Schrödinger operators with narrow channels: unusual semiclassical behavior and spectral estimates

Pavel Exner

Doppler Institute
for Mathematical Physics and Applied Mathematics
Prague

in collaboration with Diana Barseghyan

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Talk outline

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Motivation

Among many ideas we owe to Hermann Weyl semiclassical method is of the most successful as hundred years of its use demonstrates.

Nevertheless, it is not universal: there are systems with discrete spectrum for which the classically allowed phase-space volume is infinite. A classical example due to [Simon'83] is a 2D Schrödinger operator with the potential

$$V(x, y) = x^2 y^2$$

or more generally, $V(x, y) = |xy|^p$ with $p \geq 1$.

Similar behavior one can observe for Dirichlet Laplacians in regions with hyperbolic cusps – see [Geisinger-Weidl'11] for recent results and a survey. Moreover, using the dimensional-reduction technique of Laptev and Weidl one can prove spectral estimates for such operators.
A common feature of these models is that the particle motion is confined into channels narrowing towards infinity.

Potentials unbounded from below – a model

Our first aim in this talk is to show that similar behavior may occur even for Schrödinger operators with potential unbounded from below in which a classical particle can escape to infinity with an increasing velocity.

We are going to analyze the following class of operators:

$$L_p(\lambda) \,:\, L_p(\lambda)\psi = -\Delta\psi + \Big(|xy|^p - \lambda(x^2+y^2)^{p/(p+2)}\Big)\,\psi\,, \quad p \geq 1$$

on $L^2(\mathbb{R}^2)$, where (x, y) are the standard Cartesian coordinates in $\mathbb{R}^2$ and the parameter λ in the second term of the potential is non-negative; unless the value of λ is important we write it simply as L_p.

Note that $\frac{2p}{p+2} < 2$ so the operator is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ by Faris-Lavine theorem; the symbol L_p or $L_p(\lambda)$ will always mean its closure.

The subcritical case

Our first aim is to demonstrate that spectral properties of $L_p(\lambda)$ depend crucially on the value of the parameter λ and there a transition between different regimes as λ changes.

Let us start with the subcritical case which occurs for small values of λ. To characterize the smallness quantitatively we need an auxiliary operator which will be an (an)harmonic oscillator Hamiltonian on line,

$$\tilde{H}_p \,:\, \tilde{H}_p u = -u'' + |t|^p u$$

on $L^2(\mathbb{R})$ with the standard domain. Let γ_p be the minimal eigenvalue of this operator; in view of the potential symmetry we have $\gamma_p = \inf \sigma(H_p)$, where

$$H_p \,:\, H_p u = -u'' + t^p u$$

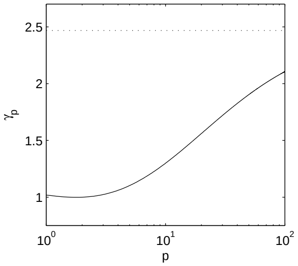
on $L^2(\mathbb{R}_+)$ with Neumann condition at $t = 0$.

The subcritical case – continued

The eigenvalue $\gamma_p = \inf \sigma(H_p)$ equals one for $p = 2$; for $p \to \infty$ where the potential becomes an infinitely deep rectangular it becomes $\gamma_\infty = \frac{1}{4}\pi^2$; it smoothly interpolates between the two values.

Since $x^p \geq 1 - \chi_{[0,1]}(x)$ it follows from the minimax principle that $\gamma_p \geq \epsilon_0 \approx 0.546$, where ϵ_0 is the ground-state energy is the corresponding rectangular potential well of depth one.

In fact, a numerical solution gives true minimum $\gamma_p \approx 0.998995$ attained at $p \approx 1.788$; in the semilogarithmic scale the plot is as follows:



The subcritical case – continued

The spectrum is naturally bounded from below and discrete if $\lambda = 0$; our aim is to show that this remains to be the case provided λ is small enough.

Theorem (E-Barseghyan'12)

For any $\lambda \in [0, \lambda_{\mathrm{crit}}]$, where $\lambda_{\mathrm{crit}} := \gamma_p$, the operator $L_p(\lambda)$ is bounded from below for $p \geq 1$; if $\lambda < \gamma_p$ its spectrum is purely discrete.

Idea of the proof: Let $\lambda < \gamma_p$. By minimax we need to estimate L_p from below by a s-a operator with a purely discrete spectrum. To construct it we employ bracketing imposing additional Neumann conditions at concentric circles of radii $n = 1, 2, \dots$.

In the estimating operators the variables decouple asymptotically and the spectral behavior is determined by the angular part of the operators.

Subcritical behavior – the proof

Specifically, in polar coordinates we get direct sum of operators acting as

$$L^{(1)}_{n,p}\psi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{1}{n^2}\frac{\partial^2\psi}{\partial\varphi^2} + \left(\frac{r^{2p}}{2^p}|\sin 2\varphi|^p - \lambda r^{2p/(p+2)}\right)\psi$$

on the annuli $G_n := \{(r,\varphi)\,:\, n-1 \le r < n,\ 0 \le \varphi < 2\pi\}$, $n = 1,2,\dots$ with Neumann conditions imposed on ∂G_n.

Obviously $\sigma(L^{(1)}_{n,p})$ is purely discrete for each $n = 1,2,\dots$, hence it is sufficient to check that $\inf\sigma(L^{(1)}_{n,p}) \to \infty$ holds as $n \to \infty$.

We estimate $L^{(1)}_{n,p}$ from below by an operator with separating variables, note that the radial part does not contribute and use the symmetry of the problem; for $\varepsilon \in (0,1)$ the question is then to analyze

$$L^{(2)}_{n,p}\,:\ L^{(2)}_{n,p}u = -u'' + \left(\frac{n^{2p+2}}{2^p}\sin^p 2x - \frac{\lambda}{1-\varepsilon}\,n^{(4p+4)/(p+2)}\right)u$$

on $L^2(0,\pi/4)$ with Neumann conditions, $u'(0) = u'(\pi/4) = 0$.

Subcritical behavior – proof continued

We have $n^2 \inf\sigma(L^{(1)}_{n,p}) \geq \inf\sigma(L^{(2)}_{n-1,p})$ if n is large enough, specifically for $n > \left(1-(1-\varepsilon)^{(p+2)/(4p+4)}\right)^{-1}$, hence it is sufficient to investigate the spectral threshold $\mu_{n,p}$ of $L^{(2)}_{n,p}$ as $n\to\infty$.

The trigonometric potential can be estimated by a powerlike one with the similar behavior around the minimum introducing, e.g.

$$L^{(3)}_{n,p} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + n^{2p+2}x^p \left(\chi_{(0,\delta(\varepsilon)]}(x) + \left(\frac{2}{\pi}\right)^p \chi_{[\delta(\varepsilon),\pi/4)}(x)\right) - \lambda'_\varepsilon\, n^{(4p+4)/(p+2)}$$

for small enough $\delta(\varepsilon)$ with Neumann boundary conditions at $x=0,\frac{1}{4}\pi$, where we have denoted $\lambda'_\varepsilon := \lambda(1-\varepsilon)^{-p-1}$.

