Dirichlet networks squeezed to graphs, with a bent tube example

Pavel Exner

in collaboration with *Claudio Cacciapuoti*

exner@ujf.cas.cz

Doppler Institute

for Mathematical Physics and Applied Mathematics

Prague

Talk overview

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- The current knowledge about the Dirichlet case
- How to get stronger results – playing around threshold resonances
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- Summary and outlook

Quantum graph concept

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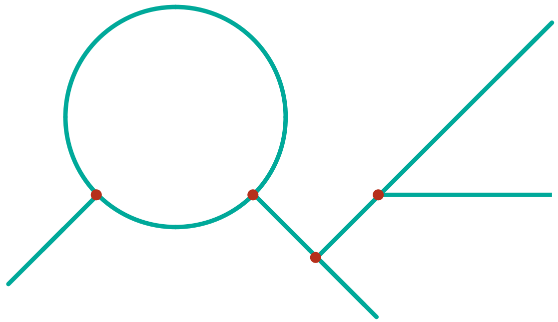
The concept extends, however, to graphs of *arbitrary shape*



Hamiltonian: $-\frac{\partial^2}{\partial x_j^2} + v(x_j)$
on graph edges,
boundary conditions at vertices

and what is important, it became *practically important* after experimentalists learned in the last two decades to fabricate tiny graph-like structure for which this is a good model

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- Moreover, from the stationary point of view a quantum graph is also equivalent to a *microwave network* built of optical cables – see [Hul et al.'04]
- In addition one can consider *generalized graphs* which consist of components of different dimensions
- Now when the microstructures reach *molecular size* quantum graphs "return" in a sense to their origin!

More remarks

- The vertex coupling is chosen to make the Hamiltonian self-adjoint, or in physical terms, to ensure *probability current conservation*. This is achieved by the method based on s-a extensions which everybody in this audience knows (at least I suppose so)

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- Graphs can support also *Dirac operators*, see [Bulla-Trenckler'90], [Bolte-Harrison'03], and also recent applications to *graphene* and its derivates

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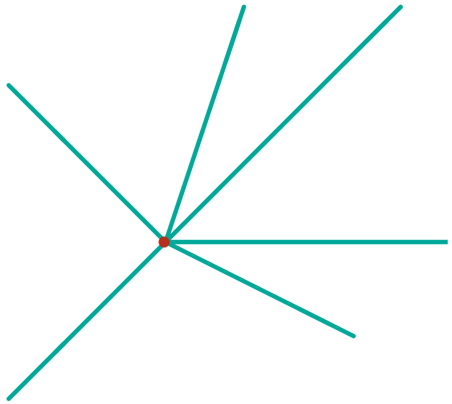
- Graphs can support also *Dirac operators*, see [Bulla-Trenckler'90], [Bolte-Harrison'03], and also recent applications to *graphene* and its derivates
- The graph literature is extensive; recall just a review [Kuchment'04], proceedings of Snowbird'05 conference, and the recent AGA Programme at INI Cambridge

Vertex coupling



The most simple example is a *star graph* with the state Hilbert space $\mathcal{H} = \bigoplus_{j=1}^n L^2(\mathbb{R}_+)$ and the particle Hamiltonian acting on $\mathcal{H}$ as $\psi_j \mapsto -\psi_j''$

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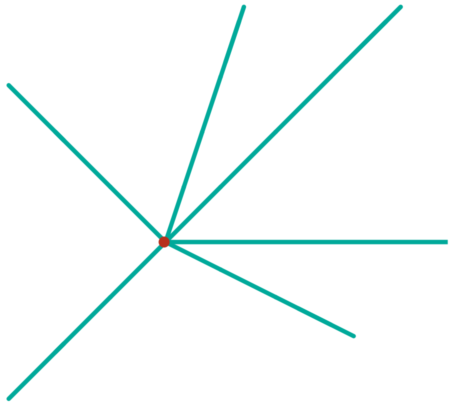

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Since it is second-order, the boundary condition involve $\Psi(0) := \{\psi_j(0)\}$ and $\Psi'(0) := \{\psi_j'(0)\}$ being of the form

$$A\Psi(0) + B\Psi'(0) = 0\,;$$

by [Kostrykin-Schrader'99] the $n \times n$ matrices A, B give rise to a self-adjoint operator if they satisfy the conditions

- $\operatorname{rank}(A, B) = n$
- AB^* is self-adjoint

Unique boundary conditions

The non-uniqueness of the above b.c. can be removed:

Proposition [Harmer'00, K-S'00]: Vertex couplings are uniquely characterized by unitary $n \times n$ matrices U such that

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One can derive them modifying the argument used in [Fülöp-Tsutsui'00] for generalized point interactions, $n = 2$

Self-adjointness requires vanishing of the boundary form,

$$\sum_{j=1}^{n} (\bar\psi_j \psi_j' - \bar\psi_j' \psi_j)(0) = 0\,,$$

which occurs *iff* the norms $\|\Psi(0) \pm i\ell\Psi'(0)\|_{\mathbb{C}^n}$ with a fixed $\ell \neq 0$ coincide, so the vectors must be related by an $n \times n$ unitary matrix; this gives $(U - I)\Psi(0) + i\ell(U + I)\Psi'(0) = 0$

Remarks

- The length parameter is not important because matrices corresponding to two different values are related by

$$U' = \frac{(\ell + \ell')U + \ell - \ell'}{(\ell - \ell')U + \ell + \ell'}$$

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- or the *on-shell scattering matrix* for a star graph of n halflines with the considered coupling which equals

$$S_U(k) = \frac{(k-1)I + (k+1)U}{(k+1)I + (k-1)U}$$

Examples of vertex coupling

- Denote by $\mathcal{J}$ the $n \times n$ matrix whose all entries are equal to one; then $U = \frac{2}{n+i\alpha}\mathcal{J} - I$ corresponds to the standard δ *coupling*,

$$\psi_j(0) = \psi_k(0) =: \psi(0)\,, \;\; j,k = 1,\dots,n\,, \;\; \sum_{j=1}^{n} \psi_j'(0) = \alpha\psi(0)$$

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- $\alpha = 0$ corresponds to the "free motion", the so-called *free boundary conditions* (better name than Kirchhoff)
- Similarly, $U = I - \frac{2}{n-i\beta}\mathcal{J}$ describes the *δ_s' coupling*

$$\psi_j'(0) = \psi_k'(0) =: \psi'(0)\,,\; j,k = 1,\dots,n\,,\; \sum_{j=1}^{n} \psi_j(0) = \beta\psi'(0)$$

with $\beta \in \mathbb{R}$; for $\beta = \infty$ we get *Neumann* decoupling

Why are vertices interesting?

