# Product formulæ and Zeno quantum dynamics

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# Product formulæ

An often used way to express exponential functions of operators is based on limit of products of their 'constituents'. Here is a classical example:

**Theorem (Trotter formula)**

*Suppose that $A$, $B$ are self-adjoint operators and $C := A+B$ is essentially self-adjoint, then the corresponding unitary groups are related by*

$$e^{it\bar{C}} = \operatorname*{s\text{-}lim}_{n\to\infty} \left(e^{itA/n} e^{itB/n}\right)^n \quad \textit{for any} \;\; t \in \mathbb{R}.$$

H. Trotter: On the product of semigroups of operators, *Proc. Amer. Math. Soc.* **10** (1959), 545–551.

The idea comes back to *Sophus Lie* who proved such a formula for matrices in 1875. His prooof, based on a telescopic estimate, was straightforward and generalize easily to operators on infinite-dimensional Hilbert spaces *as long as $A+B$ is self-adjoint*; in the more general case the original Trotter's proof was considerably more involved.

Trotter's formula has many applications; to mention just one, recall it provides a way to define rigorously the *Feynman path integral*.

# Chernoff's idea

A decade later, a significant simplification of the argument was made possible as a consequence of the following result:

**Theorem**

*For a family $\{F(t)\}_{t\geq 0}$ of linear contractions on a Banach space and the generator $A$ of a strongly continuous contraction semigroup, the following two conditions are equivalent:*

*(a) The family $\left\{\left(\lambda_0 I + \frac{I-F(\varepsilon)}{\varepsilon}\right)^{-1}\right\}_{\varepsilon>0}$ converges for some $\lambda_0 > 0$ strongly to the operator $(\lambda_0 I + A)^{-1}$ as $\varepsilon \to 0+$.*

*(b) The family $\{F(\frac{t}{n})^n\}_{n=1}^{\infty}$ converges strongly to $\mathrm{e}^{tA}$ as $n \to \infty$, uniformly on bounded intervals of $t$.*


P.R. Chernoff: Note on product formulas for operator semigroups, *J. Funct. Anal.* **2** (1968), 238–242.

P.R. Chernoff: *Product Formulas, Nonlinear Semigroups, and Addition of Unbounded Operators*, Memoirs of the American Mathematical Society, vol. 140; AMS, Providence, R.I. 1974.


Moreover, Chernoff's result opened way to various other product formulæ.

# Kato's results

**Theorem (Trotter-Kato formula, form version)**

*Let $A$, $B$ be positive self-adjoint operators. Suppose that $Q(A) \cap Q(B)$ is dense, then the form $[\phi, \psi] \mapsto (\phi, A\psi) + (\phi, B\psi)$ is closed and the self-adjoint operator $C$ associated with it satisfies*

$$\operatorname*{s-lim}_{n\to\infty} \left(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n}\right)^n = \mathrm{e}^{-tC}.$$

*Moreover, if the density assumption fails, we denote the projection to $\overline{Q(A) \cap Q(B)}$ by $P$. Then the right-hand side is then replaced by $\mathrm{e}^{-tC}P$, where $C$ is the self-adjoint operator on $P\mathcal{H}$ associated with the form sum.*

T. Kato: Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups, in *Topics in Functional Analysis* (G.C. Rotta, ed.), Academic press, New York 1978; pp 185–195.

There other extensions, for instance, to products of *nonlinear semigroups*:

T. Kato, K. Masuda: Trotter's product formula for nonlinear semigroups generated by the subdifferentials of convex functionals, *J. Math. Soc. Japan* **30** (1978), 169–178.

# Unstable quantum systems

My main aim in this talk is to demonstrate another product formula that involves *unitary groups* and *projections*, but to explain why it should be of interest we have to make first a rather *long detour*.

We turn to a motivation to quantum mechanics, in particular, to the way in which it describes behavior of *unstable systems*. There is no need to stress how important it is, world is full of such objects: among massive *elementary particles* only electron and proton (hopefully) are stable, a significant part of *atomic nuclei* decay, to say nothing of *excited states* of atoms and molecules that de-excite spontaneously, etc.

Since *Bequerel* we know that decays have a *probabilistic character* and there is no way how to explain these processes in classical physics.