We have $L^{(2)}_{n,p} \geq (1-\varepsilon)^p L^{(3)}_{n,p}$. To estimate the *rhs* by comparing the indicated potential contributions it is useful to pass to the rescaled variable $x = t\cdot n^{-(2p+2)/(p+2)}$.

Subcritical behavior – proof concluded

In this way we find that $\mu'_{n,p} := \inf \sigma(L^{(3)}_{n,p})$ satisfies

$$\frac{\mu'_{n,p}}{n^2} \to \infty \qquad \text{as} \qquad n \to \infty$$

Through the chain of inequalities we come to conclusion that $\inf \sigma(L^{(1)}_{n,p}) \to \infty$ holds as $n \to \infty$ which proves discreteness of the spectrum for $\lambda < \gamma_p$.

If $\lambda = \gamma_p$ the sequence of spectral thresholds no longer diverges but it remains bounded from below and the same is by minimax principle true for the operator $L_p(\lambda)$. $\square$

Remark

It is natural to conjecture that $\sigma(L_p(\gamma_p)) \supset \mathbb{R}_+$. It is less clear whether the critical operator can have also a negative discrete spectrum.

The supercritical case

The situation is different for large values of λ:

Theorem (E-Barseghyan'12)

The spectrum of $L_p(\lambda)$, $p \geq 1$, is unbounded below from if $\lambda > \lambda_{\mathrm{crit}}$.

Idea of the proof: Similar as above with a few differences:

- now we seek an upper bound to $L_p(\lambda)$ by a below unbounded operator, hence we impose Dirichlet conditions on concentric circles
- the estimating operators have now a nonzero contribution from the radial part, however, it is bounded by π^2 independently of n
- the estimate of the angular part is simpler; the negative λ-dependent term now outweights the anharmonic oscillator part so that $\inf \sigma(L_{n,p}^{(1,D)}) \to -\infty$ holds as $n \to \infty$ $\square$

Lower bounds to eigenvalue sums

To state the result we introduce the following quantity:

$$\alpha := \frac{1}{2}\left(1+\sqrt{5}\right)^2 \approx 5.236 > \gamma_p^{-1}$$

We denote by $\{\lambda_{j,p}\}_{j=1}^{\infty}$ the eigenvalues of $L_p(\lambda)$ arranged in the ascending order; then we can make the following claim.

Theorem (E-Barseghyan'12)

To any nonnegative $\lambda < \alpha^{-1} \approx 0.19$ there exists a positive constant C_p depending on p only such that the following estimate is valid,

$$\sum_{j=1}^{N} \lambda_{j,p} \geq C_p(1-\alpha\lambda)\frac{N^{(2p+1)/(p+1)}}{(\ln^p N+1)^{1/(p+1)}} - c\lambda N, \quad N = 1, 2, \ldots,$$

where $c = 2\left(\frac{\alpha^2}{4}+1\right) \approx 15.7$.

Proof outline

Proof is technically demanding, we just review the main steps. We denote by $\{\psi_{j,p}\}_{j=1}^{\infty}$ normalized eigenfunctions corresponding to $\{\lambda_{j,p}\}_{j=1}^{\infty}$, i.e.

$$-\Delta\psi_{j,p} + (|xy|^p - \lambda(x^2+y^2)^{p)/(p+2)})\psi_{j,p} = \lambda_{j,p}\psi_{j,p}, \quad j = 1, 2, \dots;$$

without loss of generality we may suppose that they are real-valued.

Our potential forms hyperbolic-shaped "valleys" and we have to estimate eigenfunction integrals in them. Specifically, we check that for any natural j and $\delta > 0$ one has

$$\int_1^\infty \int_0^{(1+\delta)y^{-p/(p+2)}} y^{2p/(p+2)}\psi_{j,p}^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \le 2(1+\delta)^2 \int_1^\infty \int_0^\infty \left(\frac{\partial\psi_{j,p}}{\partial x}\right)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$+2\frac{1+\delta}{\delta}\int_1^\infty \int_0^{(1+\delta)y^{-p/(p+2)}} x^p y^p \psi_{j,p}^2(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Proof outline – continued

and that for any $\varepsilon > 0$ there is a number $1 \le \theta(\varepsilon) \le 1+\delta$ such that

$$\int_1^\infty y^{p/(p+2)} \psi_{j,p}^2 \left(\frac{\theta(\varepsilon)}{y^{p/(p+2)}}, y \right) \mathrm{d}y < \frac{1}{\delta} \int_1^\infty \int_{y^{-p/(p+2)}}^{(1+\delta)y^{-p/(p+2)}} x^p y^p \psi_{j,p}^2(x,y)\, \mathrm{d}x\, \mathrm{d}y + \varepsilon\,,$$

together with the symmetry counterparts of these relations.

In combination with $\|\psi_{j,p}\| = 1$ this allows us to estimate the modulus of the attractive term by a combination of the kinetic and repulsive ones:

$$\begin{aligned} &\int_{\mathbb{R}^2} (x^2+y^2)^{\frac{p}{p+2}} \psi_{j,p}^2(x,y)\, \mathrm{d}x\, \mathrm{d}y \\ &\quad \le 2(1+\delta) \max\left\{ (1+\delta), \frac{1}{\delta} \right\} \left(\int_{\mathbb{R}^2} \left|\nabla \psi_{j,p}\right|^2 (x,y)\, \mathrm{d}x\, \mathrm{d}y \right. \\ &\qquad \left. + \int_{\mathbb{R}^2} |xy|^p \psi_{j,p}^2(x,y)\, \mathrm{d}x\, \mathrm{d}y + (1+\delta)^2 \right) + 2 \end{aligned}$$

Proof outline – continued

We choose $\delta = \frac{-1+\sqrt{5}}{2}$ and put $c := \alpha(1+\delta)^2 + 2 = 2\left(\frac{\alpha^2}{4}+1\right)$; using then the fact that $\lambda_{j,p}$ is the eigenvalue corresponding to $\psi_{j,p}$ we get

$$\int_{\mathbb{R}^2} |\nabla\psi_{j,p}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^2} |xy|^p \psi_{j,p}^2 \,\mathrm{d}x\,\mathrm{d}y \le \frac{1}{1-\alpha\lambda}(\lambda_{j,p} + c\lambda)\,, \quad j = 1, 2, \dots .$$