Apart of a general mathematical interest, there are specific reasons related to various use of such models, e.g.

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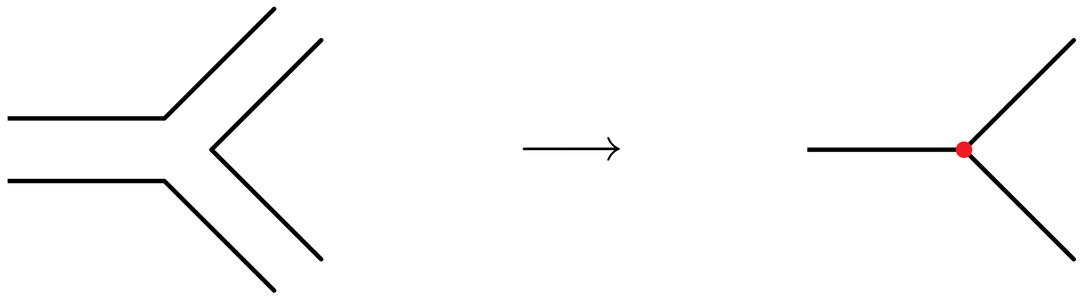
- A nontrivial vertex coupling can lead to *number theoretic properties* of graph spectrum [E'96]
- On more practical side, the conductivity of graph nanostructures is controlled typically by external fields, vertex coupling can serve the same purpose
- In particular, the generalized point interaction has been proposed as a way to realize a *qubit* [Cheon-Tsutsui-Fülöp'04]; vertices with $n > 2$ can similarly model *qudits*, etc.

Clearly, understanding of vertex couplings is needed to model physical systems by graphs and to make use of such models

A natural approach: “fat graphs”

Take a more “realistic” situation without ambiguities, such as a network of *branching tubes* and analyze its *squeezing limit*:



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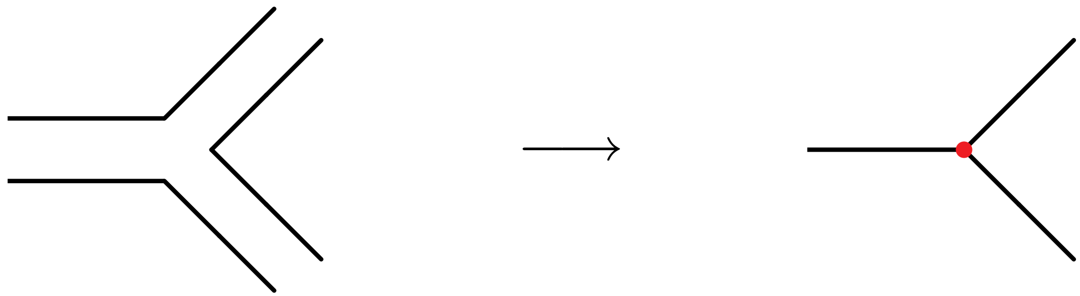

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- after a long effort the *Neumann-like case* is understood, see [Freidlin-Wentzell’93], [Freidlin’96], [Saito’01], [Kuchment-Zeng’01], [Rubinstein-Schatzmann’01], [E.-Post’05, 07], [Post’06], giving basically free b.c. only
- recently a progress achieve in the *Dirichlet case* [Post’05], [Molchanov-Vainberg’07], [Griesser’07unpub]. *This is our main topic here*

A "cheaper" alternative

One can employ approximation on the graph itself:

- δ on a star Γ through potentials with *a shrinking component*, $W_{\varepsilon,j} := \frac{1}{\varepsilon} W_j\left(\frac{x}{\varepsilon}\right)$. If they belong to $L^1(\mathbb{R}_+)$ we have

$$H_0(V + W_\varepsilon) \longrightarrow H_\alpha(V)$$

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- δ_s' coupling using additional δ’s scaled in a nonlinear way – CS technique, see [Cheon-E’04] – with extension to $2n$-parameter families [E-Turek’07]
- using extra edges and vertices properly scaled, see [E-Turek’07], one can get the $\binom{n+1}{2}$-parameter family of *time-reversal-invariant* couplings

Neumann case survey: first, the graphs

The simplest situation in [KZ'01, EP'05] (weights left out)

Let M_0 be a finite connected graph with vertices v_k, $k \in K$ and edges $e_j \simeq I_j := [0, \ell_j]$, $j \in J$; the state Hilbert space is

$$L^2(M_0) := \bigoplus_{j \in J} L^2(I_j)$$

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The form $u \mapsto \|u'\|^2_{M_0} := \sum_{j \in J} \|u'\|^2_{I_j}$ with $u \in \mathcal{H}^1(M_0)$ is associated with the operator which acts as $-\Delta_{M_0} u = -u''_j$ and satisfies free b.c.,

$$\sum_{j,\, e_j \text{ meets } v_k} u'_j(v_k) = 0$$

On the other hand, Laplacian on manifold

Consider a Riemannian manifold X of dimension $d \geq 2$ and the corresponding space $L^2(X)$ w.r.t. volume $\mathrm{d}X$ equal to $(\det g)^{1/2}\mathrm{d}x$ in a fixed chart. For $u \in C^\infty_{\mathrm{comp}}(X)$ we set

$$q_X(u) := \|\mathrm{d}u\|^2_X = \int_X |\mathrm{d}u|^2 \mathrm{d}X\,, \;\; |\mathrm{d}u|^2 = \sum_{i,j} g^{ij} \partial_i u\, \partial_j \overline{u}$$

