On the discrete spectrum of soft quantum waveguides

Pavel Exner

Doppler Institute
for Mathematical Physics and Applied Mathematics
Prague

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Whom we commemorate

A great mathematicians whose books influenced many people, and also a member of a generation whose life was no rosy garden walk indeed

Geometrically induced bound states

I am going to a topic which might sound familiar to you, at least some heard me speaking about related problems here and in other places, so there is no need for an extensive introduction

As a warm-up, just a brief reminder: let $-\Delta_{\mathrm{D}}^{\Omega}$ be the Dirichlet Laplacian in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a strip of the width $2a$ built over an infinite curve Γ without self-intersections

If Γ is straight line the spectrum is found by separation of variables, $\sigma(-\Delta_{\mathrm{D}}^{\Omega}) = \left[\left(\frac{\pi}{2a}\right)^2, \infty\right)$, and it is absolutely continuous

If, on the other hand, the curve Γ is *not* straight, but it is *asymptotically straight* – expressed in terms of suitable technical assumptions – then there are curvature-induced bound states, i.e. $\sigma_{\mathrm{disc}}(-\Delta_{\mathrm{D}}^{\Omega}) \neq \emptyset$

There is a huge number of related results involving systems in other dimensions and different geometric perturbations; for a survey and bibliography we refer to

P.E., H. Kovařík: *Quantum Waveguides*, Springer, Cham 2015

The interest in related models

Apart from a purely mathematical interest – new solutions to one of the most studied equations! – such geometrically induced bound states are of practical importance as models of various *waveguide effects*, mainly in quantum theory, but also in *electromagnetism* or *acoustics*

J.T. Londergan, J.P. Carini, D.P. Murdock: *Binding and Scattering in Two-Dimensional Systems. Applications to Quantum Wires, Waveguides and Photonic Crystals*, Springer LNP m60, Berlin 1999

E.B. Davies, L. Parnovski: Trapped modes in acoustic waveguides, *Quart. J. Mech. Appl. Math.* **51** (1998), 477–492.

From that point if view the Dirichlet boundary as a *hard wall* is naturally an idealization; in the language of quantum theory it means that the *tunneling* between different parts of the structure is forbidden

This motivated an alternative approach through *'leaky quantum wires'* which works with singular Schrödinger operators formally written as $-\Delta - \alpha\delta(x - \Gamma)$ with $\alpha > 0$, Γ being is a curve, a graph, or more generally, a complex of lower dimensionality, cf. Chapter 10 in [EK15, loc.cit.] and

P.E., T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys. A: Math. Gen.* **34** (2001), 1439–1450.

Curvature-induced states in leaky wires

We have a similar effect here; using the generalized *Birman-Schwinger principle* the following result was demonstrated in the indicated paper: consider a *non-straight, piecewise C^1-smooth curve* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ parameterized by its arc length, $|\Gamma(s) - \Gamma(s')| \le |s - s'|$, assuming that

- $|\Gamma(s) - \Gamma(s')| \ge c|s - s'|$ holds for some $c \in (0, 1)$
- Γ is *asymptotically straight*: there are $d > 0$, $\mu > \frac{1}{2}$ and $\omega \in (0, 1)$ such that

$$1 - \frac{|\Gamma(s) - \Gamma(s')|}{|s - s'|} \le d\left[1 + |s + s'|^{2\mu}\right]^{-1/2}$$

in the sector $S_\omega := \left\{(s, s') : \omega < \frac{s}{s'} < \omega^{-1}\right\}$

Theorem

Under these assumptions, $\sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = [-\frac{1}{4}\alpha^2, \infty)$ and $-\Delta_{\delta,\alpha}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$.