Subtracting a number ϱ from both sides and rewriting the first integral using Fourier-Plancherel image of $\psi_{j,p}$ we get

$$-\sum_{j=1}^{N} \int_{R^2} [\varrho - x^2 - y^2]_+ |\hat{\psi}_{j,p}|^2 \,\mathrm{d}x\,\mathrm{d}y + \sum_{j=1}^{N} \int_{R^2} |xy|^p \psi_{j,p}^2 \,\mathrm{d}x\,\mathrm{d}y \le \frac{1}{1-\alpha\lambda} \sum_{j=1}^{N} (\lambda_{j,p} + c\lambda) - N\varrho\,.$$

Proof outline – continued

Lemma (Barseghyan'09)

There is a constant C'_p such that for any orthonormal system of real-valued functions, $\Phi = \{\varphi_j\}_{j=1}^N \subset L^2(\mathbb{R}^2)$, $N = 1, 2, \dots,$ the inequality

$$\int_{\mathbb{R}^2} \rho_\Phi^{p+1}\, \mathrm{d}x\, \mathrm{d}y \le C'_p(\ln^p N + 1) \sum_{j=1}^N \int_{\mathbb{R}^2} |\xi\eta|^p |\hat{\varphi}_j|^2\, \mathrm{d}\xi\, \mathrm{d}\eta\,,$$

holds true, where $\rho_\Phi := \sum_{j=1}^N \varphi_j^2$.

We use it to estimate the second integral on the *lhs*. To the first one we apply Hölder inequality and find the minimum of the obtained expression with respect to the variable $z = \left(\int_{\mathbb{R}^2} \rho_\Phi^{p+1}\, \mathrm{d}x\, \mathrm{d}y\right)^{1/(p+1)}$

Proof outline – concluded

After a short computation we get in this way

$$C_p''(1+\ln^p N)^{1/p}\varrho^{(2p+1)/p} \geq N\varrho - \frac{1}{1-\alpha\lambda}\sum_{j=1}^{N}(\lambda_{j,p}+c\lambda)$$

with the new constant being an explicit function of C_p'.

Hence we have to find $\widetilde{g}(N) = \sup_{\varrho\geq 0}\left(N\varrho - C_p''\,\varrho^{(2p+1)/p}(1+\ln^p N)^{1/p}\right)$, in other words, the Legendre transformation of the *lhs* of the last inequality. It is straightforward to check that

$$\widetilde{g}(N) = C_p \frac{N^{(2p+1)/(p+1)}}{(1+\ln^p N)^{1/(p+1)}}$$

with the constant given by $C_p := \left(\frac{p}{(2p+1)C_p''}\right)^{p/(p+1)} \frac{p+1}{2p+1}$. This is equivalent to the sought bound concluding thus the proof. $\square$

Other eigenvalue moments

We can use the result also to derive – by far not optimal – bounds to other eigenvalue moments. Assuming $\lambda_{1,p} \le \lambda_{2,p} \le \dots$ of operator $L_p(\lambda)$ we have the inequality

$$\sum_{j=K}^{K+N} \lambda_{j,p} \ge \frac{1}{2}\, C_p(1-\alpha\lambda)\, \frac{N^{(2p+1)/(p+1)}}{(1+\ln^p N)^{1/(p+1)}}, \quad K = 1, 2, \dots .$$

Using it for $K = N$ we get a lower bound for $\lambda_{2N,p}$ which further implies

Corollary

$$\sum_{j=1}^{N} \lambda_{j,p}^{\sigma} \ge \widetilde{C}_{p,\sigma}(1-\alpha\lambda)^{\sigma}\, \frac{N^{(p(\sigma+1)+1)/(p+1)}}{(1+\ln^p N)^{\sigma/(p+1)}}$$

holds any $\sigma > 0$ *with some positive constant* $\widetilde{C}_{p,\sigma}$.

Upper bound to eigenvalue sums

Theorem (E-Barseghyan'12)

To any $p \geq 1$ there is a constant $\widetilde{C}_p$ such that

$$\sum_{j=1}^{N} \lambda_{j,p} \leq \widetilde{C}_p \frac{N^{(2p+1)/(p+1)}}{(1+\ln^p N)^{1/(p+1)}}\,, \qquad N = 1, 2, \dots$$

holds for any $0 \leq \lambda < \gamma_p$.

We note two things:

- Not surprisingly, the upper bound is valid for any subcritical value of λ
- In the case $\lambda = 0$ the asymptotics is exact up to the value of the constant while for $0 < \lambda < \alpha^{-1}$ the two bounds differs also by the lower-order term $-c\lambda N$

Proof outline

The argument is easier than in the lower-bound case and follows the idea used in [Barseghyan'09], specifically:

- we discard the negative term in the potential,
- we estimate L_p from above by the operator $\hat{H}_p = -\Delta + Q$ in $L^2(\mathbb{R}^2)$ with the potential $Q(x, y) = |xy|^p + |x|^p + |y|^p + 1$,
- we estimate the spectrum of the latter semiclassically.

Cusp-shaped regions

The above bounds are valid for any $p \geq 1$, hence it is natural to ask about the limit $p \to \infty$ describing the particle confined in a region with four hyperbolic 'horns', $D = \{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$, described by the Schrödinger operator

$$H_D(\lambda) : \; H_D(\lambda)\psi = -\Delta\psi - \lambda(x^2 + y^2)\psi$$

with a parameter $\lambda \geq 0$ and Dirichlet condition on the boundary ∂D.

Theorem (E-Barseghyan'12)

The spectrum of $H_D(\lambda)$ is discrete for any $\lambda \in [0, 1)$ and the spectral estimate

$$\sum_{j=1}^{N} \lambda_j \geq C(1-\lambda)\frac{N^2}{1 + \ln N}, \qquad N = 1, 2, \dots$$

holds true with a positive constant C.

Proof outline

One can check that for any $u \in H^1$ satisfying the condition $u|_{\partial D} = 0$ the inequality

$$\int_D (x^2+y^2)u^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \le \int_D |(\nabla u)(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y$$

is valid which in turn implies

$$H_D(\lambda) \ge -(1-\lambda)\Delta_D\,,$$

where Δ_D is the Dirichlet Laplacian on the region D.

The result then follows from the eigenvalue estimates on Δ_D known from [Simon'83], [Jakšić-Molchanov-Simon'92].

Smilansky model

The model was proposed in [Smilansky'04] to describe two simple subsystems interacting in a way which exhibits a spectral transition.

Mathematical properties of the model were analyzed in [Solomyak'04], [Evans-Solomyak'05], [Naboko-Solomyak'06]. Recently in [Guarneri'11] time evolution in such a (slightly modified) model was analyzed.

One way to describe the model is through a 2D Schrödinger operator

$$H_{\mathrm{Sm}} = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \lambda y \delta(x)\,,$$

Smilansky argued in that the behavior of the system depends crucially on the coupling parameter: if $|\lambda| > 1$ the particle can escape to infinity along the singular 'channel' in the y direction. In spectral terms, it corresponds to switch from a below bounded to below unbounded spectrum at $|\lambda| = 1$.