In QM, there is a general scheme using the following assumptions:

- the 'large' state space $\mathcal{H}$ of an *isolated system*
- projection $P$ to the subspace $P\mathcal{H} \subset \mathcal{H}$ of the *unstable system*
- time evolution $\mathrm{e}^{-iHt}$ on $\mathcal{H}$, is *not reduced by* $P$ *for any* $t > 0$

# Unstable quantum systems

We can consider *reduced evolution* $V$ : $V(t) = P\mathrm{e}^{-iHt} \upharpoonright P\mathcal{H}$ supposing that the evolution starts at $t = 0$ from a state $\psi \in P\mathcal{H}$. At some $t > 0$ we then perform *non-decay measurement*: the probability to find the state still in $\mathcal{H}$, or the *decay law* is

$$P_\psi(t) := \|V(t)\psi\|^2 = \|P\,\mathrm{e}^{-iHt}\psi\|^2\,;$$

the index $\psi$ is often dropped when it is clear from the context.

The simplest situation occurs if $\dim \mathcal{H} = 1$, that is, the unstable system is a *single state*, when

$$P(t) := |(\psi(t), \psi(0))|^2$$

A *common example*: we have $\mathcal{H} = L^2(\mathbb{R})$, the unstable state refers to *Breit-Wigner function*,

$$\psi(\lambda, 0) = \left(\frac{\Gamma}{2\pi} \frac{1}{(\lambda - \lambda_0)^2 + \frac{1}{4}\Gamma^2}\right)^{1/2},$$

and the time evolution acts as $\psi(\lambda, t) = \mathrm{e}^{-i\lambda t}\psi(\lambda, 0)$. Then the reduced evolution is obtained by *Fourier transformation* of $\psi(\cdot, 0)$, in particular,

$$P(t) = \mathrm{e}^{-\Gamma t} \quad \text{for all} \quad t \geq 0.$$

# Troubles with the exponential decay

At a glance, this corresponds to our *massive experience*, including even applications like the $C_{14}$ archeology, but there is *catch*: the result requires $\sigma(H) = \mathbb{R}$ **!** This is not what a reasonable physical model should exhibit.

Consider another example. Let $\mathcal{H} = L^2(\mathbb{R})$ with $H = -\Delta$; if unstable states are those *localized in an interval* $\Omega = (a, b) \subset \mathbb{R}$, the decay law is

$$P_\psi(t) = \int_a^b |\psi(x,t)|^2 \mathrm{d}x$$

From we already saw it is clear that $P_\psi(t) < 1$ holds *for any* $\psi \in L^2(\Omega)$ *and* $t > 0$, and one can check also that $\lim_{t\to\infty} P_\psi(t) = 0$. However, there is *no* $\psi \in L^2(\Omega)$ for which the decay law *would be exponential*

In fact, we have the following general result:

**Theorem**

*If the reduced evolution is a semigroup,* $V(t_1)V(t_2) = V(t_1 + t_2)$ *for* $t_1, t_2 > 0$, *then* $\sigma(H) = \mathbb{R}$.

K. Sinha: On the decay of an unstable particle, *Helv. Phys. Acta* **45** (1972), 619–628.

# The inverse decay problem

More generally, the knowledge of reduced evolution allows us to restore the complete dynamics of the decay:

- Given a weakly continuous contraction-valued function $V(\cdot)$ on a Hilbert space $\mathcal{G}$, one can reconstruct the triple $\{\mathcal{H},\, H,\, P\}$, uniquely up to an isomorphism under a natural *minimality condition*, such that $\mathcal{G} = P\mathcal{H}$ and $V(t) = P\mathrm{e}^{-iHt} \upharpoonright \mathcal{G}$ *if and only if* $V$ is of *positive type*, that is, $\sum_{i,j=1}^{n} (\phi_i, V(t_i - t_j)\phi_j) \geq 0$ holds for all finite combinations of vectors in $\mathcal{G}$ and arguments of the function.
- The *generalized Bochner theorem* holds: $V(\cdot)$ is weakly continuous of positive type *if and only if* there is a *positive operator-valued measure* $F$ such that $V(t) = \int_{\mathbb{R}} \mathrm{e}^{-i\lambda t} \mathrm{d}F(\lambda)$.