Soft quantum waveguides

The leaky wire model is also an idealization assuming the *zero width* of the guide; to get a more realistic model we replace the δ function by a finite *potential well*

We address the question in the simplest two-dimensional setting. Let us formulate the problem stating first the assumptions:

Let $\Gamma:\ \mathbb{R} \to \mathbb{R}^2$ be an infinite and smooth planar curve without self-intersections, parametrized by its arc length s. We introduce the signed curvature $\gamma:\ \gamma(s) = (\dot\Gamma_2\ddot\Gamma_1 - \dot\Gamma_1\ddot\Gamma_2)(s)$ and assume that

a. Γ is C^2-smooth so, in particular, $\gamma(s)$ makes sense,
b. γ is either of *compact support*, $\operatorname{supp}\gamma \subset [-s_0, s_0]$ for some $s_0 > 0$, or Γ is *C^4-smooth* and $\gamma(s)$ together with its first and second derivatives tend to zero as $|s| \to \infty$,
c. $|\Gamma(s) - \Gamma(s')| \to \infty$ holds as $|s - s'| \to \infty$.

excluding thus U-shaped curves and their various modifications

The interaction support

We can reconstruct the curve from γ, up to Euclidean transformations: putting $\beta(s_2,s_1) := \int_{s_1}^{s_2} \gamma(s)\,\mathrm{d}s$, we have

$$\Gamma(s) = \left(x_1 + \int_{s_0}^{s} \cos\beta(s_1,s_0)\,\mathrm{d}s_1,\, x_2 - \int_{s_0}^{s} \sin\beta(s_1,s_0)\,\mathrm{d}s_1\right)$$

for some $s_0 \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$. Next we define the strip Ω^a by

$$\Omega^a := \{x \in \mathbb{R}^2 :\ \mathrm{dist}(x,\Gamma) < a\},$$

in particular, $\Omega_0^a := \mathbb{R} \times (-a, a)$ corresponds to a straight line for which we use the symbol Γ_0. We assume that

- (d) Ω^a *does not intersect itself*, in particular, $a\|\gamma\|_\infty < 1$ holds for the strip halfwidth of Γ

which ensures that the points of Ω^a can be uniquely parametrized as follows,

$$x(s,u) = \left(\Gamma_1(s) - u\dot\Gamma_2(s), \Gamma_2(s) + u\dot\Gamma_1(s)\right),$$

where $N(s) = (-\dot\Gamma_2(s), \dot\Gamma_1(s))$ is the unit normal vector to Γ at the point s.

The potential 'ditch'

We will deal with Schrödinger operators having an attractive potential supported in Ω^a. To this aim, we consider

- (e) a nonnegative $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-a, a]$

(where $V \geq 0$ is assumed for convenience only) and to define

$$\tilde{V} : \Omega^a \to \mathbb{R}_+, \quad \tilde{V}(x(s,u)) = V(u), \quad \text{and} \quad H_{\Gamma,V} = -\Delta - \tilde{V}(x);$$

in view of assumption (e) the operator domain is $D(-\Delta) = H^2(\mathbb{R}^2)$

It is also useful to introduce the comparison operator on $L^2(\mathbb{R})$,

$$h_V = -\partial_x^2 - V(x)$$

with the domain $H^2(\mathbb{R})$ which has in accordance with (e) a nonempty and finite discrete spectrum such that

$$\epsilon_0 := \inf \sigma_{\text{disc}}(h_V) = \inf \sigma(h_V) \in \big(-\|V\|_\infty, 0 \big),$$

where the ground-state eigenvalue ϵ_0 is simple and the associated eigenfunction $\phi_0 \in H^2(\mathbb{R})$ can be chosen strictly positive

The essential spectrum

The spectrum of $H_{\Gamma,V}$ is easily found when Γ is straight:

$$\sigma(H_{\Gamma_0,V}) = \sigma_{\mathrm{ess}}(H_{\Gamma_0,V}) = [\epsilon_0, \infty)$$

If the ditch is straight outside a compact, or at least asymptotically straight in the sense of (b), the essential spectrum is preserved:

Proposition

Under assumptions (a)–(e) *we have* $\sigma_{\mathrm{ess}}(H_{\Gamma,V}) = [\epsilon_0, \infty)$

Proof idea: If Γ is straight outside a compact, the result is obtained by combination of Weyl's criterion and bracketing. In the other case, one brackets using strip neighborhoods of the 'tails' on Ω^a and passes to the unitarily equivalent operator

$$H_{\pm}^{(j)} = h_V^{\mathrm{N}}(u_1) \otimes (-\partial_s^2)_{\mathrm{N}} + V_\gamma(s,u)$$

$$V_\gamma(s,u) := -\frac{\gamma(s)^2}{4(1+u\gamma(s))^2} + \frac{u\ddot{\gamma}(s)}{2(1+u\gamma(s))^3} - \frac{5}{4}\,\frac{u^2\dot{\gamma}(s)^2}{(1+u\gamma(s))^4}$$

with the effective potential satisfying $V_\gamma(s,u) \to 0$ as $|s| \to \infty$. □