The closure of this form is associated with the s-a operator Δ_X which acts in fixed chart coordinates as

$$\Delta_X u = -(\det g)^{-1/2} \sum_{i,j} \partial_i((\det g)^{1/2} g^{ij}\, \partial_j u)$$

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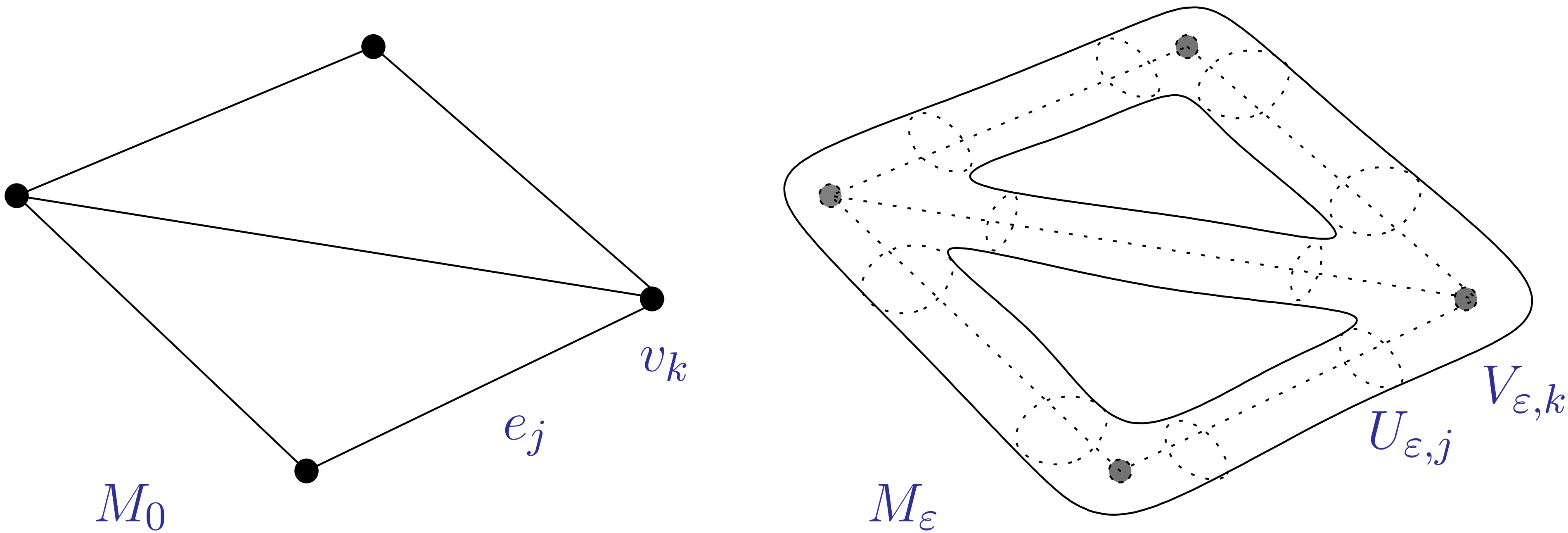
If X is compact with piecewise smooth boundary, one starts from the form defined on $C^\infty(X)$. This yields Δ_X as the *Neumann* Laplacian on X and allows us to treat “fat graphs” and “sleeves” on the same footing

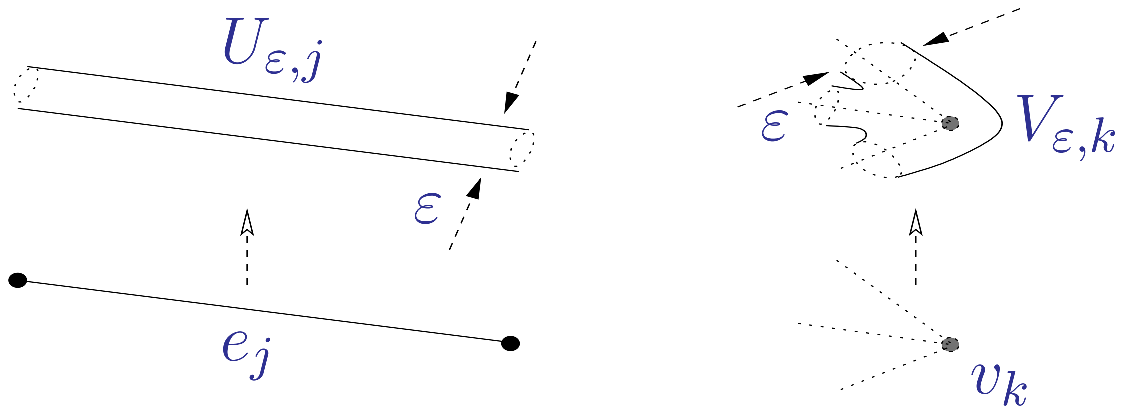
Relating the two together

We associate with the graph M_0 a family of manifolds M_ε



We suppose that M_ε is a union of compact edge and vertex components $U_{\varepsilon,j}$ and $V_{\varepsilon,k}$ such that their interiors are mutually disjoint for all possible $j \in J$ and $k \in K$

Manifold building blocks



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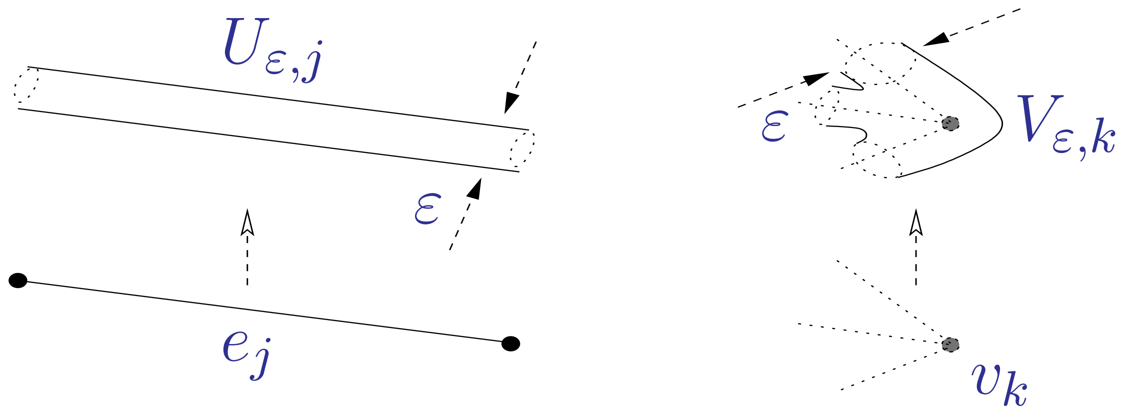

However, M_ε *need not be embedded* in some $\mathbb{R}^d$.