P.E.: *Open Quantum Systems and Feynman Integrals*, Reidel, Dordrecht 1985

The measure $F$ provides us with spectral information, in particular, we have the identity $\mathrm{supp}\, F = \sigma(H)$. This explains the above mention result since for any contraction-valued semigroup $V$ we have $\mathrm{supp}\, F = \mathbb{R}$.

P.E.: Remark on the energy spectrum of a decaying system, *Commun. Math. Phys.* **50** (1976), 1–10.

# Do the semigroup character violations matter?

The fact that the semigroup property cannot be *exactly* valid need not be anything dramatic: the difference between the actual decay law and the exponential one may be small and thus unimportant for a physicist who has to take always experimental errors into account.

The smallness was also treated mathematically in various models, e.g.

M. Demuth: Pole approximation and spectral concentration, *Math. Nachr.* **73** (1976), 65–72.

But some deviations *may be important*, in particular, in the behavior close to $t = 0$. Recall that for the Breit-Wigner function the *mean value of energy* makes no sense because $\int_{\mathbb{R}} \frac{\Gamma}{2\pi} \frac{\lambda}{(\lambda-\lambda_0)^2+\frac{1}{4}\Gamma^2}\, \mathrm{d}\lambda$ is divergent. This fact is closely related to the behavior around $t = 0$:

**Theorem**

$\dot{P}_\psi(0+) = 0$ *holds whenever* $\psi \in Q(H)$, *that is,* $\||H|^{1/2}\psi\| < \infty$.

M. Havlíček, P.E.: Note on the description of an unstable system, *Czech J. Phys.* **B23** (1973), 594–600.

# Repeated measurements

Suppose now that we perform non-decay measurements at times $t/n$, $2t/n$ ..., $t$, *all with the positive outcome*, then the resulting non-decay probability is $M_n(t) = P_\psi(t/n)P_{\psi_1}(t/n)\cdots P_{\psi_{n-1}}(t/n)$, where $\psi_{j+1}$ is the normalized projection of $\mathrm{e}^{-iHt/n}\psi_j$ on $P\mathcal{H}$ and $\psi_0 := \psi$, in particular, for $\dim P = 1$ we have

$$M_n(t) = (P(t/n))^n \quad \text{(no need to indicate } \psi\text{)}$$

Consider now the situation when the measurements are *performed frequently*, and since we watch the problem through a mathematician's eye, look what happens if $n \to \infty$. For the *exponential law* nothing happens, $M_n(t) = (\mathrm{e}^{-\Gamma t/n})^n = \mathrm{e}^{-\Gamma t}$ for any $n$

If the *initial decay rate is zero* the situation is completely different. It is straightforward to see that $\lim_{n\to\infty} f(t/n)^n = \exp\{-\dot{f}(0+)t\}$ holds if $f(0) = 1$ and the one-sided derivative $\dot{f}(0+)$ exists, and therefore

$$M(t) := \lim_{n\to\infty} M_n(t) = 1 \quad \text{if} \quad \dot{P}(0+) = 0$$

# Quantum Zeno effect

Consequently, in the *limit of infinite frequency* repeated measurements *prevent the system from decaying*. The mathematical fact was known from the 1950s as *Turing paradox*, in the context of unstable quantum systems it was considered first in

A. Beskow, J. Nilsson: The concept of wave function and the irreducible representations of the Poincaré group, II. Unstable systems and the exponential decay law, *Arkiv Phys.* **34** (1967), 561–569.

The effect was analyzed mathematically by various people after that but it attracted a lot of attention only from 1977 when Misra and Sudarshan invented a name which linked it to the *flying arrow aporia* of Zeno of Elea:

E.C.G. Sudarshan, B. Misra: The Zeno's paradox in quantum theory, *J. Math. Phys.* **18** (1977), 756–763.

Later the name *anti-Zeno effect* appeared, because we also have

$$M(t) := \lim_{n\to\infty} M_n(t) = 0 \quad \text{if} \;\; \dot{P}(0+) = -\infty\,;$$

in that case the system would decay *immediately* on the continuous observation begins – and the decay *accelerates* for finite but large measurement frequency (as we will see, $\dot{P}(0+) = -\infty$ may happen).