Asymptotic results

The 'hard-wall' and 'leaky-wire' results mentioned in the introduction provide some insight. For instance, $-\Delta - \alpha\delta(x-\Gamma)$ can be obtained as a limit of Schrödinger operators with *suitably scaled regular potentials*,

$$V_\varepsilon : \; V_\varepsilon(u) = \frac{1}{\varepsilon} V\left(\frac{u}{\varepsilon}\right)$$

obtained by scaling of a given V satisfying assumption (e). This was shown in [E-Ichinose'01, loc.cit.] and, in much greater generality, in

J. Behrndt, P. Exner, M. Holzmann, V. Lotoreichik: Approximation of Schrödinger operators with δ-interactions supported on hypersurfaces, *Math. Nachr.* **290** (2017), 1215–1248.

Since this convergence is of *norm-resolvent type* we arrive easily at

Proposition

Consider a non-straight C^2-smooth curve $\Gamma : \mathbb{R} \to \mathbb{R}^2$ such that $|\Gamma(s) - \Gamma(s')| < c|s - s'|$ holds for some $c \in (0,1)$. If the support of its signed curvature γ is noncompact, assume, in addition to (b), that $\gamma(s) = \mathcal{O}(|s|^{-\beta})$ with some $\beta > \frac{5}{4}$ as $|s| \to \infty$. Then $\sigma_{\mathrm{disc}}(H_{\Gamma,V_\varepsilon}) \neq \emptyset$ holds for all ε small enough

Asymptotic results, continued

Consider now a *flat-bottom* waveguide referring to the potential

$$V_{J,0}(u) = V_0 \chi_J(u), \quad V_0 > 0,$$

where χ_J is the indicator function of an interval $J = [-a_1, a_2] \subset [-a_0.a_0]$. Using the fact that Dirichlet condition is the limit of a *high potential wall*,

M. Demuth, M. Krishna: *Determining Spectra in Quantum Theory*, Birkhäuser, Boston 2005

B. Simon: *Functional Integration in Quantum Physics*, 2nd edition, AMS Chelsea, Providence, R.I. 2005

we can easily prove the following result:

Proposition

Suppose that Γ *is not straight and assumptions* (a)–(d) *are satisfied, then the operator* $H_{\Gamma, V_{J,0}}$ *referring to the flat-bottom potential has nonempty discrete spectrum for all* V_0 *large enough*

Birman-Schwinger analysis

This will be our main tool. Given a function V and $z \in \mathbb{C} \setminus \mathbb{R}_+$ we put

$$K_{\Gamma,V}(z) := \tilde{V}^{1/2}(-\Delta - z)^{-1}\tilde{V}^{1/2}$$

with $\tilde{V}$ defined above; we are particularly interested in the negative values of the spectral parameter, $z = -\kappa^2$ with $\kappa > 0$. In view of (e) it is a bounded operator, $L^2(\Omega^a) \to L^2(\Omega^a)$, positive for $z = -\kappa^2$. By *Birman-Schwinger principle* this operator can be used to determine the discrete spectrum of $H_{\Gamma,V}$:

Proposition

$z \in \sigma_{\mathrm{disc}}(H_{\Gamma,V})$ *holds if and only if* $1 \in \sigma_{\mathrm{disc}}(K_{\Gamma,V}(z))$. *The function* $\kappa \mapsto K_{\Gamma,V}(-\kappa^2)$ *is continuous and decreasing in* $(0,\infty)$, *tending to zero in the norm topology, that is,* $\|K_{\Gamma,V}(-\kappa^2)\| \to 0$ *holds as* $\kappa \to \infty$

Birman-Schwinger analysis, continued

Note that if g is an eigenfunction of $K_{\Gamma,V}(-\kappa^2)$ with eigenvalue one, the corresponding eigenfunction of $H_{\Gamma,V}$ is given by

$$\phi(x) = \int_{\mathrm{supp}\,\tilde{V}} G_\kappa(x,x')\,\tilde{V}(x')^{1/2} g(x')\,\mathrm{d}x',$$

where G_κ is the integral kernel of $(-\Delta+\kappa^2)^{-1}$.