A regular version of Smilansky model

A regular version requires a modification, in particular, the coupling cannot be linear in y and the profile of the channel has to change with y.

The effect leans on effective variable decoupling far from the x-axis, where the oscillator potential competes with the principal eigenvalue of the 'transverse' part of the operator equal to $\frac{1}{4}\lambda^2 y^2$.

We replacing the δ by a family of shrinking potentials whose mean matches the δ coupling constant, $\int U(x,y)\,\mathrm{d}x \sim y$. This can be achieved, e.g., by choosing $U(x,y) = \lambda y^2 V(xy)$ for a fixed function V.

This motivates us to investigate the following operator on $L^2(\mathbb{R}^2)$,

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \lambda y^2 V(xy)\chi_{\{|x|\le a\}}(y),$$

where ω, a are positive constants, $\chi_{\{|y|\le a\}}$ is the indicator function of the interval $(-a,a)$, and the potential V with $\operatorname{supp} V \subset [-a,a]$ is a nonnegative function with bounded first derivative.

A regular version of Smilansky model, continued

By Faris-Lavine theorem the operator is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ and the same is true for its generalization,

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \sum_{j=1}^{N} \lambda_j y^2 V_j(xy) \chi_{\{|x-b_j|\le a_j\}}(y)$$

with a finite number of channels, where functions V_j are positive with bounded first derivative, with the supports contained in $(b_j - a_j, b_j + a_j)$ and such that $\mathrm{supp}\, V_j \cap \mathrm{supp}\, V_k = \emptyset$ holds for $j \neq k$.

Remark

We note that the properties discussed below depend on the asymptotic behavior of the potential channels and would not change if the potential is modified in the vicinity of the x-axis, for instance, by replacing the above cut-off functions with $\chi_{|y|\ge a}$ and $\chi_{|y|\ge a_j}$, respectively.

Subcritical case

To state the result we employ a 1D comparison operator $L = L_V$,

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2 - \lambda V(x)$$

on $L^2(\mathbb{R})$ with the domain $H^2(\mathbb{R})$. What matters is the sign of its spectral threshold; since V is supposed to be nonnegative, the latter is a monotonous function of λ and there is a $\lambda_{\mathrm{crit}} > 0$ at which the sign changes.

Theorem (Barseghyan-E'13)

Under the stated assumption, the spectrum of the operator H is bounded from below provided the operator L is positive. In the multichannel case the condition reads $t_V := \min_j \inf \sigma(L_{V_j}) \geq 0$.

Proof follows the same idea we have used for the previous model. $\square$

Supercritical case

Once the transverse channel principal dominates over the harmonic oscillator contribution, the spectral behavior changes.

Theorem (Barseghyan-E'13)

Under our hypotheses, $\sigma(H) = \mathbb{R}$ *holds if* $\inf \sigma(L) < 0$. *In the multichannel case the condition reads* $t_V := \min_j \inf \sigma(L_{V_j}) < 0$.

Proof relies on construction of an appropriate Weyl sequence. □

Spectral estimates on non-straight cusps

The last result leads us naturally to the second main topic of this talk, namely more general spectral estimates for Schrödinger type operators

$$H_\Omega = -\Delta_D^\Omega - V$$

on cusp-shaped regions $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary.

In distinction to the above considerations

- we will not restrict ourselves now to the planar case assuming $d \geq 2$,
- we will suppose that $V \geq 0$ is a bounded measurable potential,
- on the other hand, we will assume that Ω can be geometrically nontrivial, either bent or twisted, and investigate the role of geometry in the spectral estimates
- since estimates for complicated regions can be obtained by combining various techniques and bracketing, we concentrate on a single cusp-shaped region, one-sided or two-sided.

A warm-up: curved planar cusps

Consider $\Omega \subset \mathbb{R}^2$ with a smooth boundary which can described it by specifying its axis and the cusp width. This will allow us to use natural curvilinear coordinates to 'straighten' the cusp translating its geometric properties into the coefficients of the resulting operator.

Specifically, we have three functions: smooth $a, b: \mathbb{R} \to \mathbb{R}^2$ and a positive continuous $f: \mathbb{R} \to \mathbb{R}^+$ such that

$$\Omega := \{(a(s) - u\dot{b}(s), b(s) + u\dot{a}(s)) : s \in \mathbb{R}, |u| < f(s)\}$$

and $\lim_{|s|\to\infty} f(s) = 0$. We will also consider Ω^+ corresponding to $s \geq 0$.

The strategy we are going to apply is first to 'straighten' Ω and then to use the dimensional-reduction technique of Laptev and Weidl.

Curved planar cusps

The reference curve $\Gamma = \{(a(s), b(s)) : \ s \in \mathbb{R}\}$ can be parametrized by its arc length in which case $\dot{a}(s)^2 + \dot{b}(s)^2 = 1.$ The signed curvature $\gamma(s)$ of Γ is given by $\gamma(s) = \dot{b}(s)\ddot{a}(s) - \dot{a}(s)\ddot{b}(s)$.

Note that Ω is determined by γ and f since

$$\left\{ \begin{array}{c} a(s) \\ b(s) \end{array} \right\} = \left\{ \begin{array}{c} a(s_0) \\ b(s_0) \end{array} \right\} + \int_{s_0}^{s} \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} \left(\int_{s_0}^{t} \gamma(\xi)\, \mathrm{d}\xi \right) \mathrm{d}t$$

with a fixed point s_0, modulo Euclidean transformations of the plane.

We adopt the following assumptions:

- $|f(s)\gamma(s)| < 1$ must hold at any point of the curve,
- the map $(s, u) \mapsto (a(s) - u\dot{b}(s), b(s) + u\dot{a}(s))$ is injective

LT-type inequalities: 2D case

The spectrum of H_Ω is discrete [Berger-Schechter'72]; our aim is to prove Lieb-Thirring-type inequalities.

Theorem (E-Barseghyan'12)

Consider the Schrödinger operator H_Ω on the region Ω. Suppose that the curvature $\gamma \in C^4$, the inequality $\|f(\cdot)\gamma(\cdot)\|_{L^\infty(\mathbb{R})} < 1$ holds true, and Ω does not intersect itself. Then for any $\sigma \geq 3/2$ we have the estimate

$$\operatorname{tr}(H_\Omega)_-^\sigma \leq \|1+f|\gamma|\|_\infty^{-2\sigma}\, L_{\sigma,1}^{\mathrm{cl}} \int_{\mathbb{R}} \sum_{j=1}^{\infty} \Bigg(-\left(\frac{\pi j}{2f(s)}\right)^2 + \|1+f|\gamma|\|_\infty^2 W^-(s) + \|1+f|\gamma|\|_\infty^2 \|\widetilde{V}(s,\cdot)\|_\infty \Bigg)_+^{\sigma+1/2} \mathrm{d}s\,,$$

where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R})}$ and $L_{\sigma,1}^{\mathrm{cl}}$ in the above formula is the usual semiclassical constant defined by

LT-type inequalities: 2D case continued

Theorem (continued)

$$L^{\mathrm{cl}}_{\sigma,1} := \frac{\Gamma(\sigma+1)}{\sqrt{4\pi}\,\Gamma(\sigma+\frac{3}{2})}\,,$$

and furthermore, we have introduced

$$W^-(s) := \frac{\gamma(s)^2}{4\,(1-f(s)|\gamma(s)|)^2} + \frac{f(s)|\ddot{\gamma}(s)|}{2\,(1-f(s)|\gamma(s)|)^3} + \frac{5f^2(s)\dot{\gamma}(s)^2}{4\,(1-f(s)|\gamma(s)|)^4}$$

and $\widetilde{V}(s,u) := V\big(a(s)-u\dot{b}(s), b(s)+u\dot{a}(s)\big)$.