It is convenient to assume that $U_{\varepsilon,j}$ and $V_{\varepsilon,k}$ depend on ε only through their metric:

- for edge regions we assume that $U_{\varepsilon,j}$ is diffeomorphic to $I_j \times F$ where F is a compact and connected manifold (with or without a boundary) of dimension $m := d-1$
- for vertex regions we assume that the manifold $V_{\varepsilon,k}$ is diffeomorphic to an ε-independent manifold V_k

Eigenvalue convergence

Let thus $U = I_j \times F$ with metric g_ε, where cross section F is a compact connected Riemannian manifold of dimension $m = d - 1$ with metric h; we assume that $\operatorname{vol} F = 1$. We define another metric $\tilde{g}_\varepsilon$ on $U_{\varepsilon,j}$ by

$$\tilde{g}_\varepsilon := \mathrm{d}x^2 + \varepsilon^2 h(y) \,;$$

the two metrics coincide up to an $\mathcal{O}(\varepsilon)$ error

This property allows us to treat manifolds embedded in $\mathbb{R}^d$ (with metric $\tilde{g}_\varepsilon$) using product metric g_ε on the edges

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The sought result now looks as follows.

Theorem [KZ'01, EP'05]: Under the stated assumptions $\lambda_k(M_\varepsilon) \to \lambda_k(M_0)$ as $\varepsilon \to 0$ (giving thus free b.c.!)

A stronger convergence

The b.c. are not the only problem. The ev convergence for finite graphs is rather weak. Fortunately, one can do better.

Theorem [Post'06]: Let M_ε be graphlike manifolds associated with a metric graph M_0, *not necessarily finite*. Under some natural uniformity conditions, $\Delta_{M_\varepsilon} \to \Delta_{M_0}$ as $\varepsilon \to 0+$ in the *norm-resolvent sense* (with suitable identification), in particular, the σ_{disc} and σ_{ess} converge uniformly in an bounded interval, and ef's converge as well.

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The *natural uniformity conditions* mean (i) existence of nontrivial bounds on vertex degrees and volumes, edge lengths, and the second Neumann eigenvalues at vertices, (ii) appropriate scaling (analogous to the described above) of the metrics at the edges and vertices.

Proof is based on an abstract convergence result.

Convergence of resonances

In a similar way we can treat convergence of resonances. As a motivating example one can think of a *“fat lasso”* graph, with the ε-squeezing setting the same as before:

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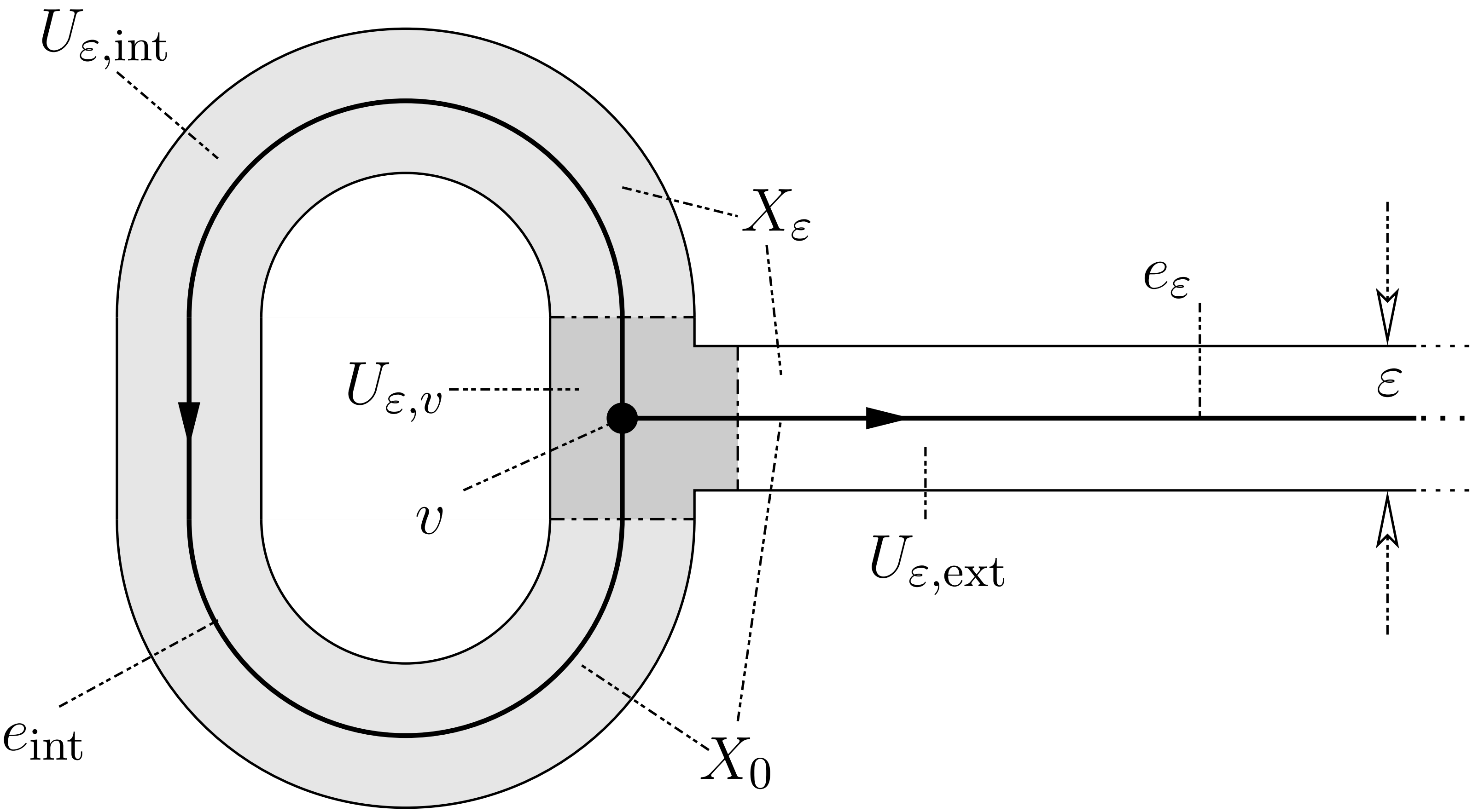