Using the knowledge of the Laplacian resolvent we can write the action of $K_{\Gamma,V}(-\kappa^2)$ explicitly: it is an integral operator with the kernel

$$K_{\Gamma,V}(x,x';-\kappa^2) = \frac{1}{2\pi}\,\tilde{V}^{1/2}(x) K_0(\kappa|x-x'|)\,\tilde{V}^{1/2}(x'),$$

where K_0 is the Macdonald function, mapping $L^2(\Omega^a)$ to itself.

In analogy with [E-Ichinose'01, loc.cit.] the idea is to treat the geometry of Ω^a as a perturbation of the straight case

Straightening the strip

First we 'straighten' strip as one does it for the 'hard-wall' waveguides. Passing from the Cartesian coordinates to s, u amounts to a unitary map $L^2(\Omega^a) \to L^2(\Omega_0^a, (1+u\gamma(s))^{1/2}\mathrm{d}s\mathrm{d}u)$; to get rid of the Jacobian, we use the unitary operator

$$L^2(\Omega^a) \to L^2(\Omega_0^a), \quad (U\psi)(s,u) = (1+u\gamma(s))^{1/2}\psi(x(s,u))$$

The operator $K_{\Gamma,V}(-\kappa^2)$ transforms to the unitarily equivalent one, $\mathcal{R}^\kappa_{\Gamma,V} := UK_{\Gamma,V}(-\kappa^2)U^{-1}$, which is an integral operator on $L^2(\Omega_0^a)$ with the kernel

$$\mathcal{R}^\kappa_{\Gamma,V}(s,u;s',u') = \frac{1}{2\pi}\, W(s,u)^{1/2} K_0(\kappa|x-x'|)\, W(s',u')^{1/2},$$

where $x = x(s,u)$, $x' = x(s',u')$, and the modified potential is

$$W(s,u) := (1+u\gamma(s))\, V(u)$$

The straight case

For the straight potential ditch we have

$$\mathcal{R}^{\kappa}_{\Gamma_0,V}(s,u;s',u') = \frac{1}{2\pi}\, V(u)^{1/2} K_0(\kappa|x_0-x_0'|)\, V(u')^{1/2},$$

where $|x_0-x_0'| = \big[(s-s')^2+(u-u')^2\big]^{1/2}$. In the s variable the operator is of convolution type, thus we have

$$(F\otimes I)\mathcal{R}^{\kappa}_{\Gamma_0,V}(F\otimes I)^{-1} = \int_{\mathbb{R}}^{\oplus} \mathcal{R}^{\kappa}_{\Gamma_0,V}(p)\,\mathrm{d}p,$$

where the fibers are integral operator on $L^2(-a,a)$ with the kernels

$$\mathcal{R}^{\kappa}_{\Gamma_0,V}(u,u';p) = V(u)^{1/2}\, \frac{\mathrm{e}^{-\sqrt{\kappa^2+p^2}|u-u'|}}{2\sqrt{\kappa^2+p^2}}\, V(u')^{1/2},$$

and $\mathcal{R}^{\kappa}_{\Gamma_0,V}(p)$ is nothing but the Birman-Schwinger operator associated with h_V referring to the parameter $z=-(\kappa^2+p^2)$

By assumption, $\epsilon_0 = \inf\sigma(h_V)$, and consequently, $-\kappa^2 = \epsilon_0 + p^2$ belongs to the spectrum of $H_{\Gamma_0,V}$ for any $p\in\mathbb{R}$ as we know already

The straight case, continued

At the same time, the operator $\mathcal{R}^{\kappa_0}_{\Gamma_0,V}$ satisfies

$$\sup \sigma(\mathcal{R}^{\kappa_0}_{\Gamma_0,V}) = 1,$$

where $\kappa_0 = \sqrt{-\epsilon_0}$, because otherwise there would be a $\tilde{\kappa} > \kappa_0$ such that $1 \in \sigma(\mathcal{R}^{\tilde{\kappa}}_{\Gamma_0,V})$, and consequently, $-\tilde{\kappa}^2 \in \sigma(H_{\Gamma_0,V})$, however, this would contradict to the already established fact that $\sigma(H_{\Gamma_0,V}) = [\epsilon_0, \infty)$