# One might naturally ask: is it mathematics only?

Many people thought so, however, quantum mechanics covers large segments of physical reality:

- apart from elementary particles, there are many more systems in *atomic and molecular physics*
- the limit may not be achievable but *sufficiently frequent* measurements can *slow down the decay* (or accelerate in case of *anti-Zeno effect* mentioned above)
- *the measurement* can take various forms, the *absorption of a photon* at some wavelength, the *release of a photon*, say, leaving an optical fiber in a prescribed mode, or others

In this way Zeno effect was first confirmed experimentally in the *cloud of $Be^+$ ions* in a Penning trap, driven by radiofrequency to an excited state (decay) and exposed to frequent UV pulses (measurement):


W. Itano, D. Heinzen, J. Bollinger, D. Wineland: Quantum Zeno effect, *Phys. Rev.* **A41** (1990), 2295–2300.

D. Leibfried, R. Blatt, C. Monroe, D. Wineland: Quantum dynamics of single trapped ions, *Rev. Mod. Phys.* **75** (2003), 281–324.

# More real life situations

Another experiment used *ultracold sodium atoms* trapped in an *optical lattice*. Their loss due to tunneling appeared to be either suppressed or enhanced by an *appropriate accelerations* of the lattice.

M. Fischer, B. Gutiérrez-Medina, M. Raizen: Observation of the quantum Zeno and anti-Zeno effects in an unstable system, *Phys. Rev. Lett.* **84** (2001), 140402.

Still another experiment employed the light used to image single atoms to modulate tunneling in an *ultracold lattice gas*.

Y.S. Patil, S. Chakram, M. Vengalattore: Measurement-induced localization of an ultracold lattice gas, *Phys. Rev. Lett.* **115** (2015), 140402.

Furthermore, quantum Zeno effect is used nowadays in *commercial atomic magnetometers*, and there is even an evidence that *birds* use it to prevent the influence of perturbations to their sensing of the Earth magnetic field.

A.T. Dellis, I.K. Kominis: The quantum Zeno effect immunizes the avian compass against the deleterious effects of exchange and dipolar interactions, *Biophysics* **107** (2012), 153–157.

Finally, a version of QZE in the framework of *open systems*, where the evolution is considered in the *Banach space* of density matrices, has been proposed as an *error correction tool* in dealing with *quantum information*.

# The anti-Zeno situation

Before passing to our main goal, let us ask under which circumstances could the anti-Zeno situation occur.

To answer this question, we need to estimate the quantity $1 - P(t)$, in other words $(\psi, P\psi) - (\psi, \mathrm{e}^{iHt}P\mathrm{e}^{-iHt}\psi)$. We rewrite it as

$$1-P(t) = 2\,\mathrm{Re}\,(\psi, P(I-\mathrm{e}^{-iHt})\psi) - \|P(I-\mathrm{e}^{-iHt})\psi\|^2\,;$$

in terms of the spectral measure $E_H$ of $H$ the right-hand side equals

$$4\int_{-\infty}^{\infty}\sin^2\frac{\lambda t}{2}\,\mathrm{d}\|E_\lambda^H\psi\|^2 - 4\left\|\int_{-\infty}^{\infty}\mathrm{e}^{-i\lambda t/2}\,\sin\frac{\lambda t}{2}\,\mathrm{d}PE_\lambda^H\psi\,\right\|^2$$

This expression is non-negative by Schwarz inequality; our aim is to find tighter upper and lower bounds to it.

Consider first the case $\dim P = 1$. Denoting $\mathrm{d}\omega(\lambda) := \mathrm{d}(\psi, E_\lambda^H\psi)$ for the sake of brevity, one can then write the expression as

$$4\int_{-\infty}^{\infty}\sin^2\frac{\lambda t}{2}\,\mathrm{d}\omega(\lambda) - 4\left|\int_{-\infty}^{\infty}\mathrm{e}^{-i\lambda t/2}\,\sin\frac{\lambda t}{2}\,\mathrm{d}\omega(\lambda)\right|^2$$

# The anti-Zeno situation, $\dim P = 1$

By spectral-measure normalization, $\int_{-\infty}^{\infty} \mathrm{d}\omega(\lambda) = 1$, this simplifies to