Convergence of resonances, continued

Let H_0, with free b.c., and H_ε will be as above. We use an *exterior complex scaling* extending to complex θ the map

$$U^\theta f := (\det D\Phi^\theta)^{1/2}(f \circ \Phi^\theta)$$

where $\Phi_e^\theta(x) := \mathrm{e}^\theta x$ on external edges, and $(\det D\Phi^\theta)^{1/2}$ equals one and $\mathrm{e}^{\theta/2}$, respectively, on $X_{0,\mathrm{int}}$ and $X_{0,\mathrm{ext}}$.

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Theorem [E.-Post'07]: Let $\lambda(0)$ be a resonance of H_0 of multiplicity $m > 0$, then for small enough $\varepsilon > 0$ there is m resonances $\lambda_1(\varepsilon), \dots, \lambda_m(\varepsilon)$ of H_ε, not necessarily distinct, which all converge to $\lambda(0)$ as $\varepsilon \to 0$. The same is true for embedded ev's of H_0, when $\operatorname{Im}\lambda_j(\varepsilon) \le 0$ holds in general.

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Remarks: (i) The above Φ^θ can have a shifted discontinuity, or be replaced by a smooth flow, with the same result
(ii) The result persists if a magnetic field is added

Back to the Dirichlet case

- *Generically* it is expected that that the limit *with the energy around the threshold* gives *Dirichlet decoupling*, but there may be *exceptional cases*

Back to the Dirichlet case

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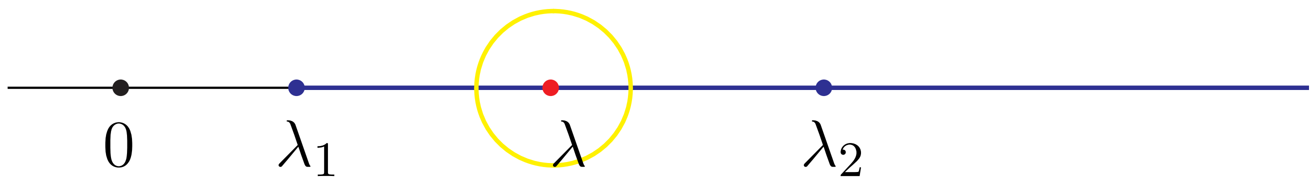
Back to the Dirichlet case

- *Generically* it is expected that that the limit *with the energy around the threshold* gives *Dirichlet decoupling*, but there may be *exceptional cases*
- if the vertex regions *squeeze faster* than the “tubes” one gets *Dirichlet decoupling* [Post’05]
- on the other hand, if you blow up the spectrum for a fixed point *separated from thresholds*, i.e.



one gets a nontrivial limit with b.c. fixed by scattering on the “fat star” [Molchanov-Vainberg’07]

More on the threshold case

As said above, generically the squeezing limit with renormalization w.r.t. the spectral threshold leads to *Dirichlet decoupled* graph – see [Molchanov-Vainberg'07] or [Dell'Antonio-Tenuta'07] for a particular case of a broken line – which is not very interesting

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A nontrivial limit may arise in the (non-generic) case when the system has a *threshold resonance*. On a general level, the claim was made by [Grieser'07], however, details are not available at the moment

On the other hand, [Albeverio-Cacciapuoti-Finco'07] provide a worked out example, the *star graph with* $n=2$. In the limit they get the family of *scale-invariant point interactions* on the line satisfying the b.c.

$$f(0+) = \alpha f(0-)\,, \quad f'(0+) = \alpha^{-1} f(0-)\,, \quad \alpha \neq 0$$

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Our main point is to show how one can get squeezing limits able to produce nonvoid discrete spectrum or resonances. The procedure we propose consists of two steps:

- choose a network collapsing to a graph in such a way that the limit Hamiltonian has a *threshold resonance*
- change the scaling properties of the vertex region slightly, typically by adding *higher order terms* in the scaling parameter

Remarks: (i) The original network Hamiltonian at that may or may not have such a resonance, depending on the limiting procedure used. (ii) The approximation follows the scheme which one uses interpreting pseudopotentials, or point interactions in dimensions two and three

How to implement the programme?

The modification of scaling properties can be achieved in various ways, e.g., one can *“wiggle” the edges angles* or *scale the vertex region* at a rate which differs from that of the “edge tubes” by a higher order term, a combination of such perturbations, etc.

Incidentally, the same effect can also be obtained by introducing *suitable potentials* into the vertex region, but a purely geometric way is probably the most interesting

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Of course, the above proposal just suggests how to *formulate a conjecture* which can be subsequently *proved* in each particular case.