One can relate relate the eigenfunction ϕ_0 of h_V to the eigenfunction g_0 of $\mathcal{R}^{\kappa}_{\Gamma_0,V}(0)$ corresponding to the unit eigenvalue. On the one hand, we have

$$g_0(u) = V^{1/2}(u)\phi_0(u),$$

on the other hand, one can write the generalized eigenfunction associated with $\inf \sigma(H_{\Gamma_0,V})$ as

$$f_0(s,u) = \phi_0(u) = \int_{-a}^{a} \frac{\mathrm{e}^{-\kappa_0|u-u'|}}{2\kappa_0}\, V(u')^{1/2}\, g_0(u')\,\mathrm{d}u'$$

Existence of bound states

Theorem

Let assumptions (a)–(e) be valid and set

$$\mathcal{C}^{\kappa}_{\Gamma,V}(s,u;s',u')$$
$$= \frac{1}{2\pi}\,\phi_0(u)V(u)\left[(1+u\gamma(s))\,K_0(\kappa|x(s,u)-x(s',u')|)\,(1+u'\gamma(s'))\right.$$
$$\left. -K_0(\kappa|x_0(s,u)-x_0(s',u')|)\right]V(u')\phi_0(u')$$

for all $(s,u),\,(s',u')\in\Omega_0^a$, *then we have* $\sigma_{\rm disc}(H_{\Gamma,V})\neq\emptyset$ *provided*

$$\int_{\mathbb{R}^2}\mathrm{d}s\mathrm{d}s'\int_{-a}^{a}\int_{-a}^{a}\mathrm{d}u\mathrm{d}u'\,\mathcal{C}^{\kappa_0}_{\Gamma,V}(s,u;s',u')>0$$

holds for $\kappa_0=\sqrt{-\epsilon_0}$.

P.E.: Spectral properties of soft quantum waveguides, *J. Phys. A: Math. Theor.* **53** (2020), 355302.

In contrast to the asymptotic results above, this one has a quantitative character

Bound state existence, proof sketch

The idea is to treat the geometry of the system, translated into the coefficients of the operator, as a perturbation of the straight case. As the essential spectrum is preserved, it is enough to find $\psi_\eta \in L^2(\Omega_0^a)$ such that

$$(\psi, \mathcal{R}_{\Gamma,V}^{\kappa_0}\psi) - \|\psi\|^2 > 0. \tag{1}$$

A trial function combines the generalized eigenfunction, associated with the edge of the spectrum, with a mollifier which makes it an element of the Hilbert space. Inspect first the effect of the mollifier for $\Gamma = \Gamma_0$:

Lemma

Let $\psi_\eta \in L^2(\Omega_0^a)$ be of the form $\psi_\eta(s,u) = h_\eta(s)g_0(u)$ with $h_\eta(s) = h(\eta s)$, where $h \in C_0^\infty(\mathbb{R})$ and $h(s) = 1$ in the vicinity of $s = 0$. Then

$$(\psi_\eta, \mathcal{R}_{\Gamma_0,V}^{\kappa_0}\psi_\eta) - \|\psi_\eta\|^2 = \mathcal{O}(\eta) \quad \text{as } \eta \to 0.$$

Bound state existence, proof sketch

We can rewrite the expression to be estimated into the form

$$\int_{-a}^{a}\int_{-a}^{a} g_0(u)V(u)^{1/2}\Big[\int_{\mathbb{R}} |\hat{h}_\eta(p)|^2\, \frac{\mathrm{e}^{-\sqrt{\kappa_0^2+p^2}|u-u'|}}{2\sqrt{\kappa_0^2+p^2}}\,\mathrm{d}p - \|h_\eta\|^2\frac{\mathrm{e}^{-\kappa_0|u-u'|}}{2\kappa_0}\Big]$$

$$\times V(u')^{1/2}g_0(u')\,\mathrm{d}u\mathrm{d}u',$$

and it is enough to check that the square bracket is $\mathcal{O}(\eta)$ as $\eta\to 0$.