Note that the sum is finite; the estimate looks like 1D LT inequality with potential consisting of three parts: the transverse energy together with multiples of the potential $\widetilde{V}(\cdot)$ and curvature-induced potential $W^-(\cdot)$.

Proof outline

Using the 'straightening' transformation we infer that H_Ω is unitarily equivalent to the operator H_0 on $L^2(\Omega_0)$ acting as

$$(H_0\psi)(s,u) = -\frac{\partial}{\partial s}\left(\frac{1}{(1+u\gamma(s))^2}\frac{\partial\psi}{\partial s}(s,u)\right) - \frac{\partial^2\psi}{\partial u^2}(s,u) + ((W-\widetilde{V})\psi)(s,u)$$

and Dirichlet b.c. at $u = \pm f(s)$, where $\Omega_0 = \{(s,u):\ s\in\mathbb{R},\ |u| < f(s)\}$.

We introduce the operator H_0^- defined on the domain $\mathcal{H}_0^2(\Omega_0)$ in $L^2(\Omega_0)$ by

$$H_0^- = -\Delta_D^{\Omega_0} - \|1+f|\gamma|\|_\infty^2\,(W^- + \widetilde{V})\,,$$

then by a simple minimax estimate we have

$$H_0 \geq \|1+f|\gamma|\|_\infty^{-2}\ H_0^-\,,$$

and consequently, it is enough to establish an appropriate bound on the trace of the (negative part of the) operator $(H_0^-)^\sigma$.

Proof outline – continued

Next comes a dimensional reduction analogous to that in [Weidl'08].
We denote by $H(s, \widetilde{V}, W^-)$ the negative part of Sturm-Liouville operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \|1 + f|\gamma|\|_\infty^2 \left(W^- + \widetilde{V}\right)$$

defined on $C_0^\infty(-f(s), f(s))$ with Dirichlet b.c. at $u = \pm f(s)$.

Extending it to the whole line as an operator acting as zero outside $[-f(s), f(s)]$ we derive easily the inequality

$$\|\nabla g\|^2_{L^2(\Omega_0)} + \|\nabla v\|^2_{L^2(\widehat{\Omega}_0)} - \|1 + f|\gamma|\|_\infty^2 \int_{\Omega_0} \left(W^- + \widetilde{V}\right)(s,u)|g(s,u)|^2 \,\mathrm{d}s\,\mathrm{d}u$$
$$\geq \int_{\mathbb{R}^2} \left|\frac{\partial h}{\partial s}(s,u)\right|^2 \mathrm{d}s\,\mathrm{d}u + \int_{\mathbb{R}} \left\langle\, H(s, \widetilde{V}, W^-)\, h(s,\cdot),\; h(s,\cdot) \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s\,.$$

which holds true for any function $g \in C_0^\infty\left(\mathbb{R}^2 \backslash \partial\Omega_0\right)$

Proof outline – continued

The *lhs* is the quadratic form corresponding to $H_0^- \oplus \left(-\Delta_D^{\Omega_0^c}\right)$, while the *rhs* is associated with $-\frac{\partial^2}{\partial s^2} \otimes I_{L^2(\mathbb{R})} + H\left(s, \widetilde{V}, W^-\right)$ defined on $\mathcal{H}^1\left(\mathbb{R}, L^2(\mathbb{R})\right)$. By minimax we infer that

$$\mathrm{tr}\,(H_0^-)_-^\sigma \le \mathrm{tr}\left(-\frac{\partial^2}{\partial s^2} \otimes I_{L^2(\mathbb{R})} + H(s, \widetilde{V}, W^-)\right)_-^\sigma, \quad \sigma > 0\,.$$

Now we use the Lieb-Thirring inequality for operator-valued potentials proved in [Laptev-Weidl'00] obtaining

$$\mathrm{tr}\,(H_0^-)_-^\sigma \le L_{\sigma,1}^{\mathrm{cl}} \int_{\mathbb{R}} \mathrm{tr}\left(H\left(s, \widetilde{V}, W^-\right)\right)_-^{\sigma+1/2} \mathrm{d}s\,, \quad \sigma \ge 3/2$$

with the standard semiclassical constant $L_{\sigma,1}^{\mathrm{cl}}$.

Proof outline – concluded

Now we define Sturm-Liouville operator $L_f(s)$ on $L^2\,(-f(s), f(s))$ acting as

$$L(s,\widetilde{V},W^-) = -\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \|1+f|\gamma|\|_\infty^2 \left(W^-(s) + \|\widetilde{V}(s,\cdot)\|_\infty \right)$$

with Dirichlet b.c. at $u=\pm f(s)$. By the last inequality and minimax principle we infer that

$$\mathrm{tr}\,(H_0^-)_-^\sigma \le L_{\sigma,1}^{\mathrm{cl}} \int_{\mathbb{R}} \mathrm{tr}\left(L_f\left(s,\widetilde{V},W^-\right)\right)_-^{\sigma+1/2}$$

holds for any $\sigma \ge 3/2$.

It remains to find the negative eigenvalues $-\mu_j(s)$, $j=1,2,\dots$, of $L_f(s,\widetilde{V},W^-)$ which is easy because the potential is independent of u; this concludes the proof. $\square$

One-sided and 'thick' cusps

By Dirichlet bracketing the above result easily implies

Corollary

Consider the operator H_{Ω^+} on the one-sided cusp Ω^+. The inequality holds again with integration variable running now over the interval $(0,\infty)$.

Remark

Note that we have not used the condition $\lim_{|s|\to\infty} f(s) = 0$. If it is not satisfied, the spectrum of H_Ω may not be purely discrete; the proved inequality remains valid as long as we stay below $\inf \sigma_{\mathrm{ess}}(H_\Omega)$.

Comparison with phase space estimate

While Berezin-Lieb-Yau inequalities for Dirichlet Laplacians correctly describe semiclassical behavior of the spectrum at high energies, we want to show now that if the region has 'thin' parts there may exist an intermediate interval of energies where the above estimates are considerably stronger than the BLY bound.