We are going to do that in the simplest nontrivial case, i.e. in the setting of the paper [Albeverio-Cacciapuoti-Finco’07] mentioned above

Bent waveguide

We will consider a result of “fattening” a smooth curve $C := \{(x,y) \in \mathbb{R}^2 |\, x = \gamma_1(s),\, y = \gamma_2(s),\, s \in \mathbb{R}\}$ parameterized by its arc length, $\gamma_1'^2 + \gamma_2'^2 = 1$. As usual denote by we introduce the *signed curvature* of C,

$$\gamma(s) := \gamma_2'(s)\gamma_1''(s) - \gamma_1'(s)\gamma_2''(s)\,;$$

we suppose that the curve C is not self-intersecting and γ is compactly supported, so that C consists of two straight half lines joined by a smooth curve. In particular, the *bending angle* of C is $\theta = \int_{\mathbb{R}} \gamma(s)\, ds$.

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$$\Omega := \{(x, y) \in \mathbb{R}^2 | \, x = \gamma_1(s) - u\gamma_2'(s), y = \gamma_2(s) + u\gamma_1'(s), s \in \mathbb{R}, u \in (-d, d)\} \, ;$$

we assume that $d\|\gamma\|_\infty < 1$. The Dirichlet Laplacian $-\Delta_\Omega^D$ on Ω with $\mathcal{D}(L_0) := C_0^\infty(\Omega)$ is defined as usual

Scaling and "wiggling" of Ω

We assume now that the strip changes its shape and width as a function of $\varepsilon \in (0, 1]$ according to

$$\gamma_\varepsilon(s) := \frac{\sqrt{\lambda(\varepsilon)}}{\varepsilon}\gamma\left(\frac{s}{\varepsilon}\right), \quad d_\varepsilon := \varepsilon^\alpha d\,, \quad \text{with } \alpha > 1\,,$$

where $\lambda(\varepsilon)$ is a fixed function to be specified; it replaces $\lambda(\varepsilon) = 1$ used in [Albeverio-Cacciapuoti-Finco'07].

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where $\lambda(\varepsilon)$ is a fixed function to be specified; it replaces $\lambda(\varepsilon) = 1$ used in [Albeverio-Cacciapuoti-Finco'07]. We choose it to be real, positive and analytic near the origin, with the expansion

$$\lambda(\varepsilon) = 1 + \lambda_1\varepsilon + \mathcal{O}(\varepsilon^2)\,.$$

It means that the shape changes slightly with respect to ε, in particular, the bending angle of the strip Ω_ε is

$$\theta_\varepsilon = \int_{\mathbb{R}} \gamma_\varepsilon(s)\,ds = \theta\sqrt{\lambda(\varepsilon)} = \theta\Big(1 + \frac{1}{2}\lambda_1\varepsilon\Big) + \mathcal{O}(\varepsilon^2)\,.$$

“Straightening” of Ω_ε

Conventionally we pass to an operator on $\Omega' = \mathbb{R} \times (-d, d)$:

Proposition: Let γ be piecewise C^2 and compactly supported with γ', γ'' are bounded. Then $-\Delta^D_{\Omega_\varepsilon}$ is unitarily equivalent to the closure of $H_{0\varepsilon}$ acting on $L^2(\Omega')$ as

$$H_{0\varepsilon} = -\frac{\partial}{\partial s} \frac{1}{(1+\varepsilon^{\alpha-1} u \sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^2} \frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2\alpha}} \frac{\partial^2}{\partial u^2} + \frac{1}{\varepsilon^2} V_\varepsilon(s,u)$$

with $V_\varepsilon(s,u)$ in the effective potential given by

$$\begin{aligned} V_\varepsilon(s,u) \;&=\; -\frac{\lambda(\varepsilon)\gamma(s/\varepsilon)^2}{4(1+\varepsilon^{\alpha-1} u \sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^2} + \frac{\varepsilon^{\alpha-1} u \sqrt{\lambda(\varepsilon)}\gamma''(s/\varepsilon)}{2(1+\varepsilon^{\alpha-1} u \sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^3} \\ &\quad -\frac{5}{4} \frac{\varepsilon^{2\alpha-2} u^2 \lambda(\varepsilon)\gamma'(s/\varepsilon)^2}{(1+\varepsilon^{\alpha-1} u \sqrt{\lambda(\varepsilon)}\gamma(s/\varepsilon))^4} \end{aligned}$$

and $\mathcal{D}(H_{0,\varepsilon}) = \{\psi \in L^2(\Omega') \,|\, \psi \in C^\infty(\Omega')\,,\ \psi(s, \pm d) = 0\,,\ H_{0\varepsilon}\psi \in L^2(\Omega')\}$.

More preliminaries

We will employ transverse modes, i.e. the normalized functions $\phi_n(u)$ solving $-\varepsilon^{-2\alpha}\phi_n''(u) = E_{\varepsilon,n}\phi_n(u)$ with the boundary conditions $\phi_n(\varepsilon^\alpha d) = \phi_n(-\varepsilon^\alpha d) = 0$. The corresponding eigenvalues $E_{\varepsilon,n}$ are

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The resolvent of H_ε admits an integral representation with the kernel $(H_\varepsilon - z)^{-1}(s,u,s',u')$ for every $z \in \rho(H_\varepsilon)$. We define the projection on the normal-modes eigenspaces as

$$\overline{R}^{\varepsilon}_{n,m}(k^2, s, s') := \int_{-d}^{d} du\, du'\, \phi_n(u)(H_\varepsilon - k^2 - E_{\varepsilon,m})^{-1}(s,u,s',u')\phi_m(u'$$

The operators $\overline{R}^{\varepsilon}_{n,m}(k^2)$ are bounded operator-valued analytic functions of k^2 for all $k^2 \in \mathbb{C}\backslash\mathbb{R}$ and $\mathrm{Im}\, k > 0$.