We have $\hat{h}_\eta(p)=\frac{1}{\eta}\,\hat{h}\big(\frac{p}{\eta}\big)$, which allows us to rewrite the first term as

$$\frac{1}{\eta}\int_{\mathbb{R}} |\hat{h}(\zeta)|^2\, \frac{\mathrm{e}^{-\sqrt{\kappa_0^2+\eta^2\zeta^2}|u-u'|}}{2\sqrt{\kappa_0^2+\eta^2\zeta^2}}\,\mathrm{d}\zeta == \frac{1}{\eta}\Big(\frac{\mathrm{e}^{-\kappa_0|u-u'|}}{2\kappa_0}+\mathcal{O}(\eta^2)\Big),$$

and using further the relation $\|h_\eta\|^2=\frac{1}{\eta}\,\|h\|^2$ we prove the lemma.

Bound state existence, proof concluded

Consider now the difference of the Birman-Schwinger operators

$$\mathcal{D}^{\kappa}_{\Gamma,V} := \mathcal{R}^{\kappa}_{\Gamma,V} - \mathcal{R}^{\kappa}_{\Gamma_0,V}$$

which is an integral operator with the kernel

$$\mathcal{D}^{\kappa}_{\Gamma,V}(s,u;s',u') = \frac{1}{2\pi}\Big(W(s,u)^{1/2} K_0(\kappa|x(s,u)-x(s',u')|)\, W(s',u')^{1/2} \\ - V(u)^{1/2} K_0(\kappa|x_0(s,u)-x_0(s',u')|)\, V(u')^{1/2}\Big)$$

By BS principle, a bound state existence requires $\sup\sigma(\mathcal{R}^{\kappa_0}_{\Gamma_0,V}) > 1$, and that happens if

$$\lim_{\eta\to 0}(\psi_\eta, \mathcal{D}^{\kappa_0}_{\Gamma,V}\psi_\eta) > 0.$$

Using the choice of the function h_η, this is equivalent to

$$\int_{\mathbb{R}^2} \mathrm{d}s\mathrm{d}s' \int_{-a}^{a}\int_{-a}^{a} \mathrm{d}u\mathrm{d}u'\, g_0(u)\, \mathcal{D}^{\kappa_0}_{\Gamma,V}(s,u;s',u')\, g_0(u') > 0,$$

which is nothing else than the condition stated in the theorem. □

Distances involved

To use the theorem one has to compare point distances in the straight strip,

$$|x_0(s,u) - x_0(s',u')| = \left[(s-s')^2 + (u-u')^2\right]^{1/2}$$

with those is the curved one,

$$\begin{aligned} &|x(s,u) - x(s',u')|^2 \\ &= |\Gamma(s) - \Gamma(s')|^2 + u^2 + u'^2 - 2uu' \cos\beta(s,s') + 2(u\cos\beta(s,s') - u') \int_{s'}^{s} \sin\beta(\xi,s')\,\mathrm{d}\xi, \end{aligned}$$

where the first term on the right-hand side equals

$$|\Gamma(s) - \Gamma(s')|^2 = \int_{s'}^{s} \int_{s'}^{s} \cos\beta(\xi,\xi')\,\mathrm{d}\xi\,\mathrm{d}\xi'.$$

Note that $|\Gamma(s) - \Gamma(s')| < |\Gamma_0(s) - \Gamma_0(s')| = |s - s'|$ holds if the bend is nontrivial

This property was decisive in the leaky wire case, [E-Ichinose'01, loc.cit.].

One more existence result

Proposition

Let $\mathcal{V}_{\epsilon_0}$ be the family of potentials V satisfying assumptions (d), (e), and $\inf \sigma(h_V) \le \epsilon_0$. *Then to any $\epsilon_0 > 0$ there exists an $a_0 = a_0(\epsilon_0)$ such that* $\sigma_{\rm disc}(H_{\Gamma,V}) \neq \emptyset$ *holds for all $V \in \mathcal{V}_{\epsilon_0}$ with* $\operatorname{supp} V \subset [-a_0, a_0]$.