The standard way to study spectra of Dirichlet Laplacians below a fixed value of energy is to consider $-\Delta_D^\Omega - V$ with a constant $V(s,u) = \Lambda > 0$ and to investigate its negative spectrum.

Consider the cusp-shaped region Ω satisfying the conditions

$$\|f\gamma\|_\infty < c < \frac{-\pi - 1 + \sqrt{(\pi+1)^2 + 4\pi}}{2} \approx 0.655$$

and $\max\{\|f\dot\gamma\|_\infty,\, \|f\ddot\gamma\|_\infty\} < 1$ as an example.

Comparison with phase space estimate – cont'd

The theorem proved above implies

$$\mathrm{tr}\,(H_\Omega)_-^\sigma \le \frac{8}{\pi}\left(\frac{c^2}{4(1-c)^2\alpha_c^2}+1\right)^{\sigma+1} L^{\mathrm{cl}}_{\sigma,1}\,\Lambda^{\sigma+1}\int_{f(s)\ge\alpha_c\Lambda^{-1/2}} f(s)\,\mathrm{d}s$$

with $\alpha_c^2 := \frac{\pi^2-c^2(1+c)^2/(1-c)^2}{4(1+c)^2}$; note that the curvature is present in this estimate through the constant c only.

Our aim is to show that such an estimate can be stronger than the phase-space bound mentioned above ,

$$\mathrm{tr}\,(H_{\Omega'})_-^\sigma \le L^{\mathrm{cl}}_{\sigma,1}\Lambda^{\sigma+1}\,\mathrm{vol}\,(\Omega')\,, \quad \sigma\ge 1\,.$$

We will construct a straight cusp example, $c=0$, starting from an unbounded cusped region Ω and passing subsequently to cut-off regions $\Omega'\subset\Omega$ such that $\mathrm{tr}\,(H'_\Omega)_-^\sigma \le \mathrm{tr}\,(H_\Omega)_-^\sigma$ holds for $\sigma\ge 0$.

Comparison with phase space estimate – cont'd

We choose an arbitrary $\alpha > 0$ and a natural number N and set

$$f_{\alpha,N}(x) := \begin{cases} \frac{\pi}{2}\, x^{-1-\alpha} & \text{for} \quad |x| > N \\ \frac{\pi}{2}\, N^{-1-\alpha} & \text{for} \quad |x| \le N \end{cases}$$

Consider now the finite region $\Omega'_{\alpha,N} := \{|x| < 2^{1/\alpha}N,\ |y| < f_{\alpha,N}(x)\}$. Using our result in combination with Dirichlet bracketing we get

$$\mathrm{tr}(H_{\Omega'_{\alpha,N}})_-^\sigma \le 4\, L^{\mathrm{cl}}_{\sigma,1} \Lambda^{\sigma+1}\, N^{-\alpha} \quad \text{for} \quad \sigma \ge 3/2\,.$$

On the other hand, the phase-space estimate gives

$$\mathrm{tr}\,(H_{\Omega'})_-^\sigma \le 2\pi\, L^{\mathrm{cl}}_{\sigma,1}\, N^{-\alpha}\, \Lambda^{\sigma+1} \left(\frac{1}{2\alpha} + 1\right) \quad \text{for} \quad \sigma \ge 1\,.$$

Given $\sigma \ge 3/2$, this can be made much larger than the *rhs* of the first inequality by choosing α small; for N large the difference between the two estimates persists over a large energy interval.

Curved circular cusps in $\mathbb{R}^d$

Now we generalize the above result to cusps in $\mathbb{R}^d$, $d \geq 3$, of a circular cross-section. The main trick is again to choose suitable curvilinear coordinates which would allow us to 'straighten' such a region; we shall follow the argument of [Chenaud-Duclos-Freitas-Krejčiřík'05]

We suppose that the region axis is a unit-speed C^{d+2}-smooth curve $\Gamma : \mathbb{R} \to \mathbb{R}^d$ which possesses a positively oriented Frenet frame, i.e. a d-tuple $\{e_1, \dots, e_d\}$ of functions such that

1. (i) $e_1 = \dot\Gamma$,
2. (ii) $e_i \in C^1(\mathbb{R}, \mathbb{R}^d)$ holds for any $j = 1, \dots, d$,
3. (iii) $\dot e_i(s)$ lies in the span of $e_1(s), \dots, e_{i+1}(s)$ for any $j = 1, \dots, d-1$.

A sufficient condition for existence of such a frame is that the vector values of the derivatives $\dot\Gamma(s), \ddot\Gamma(s), \dots, \Gamma^{(d-1)}(s)$ are linearly independent for all $s \in \mathbb{R}$; note that this is always satisfied if $d = 2$.

Coordinates in curved circular cusps

We have Frenet-Serret formulæ, $\dot{e}_i = \sum_{j=1}^{d-1} \mathcal{K}_{ij} e_j$, where

$$\mathcal{K} = \begin{pmatrix} 0 & \kappa_1 & \dots & 0 \\ -\kappa_1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \kappa_{d-1} \\ 0 & \dots & -\kappa_{d-1} & 0 \end{pmatrix},$$

where $\kappa_i : \mathbb{R} \to \mathbb{R}$ is the *i*-th curvature of Γ. Under (i)–(iii) these curvatures are continuous functions of the arc-length parameter $s \in \mathbb{R}$.

Consider a $(d-1) \times (d-1)$ matrix function $\mathcal{R} = (\mathcal{R}_{\mu,\nu})$ determined by

$$\dot{\mathcal{R}}_{\mu\nu} + \sum_{\rho=2}^{d} \mathcal{R}_{\mu,\rho} \mathcal{K}_{\rho,\nu} = 0\,, \quad \mu, \nu = 2, \dots, d\,,$$

with an initial conditions at $s_0 \in \mathbb{R}$ such that $\mathcal{R}(s_0)$ is a rotation matrix in $\mathbb{R}^{d-1}$, i.e. $\det \mathcal{R}(s_0) = 1$ and $\sum_{\rho=2}^{d} \mathcal{R}_{\mu,\rho}(s_0) \mathcal{R}_{\nu,\rho}(s_0) = \delta_{\mu,\nu}$.

Coordinates in curved circular cusps – cont'd

Next we associate with $\mathcal{R}(\cdot)$ a $d \times d$ matrix function given by

$$(\mathcal{R}_{ij}(s)) := \begin{pmatrix} 1 & 0 \\ 0 & (\mathcal{R}_{\mu,\nu}(s)) \end{pmatrix}$$

and define the moving frame $\{\widetilde{e}_1, \dots, \widetilde{e}_d\} \subset \mathbb{R}^d$ along the curve Γ by

$$\widetilde{e}_i := \sum_{j=1}^{d} \mathcal{R}_{ij} e_j \,;$$

we call it the Tang frame (relative to the given Frenet frame).