Threshold resonances

We say that a 1D Schrödinger operator $\overline{H} = -\frac{d^2}{ds^2} + \overline{V}(s)$ has a *zero energy resonance* if there is a function $\psi_r \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$ solving $\overline{H}\psi_r = 0$ in the sense of distributions. In particular, if

$$\int_{\mathbb{R}} \overline{V}(s)\, ds \neq 0 \quad \text{and} \quad e^{a|\cdot|}\overline{V} \in L^1(\mathbb{R})$$

for some $a > 0$, then exactly one of the following situations can occur [Bolle-Gesztesy-Wilk'85]:

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for some $a > 0$, then exactly one of the following situations can occur [Bolle-Gesztesy-Wilk'85]:

Case I: $\overline{H}$ has no zero energy resonance

Case II: there is such a resonance; then ψ_r can be chosen real and the numbers $c_2 = -\frac{1}{2}\int_{\mathbb{R}} s\,\overline{V}(s)\psi_r(s)\, ds$ and

$$c_1 = \left[\int_{\mathbb{R}} \overline{V}(s) ds\right]^{-1} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{V}(s)\frac{|s-s'|}{2}\overline{V}(s')\psi_r(s')\, ds\, ds'$$

cannot not vanish simultaneously

Point interactions

Let us list the generalized point interactions we need to state the result. The first is the Dirichlet-decoupled operator $\overline{H}^d$ with $\mathcal{D}(\overline{H}^d) := \{f \in H^2(\mathbb{R} \setminus 0) \cap H^1(\mathbb{R}) |\, f(0) = 0\}$.

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The other is a point-interaction Hamiltonian $\overline{H}^r$ acting again as $\overline{H}^r f = -f''$ but on the domain

$$\begin{aligned} \mathcal{D}(\overline{H}^r) \;&:=\; \Big\{ f \in H^2(\mathbb{R} \setminus 0) |\, (c_1 + c_2) f(0^+) = (c_1 - c_2) f(0^-)\,, \\ &\qquad (c_1 - c_2) f'(0^+) = (c_1 + c_2) f'(0^-) + \frac{\hat{\lambda}}{c_1 + c_2} f(0^-) \Big\} \quad \text{for } c_2 \neq -c_1\,; \\ \mathcal{D}(\overline{H}^r) \;&:=\; \Big\{ f \in H^2(\mathbb{R} \setminus 0) |\, f(0^-) = 0\,,\; f'(0^+) = \frac{\hat{\lambda}}{4c_1^2} f(0^+) \Big\} \quad \text{for } c_2 = -c_1\,, \end{aligned}$$

where we put

$$\hat{\lambda} := \lambda_1 \int_{\mathbb{R}} \overline{V}(s) (\psi_r(s))^2\, ds\,.$$

Point interactions, continued

The family $\overline{H}^r$ obviously depends on two real parameters. We can write the b.c. involved in the standard form $(U-I)\Psi(0) + i(U+I)\Psi'(0) = 0$ with $\Psi := (f(0^+), f(0^-))^T$, $\Psi' := (f'(0^+), -f'(0^-))^T$ and the 2×2 unitary matrix

$$U := \frac{1}{2(c_1^2+c_2^2)+i\hat\lambda} \begin{pmatrix} -4c_1c_2 - i\hat\lambda & 2(c_1^2-c_2^2) \\ 2(c_1^2-c_2^2) & 4c_1c_2 - i\hat\lambda \end{pmatrix}.$$

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The spectral and scattering properties of these interactions are easily found [AGHHE'05]. In particular, for $\lambda_1 = 0$ these boundary conditions define the “scale invariant” point interaction Hamiltonian mentioned above. On the other hand, we have here also the *standard δ interaction* of coupling strength $\hat{\lambda}$ corresponding to $c_1 = 1$ and $c_2 = 0$

The resolvents

They can be expressed through free resolvent kernel,

$$G_k(s-s') = \frac{i}{2k} e^{ik|s-s'|}, \quad k^2 \in \mathbb{C}\backslash\mathbb{R}^+,\ \operatorname{Im} k > 0\,.$$

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By Krein's formula the Dirichlet-decoupled resolvent kernel is $\overline{R}^d(k^2,s,s') = G_k(s-s') + 2ikG_k(s)G_k(s')$ for $\mathrm{Im}\,k>0$ and $k^2 \in \mathbb{C}\backslash\mathbb{R}^+$, while the kernel of $\overline{R}^r(k^2) := (\overline{H}^r - k^2)^{-1}$ equals

$$\overline{R}^r(k^2;s,s') = G_k(s-s') + 2ik\frac{2kc_2^2+i\hat\lambda}{2k(c_1^2+c_2^2)+i\hat\lambda} G_k(s)G_k(s')$$

$$+\frac{4ic_2^2}{2k(c_1^2+c_2^2)+i\hat\lambda} G_k'(s)G_k'(s') + \frac{4kc_1c_2}{2k(c_1^2+c_2^2)+i\hat\lambda}\left[G_k(s)G_k'(s') + G_k'(s)G_k\right.$$

for $k^2 \in \rho(\overline{H}^r)\,,\ \mathrm{Im}\,k>0$.

The main result

Theorem [Cacciapuoti-E'07]: Suppose that for $\varepsilon \in (0, \varepsilon_0]$ the curve C_ε has no self-intersections, γ is piecewise C^2 with a compact support, and γ', γ'' are bounded. Assuming $\alpha > 5/2$, we have:

(i) If $-\frac{d^2}{ds^2} - \frac{1}{4}\gamma^2(s)$ has no zero energy resonance, then we have in the operator-norm sense

$$\mathrm{u}-\lim_{\varepsilon\to 0} \overline{R}^{\varepsilon}_{n,m}(k^2) = \delta_{n,m}\overline{R}^{d}(k^2)\,, \quad k^2 \in \mathbb{C}\backslash\mathbb{R},\;\; \mathrm{Im}\, k > 0\,.$$

(ii) On the other hand, if there is such a resonance, then

$$\mathrm{u}-\lim_{\varepsilon\to 0} \overline{R}^{\varepsilon}_{n,m}(k^2) = \delta_{n,m}\overline{R}^{r}(k^2)\,, \quad k^2 \in \rho(\overline{H}^{r}),\;\; \mathrm{Im}\, k > 0\,,$$

where c_1, c_2 and $\hat\lambda$ are defined as above with $\overline{V} := -\frac{1}{4}\gamma^2$