It is sufficient to consider $\inf \sigma(h_V) = \epsilon_0$ since the family $\{h_{\lambda V} :\; \lambda > 0\}$ is monotonous with the same essential spectrum. $\sigma_{\rm disc}(H_{\Gamma,V}) \neq \emptyset$ hold if

$$\frac{1}{2\pi} \int_{-a}^{a} \int_{-a}^{a} \phi_0(u) V(u)\, F(u,u')\, V(u')\phi_0(u)\, \mathrm{d}u \mathrm{d}u' > 0,$$

where

$$F(u,u') := \int_{\mathbb{R}^2} \left[(1+u\gamma(s))\, K_0(\kappa_0 |x(s,u) - x(s',u')|)\, (1+u'\gamma(s')) - K_0(\kappa_0 |x_0(s,u) - x_0(s',u')|) \right] \mathrm{d}s \mathrm{d}s'$$

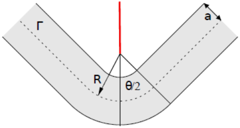
The function $F(\cdot,\cdot)$ is well defined, continuous, and we have

$$F(0,0) = \int_{\mathbb{R}^2} \left[K_0(\kappa_0 |\Gamma(s) - \Gamma(s')|) - K_0(\kappa_0 |s-s'|) \right] \mathrm{d}s \mathrm{d}s' > 0.$$

By continuity there is a neighborhood $(-a_0, a_0) \times (-a_0, a_0)$ of $(0,0)$ on which $F(u,u')$ is positive, and that in combination with the positivity of $\phi_0 V$ concludes the proof. □

Comments

Birman-Schwinger principle is not the only tool available; a natural alternative is to employ a *variational method*. In this way the bound state existence was proved for *bookcover-shaped* potential ditches



Source: the cited paper

S. Kondej, D. Krejčiřík, J. Kříž: Soft quantum waveguides with a explicit cut locus, arXiv:2007.10946

This is not the end of the story, many questions remain open, for instance

- The existence of bound states in *polygonal channels*. There is such a result obtained variationally for *crossed channels* but this fact alone does not allow us to make conclusions about a single broken channel.

  S. Egger, J. Kerner, K. Pankrashkin: Bound states of a pair of particles on the half-line with a general interaction potential, *J. Spect. Theory*, to appear; arXiv:1812.06500

More problems

- Tubular potential channels *in three dimensions*. One expects the validity of asymptotic results similar to those discussed above. If the channel profile lacks the rotational symmetry with respect to its axis Γ, one expects additional effects coming from the channel *torsion* giving rise repulsion in analogy with

  T. Ekholm, H. Kovařík, D. Krejčiřík: A Hardy inequality in twisted waveguides, *Arch. Rat. Mech. Anal.* **188**(2008), 245–264.

- *Local perturbations* of potential channels, coming from variation either of their depth or width. This is easy if such a perturbation is 'sign-definite', in general it may be harder.
- Potential channels of a more complicated geometry, in first place *branched ones* built over a metric graph. Of course, to have the problem well defined one must specify the potential in the vicinity of the graph vertices because the spectrum would depend on it.
- One can ask about the *number of eigenvalues* and their properties in dependence on the system geometry. Of particular interest are the *weakly bound states* corresponding to mild geometric perturbations.

More problems

- Another question concerns *scattering* in a bent or locally perturbed potential channel including possible *resonance effects* in narrow and sufficiently deep channels.
- Another extension to three dimensions concerns *potential layers*, that is potentials of a fixed transverse profile built over an infinite surface Σ in $\mathbb{R}^3$. One can again establish the discrete spectrum existence for potential layers with the profile deep enough, while in the regime different from the asymptotic one, the question is open.
- For layers the spectrum may depend on the *global* geometry of the interaction support. An example of a *conical* potential layer was found recently, properties of more general layers are of interest.

S. Egger, J. Kerner, K. Pankrashkin: Discrete spectrum of Schrödinger operators with potentials concentrated near conical surfaces, *Lett. Math. Phys.* **110** (2020), 945–968.

More problems

- Another question concerns the influence of *external fields*. In a two-dimensional hard-wall tube we have a Hardy-type inequality that prevents the existence of weakly bound states; the question is whether this extends to soft waveguides. On the other hand, with a homogeneous magnetic field, the question is about *stability of edge currents* with respect to various perturbations.

T. Ekholm, H. Kovařík: Stability of the magnetic Schrödinger operator in a waveguide, *Comm. PDE* **30** (2005), 539–565.