Given the Tang frame we can characterize points in the vicinity of Γ by means of the corresponding Cartesian coordinates $u_2, \dots, u_d$ in the normal plane to Γ at each point of the curve,

$$x(s, u_2, \dots, u_d) := \Gamma(s) + \sum_{\nu=2}^{d} \widetilde{e}_\mu(s) u_\mu \,,$$

in particular, $|u| = \left(\sum_{\nu=2}^{d} u_\nu^2\right)^{1/2}$ measures the radial distance from Γ.

Unitary equivalence

Given a function $f : \mathbb{R} \to \mathbb{R}_+$ satisfying $\lim_{|s|\to\infty} f(s) = 0$ we define

$$\Omega := \{x(s, u_2, \dots, u_d) :\ s \in \mathbb{R},\ |u| < f(s)\}\,;$$

we again assume that $(s, u_2, \dots, u_d) \mapsto x(s, u_2, \dots, u_d)$ is injective.

Consider again $H_\Omega = -\Delta_D^\Omega - V$ on $L^2(\Omega)$ with a bounded measurable $V \geq 0$. Using the described coordinates one can check that H_Ω is unitarily equivalent to the operator on $L^2(\Omega_0)$, where Ω_0 is the straightened region, $\Omega_0 = \{(s, u_1, \dots, u_{d-1}) :\ s \in \mathbb{R},\ |u| < f(s)\}$, acting as

$$H_0 = -\partial_1 \frac{1}{h^2} \partial_1 - \sum_{\mu=2}^{d} \partial_\mu^2 + W - \widetilde{V}$$

with Dirichlet b.c. at the boundary of the disc, $|u| = f(s)$. Here ∂_1, ∂_μ are the usual shorthands for $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial u_\mu}$, respectively.

Unitary equivalence – continued

Furthermore, $\widetilde{V}(s,u_1,\dots,u_{d-1}) := V(x(s,u_1,\dots,u_{d-1}))$ and the curvature-induced part of the potential equals

$$W := -\frac{1}{4}\frac{\kappa_1^2}{h^2} + \frac{1}{2}\frac{h_{11}}{h^3} - \frac{5}{4}\frac{h_1^2}{h^4}\,,$$

where $h(s,u_2,\dots,u_d) := 1 - \kappa_1(s)\sum_{\mu=2}^{d}\mathcal{R}_{\mu 2}(s)u_\mu$.

The derivatives with respect to s are given explicitly by

$$h_1(\cdot,u) = \sum_{\mu,\alpha=2}^{d} u_\mu \mathcal{R}_{\mu,\alpha}\dot{\mathcal{K}}_{\alpha,1} - \sum_{\substack{\mu,\alpha=2,\dots,d\\ \beta=1,\dots,d}} u_\mu \mathcal{R}_{\mu,\alpha}\mathcal{K}_{\alpha,\beta}\mathcal{K}_{\beta,1}\,,$$

$$h_{11}(\cdot,u) = \sum_{\mu,\alpha=2}^{d} u_\mu \mathcal{R}_{\mu,\alpha}\ddot{\mathcal{K}}_{\alpha,1} - \sum_{\substack{\mu,\alpha=2,\dots,d\\ \beta=1,\dots,d}} u_\mu \mathcal{R}_{\mu,\alpha}\left(\dot{\mathcal{K}}_{\alpha,\beta}\mathcal{K}_{\beta,1} + 2\mathcal{K}_{\alpha,\beta}\dot{\mathcal{K}}_{\beta,1}\right)$$

$$+ \sum_{\substack{\mu,\alpha=2,\dots,d\\ \beta,\gamma=1,\dots,d}} u_\mu \mathcal{R}_{\mu,\alpha}\mathcal{K}_{\alpha,\beta}\mathcal{K}_{\beta,\gamma}\mathcal{K}_{\gamma,1}\,.$$

Spectral estimate

Theorem (E-Barseghyan'12)

Consider $H_\Omega = -\Delta_D^\Omega - v$ *with a bounded measurable* $V \geq 0$ *corresponding to* Ω *which is not self-intersecting; we assume that it is determined by a* C^{d+2}*-smooth curve* Γ *and a function* f *satisfying the condition* $\|\kappa_1(\cdot)f(\cdot)\|_\infty < 1$, *where* κ_1 *is the first curvature of* Γ. *Then for the negative spectrum of* H_Ω *the following inequality holds true,*

$$\mathrm{tr}\,(H_\Omega)_-^\sigma \leq \|1+f|\kappa_1|\|_\infty^{-2\sigma}\; L_{\sigma,1}^{\mathrm{cl}} \int_{\mathbb{R}} \sum_{k,m=0,1,\dots} \Bigg(-\left(\frac{j_{k+(d-3)/2,m}}{f(s)}\right)^2 + \|1+f|\kappa_1|\|_\infty^2 \left(W^-(s) + \|\widetilde{V}(s,\cdot)\|_\infty \right)\Bigg)_+^{\sigma+1/2} \mathrm{d}s\,,$$

where $L_{\sigma,1}^{\mathrm{cl}}$ *is the LT constant,* W^- *is the (explicit) s-dependent bound to* $W(s,u)_-$, $\widetilde{V}(s,\cdot)$ *is defined above, and* $j_{l,m}$ *is the m-th positive zero of the first-kind Bessel function* J_l.

Proof outline

The scheme is the same as for $d = 2$. We introduce the operator

$$H_0^- = -\sum_{\mu=1}^{d} \partial_\mu^2 - \|1 + f|\kappa_1|\|_\infty^2 \left(W^- + \widetilde{V}\right)$$

and prove the inequality

$$H_0 \geq \|1 + f|\kappa_1|\|_\infty^{-2} \, H_0^- \, ;$$

then we apply again the dimension reduction technique, which gives

$$\mathrm{tr}\,(H_0^-)_-^\sigma \leq L_{\sigma,1}^{\mathrm{cl}} \int_{\mathbb{R}} \mathrm{tr}\left(L\left(s, \widetilde{V}, W^-\right)\right)_-^{\sigma+1/2} \mathrm{d}s$$

for $\sigma \geq 3/2$, where the operator in $L^2(D_{f(s)})$ at the *rhs* is given by

$$L\left(s, \widetilde{V}, W^-\right) := -\Delta_D^{D_{f(s)}} - \|1 + f|\kappa_1|\|_\infty^2 \left(W^-(s) + \|\widetilde{V}(s,\cdot)\|_\infty\right). \quad \square$$

Twisted cusps in $\mathbb{R}^3$

Consider now another type of nontrivial cusp geometry: the axis will be straight but the cross section will be non-circular. Having an open connected set $\omega_0 \subset \mathbb{R}^2$ and a function $f:\, \mathbb{R} \to \mathbb{R}_+$ satisfying $\lim_{|s|\to\infty} f(s) = 0$, we set

$$\omega_s := f(s)\omega_0\,;$$

it is used to define a straight cusped region determined by ω_0, f as $\Omega_0 := \{(s,x,y):\; s\in\mathbb{R},\, (x,y)\in\omega_s\}$.