Sketch of the proof

The key element is the scaling analysis of 1D Hamiltonians,

$$\overline{H}_\varepsilon := -\frac{d^2}{ds^2} + \frac{\lambda(\varepsilon)}{\varepsilon^2}\overline{V}\left(\frac{s}{\varepsilon}\right), \quad s \in \mathbb{R},$$

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We suppose that $\lambda(\varepsilon) = 1 + \sum_{n=1}^{\infty} \lambda_n \varepsilon^n$ near the origin and factorize conventionally, $u(s) := \operatorname{sgn}[\overline{V}(s)]|\overline{V}(s)|^{1/2}$ and $v(s) := |\overline{V}(s)|^{1/2}$, which allows us to write

$$(\overline{H}_\varepsilon - k^2)^{-1} = G_k - \frac{\lambda(\varepsilon)}{\varepsilon} A_\varepsilon(k) T_\varepsilon(k) C_\varepsilon(k)$$

with $T_\varepsilon(k) = [1 + \lambda(\varepsilon) u G_{\varepsilon k} v]^{-1}$ for appropriate values of k, $A_\varepsilon(k;s,s') = G_k(s-\varepsilon s')v(s')$ and $C_\varepsilon(k;s,s') = u(s)G_k(\varepsilon s - s')$

Sketch of the proof, continued

To find the behaviour of $T_\varepsilon(k)$ as $\varepsilon \to 0$ we introduce the projections P, of dimension one, and Q by

$$P := \frac{1}{(v,u)}(v,\cdot\,)u\,, \qquad Q := 1 - P\,;$$

recall that $(v,u) = \int_{\mathbb{R}} \overline{V}(s)ds \neq 0$ holds by assumption

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Then the operator $T_\varepsilon(k)$ can be written as

$$T_\varepsilon(k) = \left[1 + \frac{i(v,u)}{2\varepsilon k}P + \widetilde{M}_\varepsilon(k)\right]^{-1}$$

where the regular part $\widetilde{M}_\varepsilon(k) \in \mathcal{B}(L^2, L^2)$ expresses, after some computation, as an explicit power series in ε

Sketch of the proof, continued

Presence of a zero-energy resonance is equivalent to the existence of a function $\varphi_0 \in L^2(\mathbb{R})$ which satisfies the relation $\varphi_0 + QM_0Q\varphi_0 = 0$ where M_0 is the absolute term in the series. If such a φ_0 exists, it can be chosen real, then the parameters c_1, c_2 and $\hat{\lambda}$ are related to φ_0 by

$$c_1 = \frac{(v, m_0\varphi_0)}{(v, u)}\,, \quad c_2 = \frac{1}{2}((\cdot)v, \varphi_0)\,, \quad \hat{\lambda} = \lambda_1(\tilde{\varphi}_0, \varphi_0)$$

with $\tilde{\varphi}_0(s) := \operatorname{sgn}[\overline{V}(s)]\varphi_0$ and $m_0(s, s') == u(s)|s - s'|v(s')$; the relation $u(s)\psi_r(s) = -\varphi_0(s)$ holds a.e.

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Using these notions and results of [BGW'85] we get after straightforward but tedious computations the following asymptotic expansion for $T_\varepsilon(k)$:

Sketch of the proof, continued

Lemma: Suppose in addition that $k \neq -i\hat{\lambda}/(2(c_1^2+c_2^2))$ holds in the case II. Then for all ε small enough the operator $T_\varepsilon(k)$ has the following norm-convergent series expansions

$$T_\varepsilon(k) = \sum_{n=p}^{\infty} \varepsilon^n t_n(k)\,,$$

where $p=0$ in the case I and $p=-1$ in the case II. Moreover, we have $(v, t_0 u)=0,\ ((\cdot)v, t_0 u) = (v, t_0(\cdot)u) = 0$, and $(v, t_1 u) = -2ik$ in case I, while in case II it holds

$$t_{-1}u = t_{-1}^* v = 0\,;\ ((\cdot)v, t_{-1}(\cdot)u) = -\frac{4ic_2^2}{2k(c_1^2+c_2^2)+i\hat{\lambda}}\,,$$

$$(v, t_0 u) = 0\,;\ ((\cdot)v, t_0 u) = (v, t_0(\cdot)u) = \frac{4kc_1c_2}{2k(c_1^2+c_2^2)+i\hat{\lambda}}\,,$$

$$(v, t_1 u) = -2ik\frac{2kc_2^2+i\hat{\lambda}}{2k(c_1^2+c_2^2)+i\hat{\lambda}}\,.$$

Sketch of the proof, conclusion

Now one can proceed as in [ACF'07]. The above lemma and the explicit formulæ for the point-interaction Green function allow us to verify that $(\overline{H}_\varepsilon - k^2)^{-1}$ converges in the norm resolvent sense to $\overline{R}^d(k^2)$ and $\overline{R}^r(k^2)$, respectively

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In the final step we compare the "straightened" operator with its leading term w.r.t. the transverse variable,

$$H_{0\varepsilon}^{\gamma} := -\frac{\partial^2}{\partial s^2} - \frac{1}{\varepsilon^{2\alpha}}\frac{\partial^2}{\partial u^2} - \frac{\lambda(\varepsilon)}{\varepsilon^2}\frac{\gamma(s/\varepsilon)^2}{4}$$

with the appropriate domain; their resolvents satisfy

$$\mathrm{u} - \lim_{\varepsilon \to 0} \left(R_{n,m}^{\varepsilon}(k^2) - R_{n,m}^{\gamma;\varepsilon}(k^2)\right) = 0\,, \quad k^2 \in \mathbb{C}\setminus\mathbb{R},\ \mathrm{Im}\, k > 0\,;$$

putting these claims together we get the sought result. $\square$

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- A lot of work remains to be done in these problems

The talk was based on

[CE07] C. Cacciapuoti, P.E.: Nontrivial edge coupling from a Dirichlet network squeezing: the case of a bent waveguide, *J. Phys. A: Math. Theor.* **A40** (2007), F511-F523

see also, e.g.

[CE04] P. Kuchment: Quantum graphs I. Some basic structures, *Waves in Random media* **14** (2004), S107-S128

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