- In *periodic waveguides* one is interested primarily in the absolute continuity of the spectrum and the existence of spectral gaps. The latter would be for sure true with profiles deep an narrow enough, the former remain an open problem even for periodic leaky wires.
- A completely new area opens when we consider a system of *many particles* interacting mutually, for instance, due the charges they carry, confined in a soft waveguide.

One more problem

Another question one may pose concerns the *spectral optimization* in analogy with what is known in Dirichlet and δ potential cases

P.E., E.M. Harrell, M. Loss: Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature, in *Mathematical Results in Quantum Mechanics*, Birkhäuser, Basel 1999; pp. 47–53.

P.E., E.M. Harrell, M. Loss: Inequalities for means of chords, with application to isoperimetric problems, *Lett. Math. Phys.* **75** (2006), 225–233; addendum **77** (2006), 219.

Let Γ be a C^2-smooth *loop* without self-intersections of a *fixed length L*. For small enough positive $d_\pm$ the map $[0,L)\times\mathcal{J}\ni(s,u)\mapsto\Gamma(s)+u\nu(s)$, where $\mathcal{J}=[-d_-,d_+]$ and $\nu=(-\dot\Gamma_2,\dot\Gamma_1)$ is the normal to Γ, is *bijective*.

We consider operators $H_{\gamma,\mu}$ corresponding the measure-type interaction

$$\mu(M):=\int_0^L\int_{-d_-}^{d_+}\chi_M\big(\Gamma(s)+u\nu(s)\big)\,(1+u\gamma(s))\,\mathrm{d}\mu_\perp(t)\mathrm{d}s,$$

where the positive transverse measure $\mu_\perp$ describes either a regular attractive potential channel, $\mu_\perp(u)=V(u)\mathrm{d}u$, or a δ potential.

Shape optimization

We define $H_{\Gamma,\mu}$ as the self-adjoint operator associated with the form

$$h_{\Gamma,\mu}[\psi] := \|\nabla\psi\|^2 - \int_{\mathbb{R}^2} |\psi|^2 \mathrm{d}\mu, \quad \mathrm{dom}\, h_{\Gamma,\mu} = H^1(\mathbb{R}^2).$$

It is not difficult to check that the essential spectrum of $H_{\Gamma,\mu}$ is $[0,\infty)$ and $\sigma_{\mathrm{disc}}(H_{\Gamma,\mu}) \neq \emptyset$. Let $\mathcal{C}$ be a *circle of radius* $\frac{L}{2\pi}$. By $\mu_\circ$ we denote the corresponding measure generated by $\mu_\perp$ and giving rise to operator $H_{\Gamma,\mu_\circ}$.

Theorem

The lowest eigenvalues $\lambda_1(\mu)$ and $\lambda_1(\mu_\circ)$, respectively, of $H_{\Gamma,\mu}$ and of $H_{\Gamma,\mu_\circ}$ satisfy the inequality

$$\lambda_1(\mu) \leq \lambda_1(\mu_\circ).$$

We *conjecture* that the inequality is strict unless Γ and $\mathcal{C}$ are congruent.

Another optimization result

The claim follows by a simple *variational argument*: the appropriate trial function is obtained using the lowest eigenfunction of $H_{\Gamma,\mu_\circ}$ and *'transplanting'* it to the parallel coordinates.

P.E., V. Lotoreichik: Optimization of the lowest eigenvalue of a soft quantum ring, arXiv:2011.02257 [math-ph]

One can also optimize with respect to the *channel profile*:

Theorem

Put $\alpha := \mu_\perp(\mathcal{J})$ and consider Schrödinger operators $H_{\Gamma,\alpha\delta_u}$, then the lowest eigenvalues $\lambda_1(\mu)$ and $\lambda_1(\alpha\delta_u)$ of $H_{\Gamma,\mu}$ and of $H_{\Gamma,\alpha\delta_u}$, respectively, satisfy the inequality

$$\lambda_1(\mu) \geq \min_{u\in\mathcal{J}} \lambda_1(\alpha\delta_u).$$

This is again easy to prove variationally; one has to check that the function $\mathcal{J} \ni t \mapsto \|\psi|_u\|^2$ is continuous so that it attains its maximum value at some $t_\star = t_\star(\mu) \in \mathcal{J}$.

It remains to say

Спасибо за внимание!