Next we twist the region. We fix a C^1-smooth function $\theta:\mathbb{R}\to\mathbb{R}$ with bounded derivative, $\|\dot\theta\|_\infty < \infty$, and introduce the region Ω_θ as the image

$$\Omega_\theta := \mathfrak{L}_\theta(\Omega_0)\,,$$

where the map $\mathfrak{L}_\theta:\, \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathfrak{L}_\theta(s,x,y) := (s, x\cos\theta(s) + y\sin\theta(s), -x\sin\theta(s) + y\cos\theta(s))\,.$$

Twisted cusps in $\mathbb{R}^3$ – continued

We are interested in the situation nontrivial twist situation, i.e.

1. the function θ is not constant ,
2. ω_0 is not rotationally symmetric w.r.t. the origin in $\mathbb{R}^2$.

We need a few more preliminaries. Define $\varrho := \sup_{(x,y)\in\omega_0} \sqrt{x^2+y^2}$ and assume

$$\varrho \|f\dot{\theta}\|_\infty < 1\,.$$

Next we set $\widetilde{V}(s,x,y) := V(\mathfrak{L}_\theta(s,x,y))$ and finally, we introduce the operator

$$L_{\mathrm{trans}} := -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \quad \mathrm{Dom}\big(L_{\mathrm{trans}}\big) = \mathcal{H}_0^1(\omega_0)\,,$$

of angular momentum component canonically associated with rotations in the transverse plane.

Spectral estimate

Theorem (E-Barseghyan'12)

Consider $H_{\Omega_\theta} = -\Delta_D^{\Omega_\theta} - V$ *referring to the region* Ω_θ *defined above with a potential* $V \geq 0$ *which is bounded and measurable. Under the assumption* $\varrho\|f\dot\theta\|_\infty < 1$ *the negative spectrum of* H_{Ω_θ} *satisfies the inequality*

$$\mathrm{tr}\big(H_D^{\Omega_\theta}\big)_-^\sigma \leq L_{\sigma,1}^{\mathrm{cl}} \left(1 - \varrho\|f\dot\theta\|_\infty\right)^\sigma \int_{\mathbb{R}} \sum_{j=1}^\infty \left(-\frac{\lambda_{0,j}(s)}{f^2(s)} + \frac{\|\widetilde{V}(s,\cdot)\|_\infty}{1-\varrho\|f\dot\theta\|_\infty}\right)_+^{\sigma+1/2} \mathrm{d}s$$

for $\sigma \geq 3/2$, *where* $L_{\sigma,1}^{\mathrm{cl}}$ *is the LT constant and* $\lambda_{0,j}(s)$, $j = 1, 2, \dots,$ *are the eigenvalues of the operator*

$$H_{f,\theta}(s) := -\Delta_D^{\omega_0} + f^2(s)\dot\theta^2(s)L_{\mathrm{trans}}^2$$

defined on the domain $(\mathcal{H}^2 \cap \mathcal{H}_0^1)(\omega_0)$ *in* $L^2(\omega_0)$.

Proof outline

Similar as above: we pass to unitarily equivalent operator H_0 on $L^2(\Omega_0)$ using $U_\theta\psi := \psi\circ\mathfrak{L}_\theta$ and employ the inequality $H_0 \geq \big(1-\varrho\|f\dot\theta\|_\infty\big)H_0^-$, where

$$H_0^- = -\Delta_D^{\Omega_0} + \dot\theta^2(s)L_{\mathrm{trans}}^2 - \frac{1}{1-\varrho\|f\dot\theta\|_\infty}\|\widetilde{V}(s,\cdot)\|_\infty$$

with the domain $\mathcal{H}_0^2(\Omega_0)$.

Denoting by $H(s,\widetilde{V})$ the negative part of 2D Schrödinger operator

$$-\Delta_D^{\omega_s} + \dot\theta^2(s)L_{\mathrm{trans}}^2 - \frac{1}{1-\varrho\|f\dot\theta\|_\infty}\|\widetilde{V}(s,\cdot)\|_\infty$$

defined on $\mathcal{H}_0^2(\omega_s)$, we prove the inequality

$$\mathrm{tr}\,\big(H_0^-\big)_-^\sigma \leq L_{\sigma,1}^{\mathrm{cl}}\int_{\mathbb{R}} \mathrm{tr}\,H(s,\widetilde{V})_-^{\sigma+1/2}\,\mathrm{d}s\,,\quad \sigma\geq 3/2$$

and find the spectrum of the subintegral operator. $\square$

Summary

We have investigated spectral behavior of Schrödinger operators and Dirichlet Laplacians confining the particle to 'thin channels', specifically

- We have demonstrated occurrence of different spectral regime for the operator $-\Delta + \left(|xy|^p - \lambda(x^2+y^2)^{p/(p+2)}\right)$ in $L^2(\mathbb{R}^2)$
- in the subcritical case we proved a spectral estimate for small enough value of the parameter λ
- the analogous result was obtained in the hard-wall case, $p = \infty$
- a similar spectral transition has been demonstrated in a regular version of Smilansky model
- spectral estimates have been derived for Schrödinger operators in bent circular cusp-shaped regions in $\mathbb{R}^d$
- analogous spectral estimates have been obtained for twisted non-circular cusps in $\mathbb{R}^3$

Some open questions

- Are there eigenvalue-sum estimates for λ between $\alpha^{-1} \approx 0.19$ and $\gamma_p \gtrsim 0.999$?
- Find better estimates for other eigenvalue moments
- One expects that $\mathbb{R}_+ \subset \sigma(L_p(\lambda_{\mathrm{crit}})$. Are there negative eigenvalues?
- Prove that $\sigma(L_p(\lambda_{\mathrm{crit}})) = \mathbb{R}$ for $\lambda > \lambda_{\mathrm{crit}}$
- Analyze the negative discrete spectrum for subcritical regular Smilansky model
- Find the time evolution in this model, in particular, its change when we pass from subcritical to supercritical regime
- Spectral estimates for cusps which are both bent and twisted

The talk was based on

[EB12a] P.E., D. Barseghyan: Spectral estimates for a class of Schrödinger operators with infinite phase space and potential unbounded from below, *J. Phys. A: Math. Theor.* **A45** (2012), 075204.

[EB12b] P.E., D. Barseghyan: Spectral estimates for Dirichlet Laplacians and Schrödinger operators on geometrically nontrivial cusps, *J. Spectral Theory*, to appear; `arXiv: 1203.2098 [math-ph]`.

[EB13] D. Barseghyan, P.E.: A regular version of Smilansky model, submitted; `arXiv: 1308.4249 [math-ph]`.

It remains to say

Merci pour votre attention!