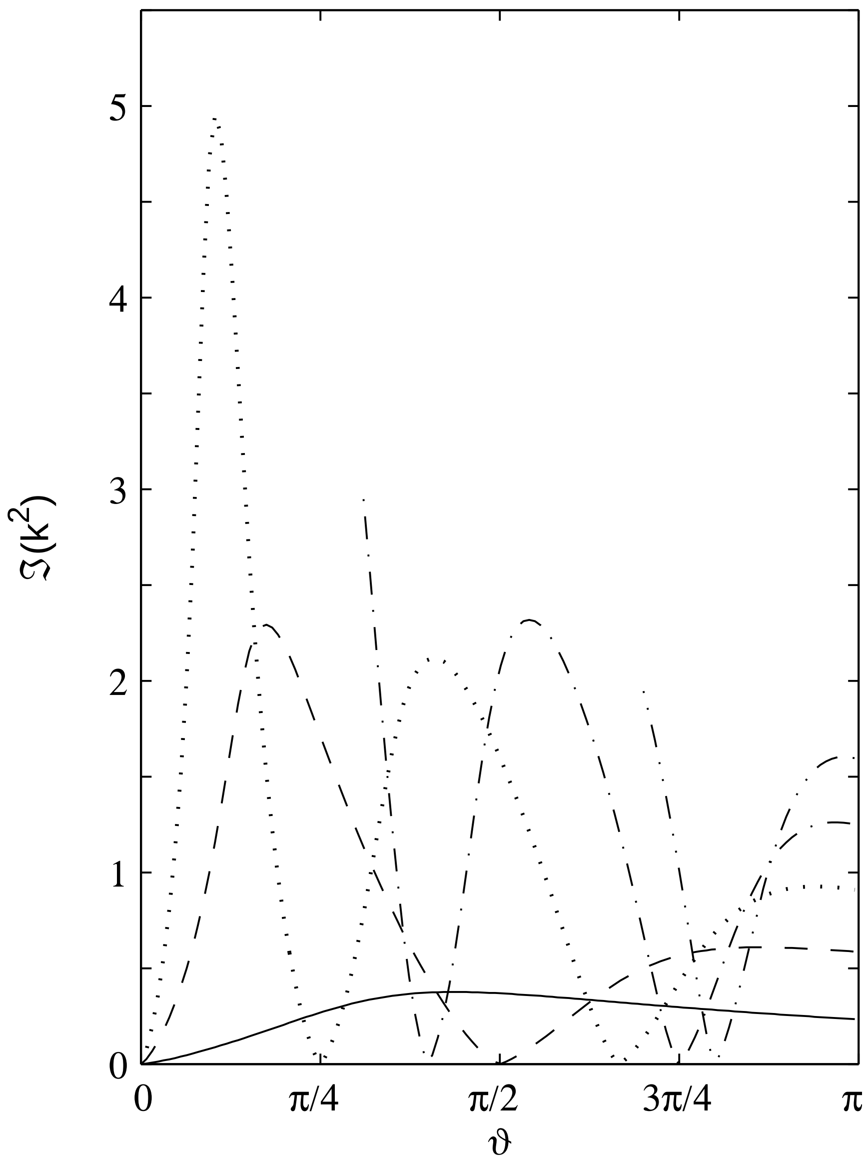
A further analysis of the spectral condition gives

Proposition: The eigenvalue and resonance curves for H^+ are *analytic* everywhere except at $(\vartheta, k) = (\frac{n+1-2\ell}{n}\pi, n)$, where $n \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, $\ell \leq \left[\frac{n+1}{2}\right]$. Moreover, the real solution in the n-th spectral gap is given by a function $\vartheta \mapsto k$ which is *real-analytic*, except at the points $\frac{n+1-2\ell}{n}\pi$. Similar claims can be made for the odd part for H^-.

Imaginary parts of H^+ resonances, $\alpha = 3$



More on the angle dependence

For simplicity we take H^+ only, the results for H^- are analogous. Ask about the behavior of the curves at the points whe they touch bands and where eigenvalues and resonances may cross

If $\vartheta_0 := \frac{n+1-2\ell}{n}\pi > 0$ is such a point we find easily that in is vicinity we have

$$k \approx k_0 + \sqrt[3]{\frac{\alpha}{4}\frac{k_0}{\pi}}\,|\vartheta - \vartheta_0|^{4/3}$$

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However, H^+ has an eigenvalue near $\vartheta_0 = 0$ also in the gaps adjacent to even numbers, when the curve starts at $(0, k_0)$ for k_0 solving $|\cos k\pi + \frac{\alpha}{4k}\sin k\pi| = 1$ in $(n, n+1)$, n even

Even threshold behavior

Proposition: Suppose that $n \in \mathbb{N}$ is even and k_0 is as described above, i.e. k_0^2 is the right endpoint of the spectral gap adjacent to n^2. Then the behavior of the solution in the vicinity of $(0, k_0)$ is given by

$$k = k_0 - C_{k_0,\alpha} \cdot \vartheta^4 + \mathcal{O}(\vartheta^5)\,,$$

where $C_{k_0,\alpha} := \frac{k_0^2}{8\pi} \cdot \left(\frac{\alpha}{4}\right)^3 (k_0\pi + \sin k_0\pi)^{-1}$

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Remark: Notice that the fourth-power is the same as for the ground state of a *slightly bent Dirichlet tube* despite the fact that the dynamics is completely different in the two cases

Even threshold behavior

Proposition: Suppose that $n \in \mathbb{N}$ is even and k_0 is as described above, i.e. k_0^2 is the right endpoint of the spectral gap adjacent to n^2. Then the behavior of the solution in the vicinity of $(0, k_0)$ is given by

$$k = k_0 - C_{k_0,\alpha} \cdot \vartheta^4 + \mathcal{O}(\vartheta^5)\,,$$

where $C_{k_0,\alpha} := \frac{k_0^2}{8\pi} \cdot \left(\frac{\alpha}{4}\right)^3 (k_0\pi + \sin k_0\pi)^{-1}$

Remark: Notice that the fourth-power is the same as for the ground state of a *slightly bent Dirichlet tube* despite the fact that the dynamics is completely different in the two cases

Remark: Note that the analogous problem for *bent leaky wires* studied in [E-Ichinose'01] remains open.

The results discussed here come from

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and also

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It remains to say

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