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sin^2\frac{\lambda t}{2} + \sin^2\frac{\mu t}{2}\right)\mathrm{d}\omega(\lambda)\mathrm{d}\omega(\mu) - 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\frac{(\lambda-\mu)t}{2}\,\sin\frac{\lambda t}{2}\,\sin\frac{\mu t}{2}\,\mathrm{d}\omega(\lambda)\mathrm{d}\omega(\mu),$$

or equivalently

$$1-P(t) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\frac{(\lambda-\mu)t}{2}\,\mathrm{d}\omega(\lambda)\mathrm{d}\omega(\mu).$$

Thus we have to estimate the integrated function. Let us fix $\alpha \in (0, 2]$. Using $|x|^\alpha \geq |\sin x|^\alpha \geq \sin^2 x$ together with $|\lambda-\mu|^\alpha \leq 2^\alpha(|\lambda|^\alpha + |\mu|^\alpha)$ we infer from the above formula

$$\begin{aligned}\frac{1-P(t)}{t^\alpha} &\leq 2^{1-\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda-\mu|^\alpha\mathrm{d}\omega(\lambda)\mathrm{d}\omega(\mu)\\ &\leq 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|\lambda|^\alpha + |\mu|^\alpha)\mathrm{d}\omega(\lambda)\mathrm{d}\omega(\mu) \leq 4\langle|H|^\alpha\rangle_\psi\end{aligned}$$

Hence $1 - P(t) = \mathcal{O}(t^\alpha)$ if $\psi \in D(|H|^{\alpha/2})$. If this is true for some $\alpha > 1$ we have *Zeno effect* – which is a slightly weaker sufficient condition than the above mentioned one.

# The anti-Zeno situation, $\dim P = 1$

By negation, $\psi \notin D(|H|^{1/2})$ is a *necessary condition* for the anti-Zeno effect. Notice that in the particular case $\psi \in \mathcal{H}_{\rm ac}(H)$ the same follows from *Lipschitz regularity*, since $P(t) = |\hat\omega(t)|^2$ and $\hat\omega$ is bounded and uniformly $\alpha$-Lipschitz if and only if $\int_{\mathbb{R}} \omega(\lambda)(1+|\lambda|^\alpha)\,\mathrm{d}\lambda < \infty$.

To find a *sufficient condition* we note that for $\lambda, \mu \in [-1/t, 1/t]$ there is a positive $C$ independent of $t$ such that

$$\left|\sin\frac{(\lambda-\mu)t}{2}\right| \geq C|\lambda-\mu|t\,;$$

one can make the constant explicit but it is not necessary. Consequently, we have the estimate

$$1-P(t) \geq 2C^2t^2 \int_{-1/t}^{1/t} \mathrm{d}\omega(\lambda) \int_{-1/t}^{1/t} \mathrm{d}\omega(\mu)(\lambda-\mu)^2$$

which in turn implies

$$\frac{1-P(t)}{t} \geq 4C^2t \left\{ \int_{-1/t}^{1/t} \lambda^2\,\mathrm{d}\omega(\lambda) \int_{-1/t}^{1/t} \mathrm{d}\omega(\lambda) - \left(\int_{-1/t}^{1/t} \lambda\,\mathrm{d}\omega(\lambda)\right)^2 \right\}$$

# The anti-Zeno situation, $\dim P = 1$

It implies that anti-Zeno effect occurs if the right-hand side diverges as $t \to 0$, in other words, if the inequality

$$\int_{-N}^{N} \lambda^2 \, \mathrm{d}\omega(\lambda) \int_{-N}^{N} \mathrm{d}\omega(\lambda) - \left( \int_{-N}^{N} \lambda \, \mathrm{d}\omega(\lambda) \right)^2 \geq cN^{\alpha}$$

holds for any $N$ and some $c > 0,\ \alpha > 1$.

To see what it means, consider the following *example*: let $H$ be bounded from below and $\psi \in \mathcal{H}_{\mathrm{ac}}(H)$ s.t. $\omega(\lambda) \approx c\lambda^{-\beta}$ as $\lambda \to +\infty$ for some $c > 0$ and $\beta \in (1, 2)$. While $\int_{-N}^{N} \omega(\lambda) \, \mathrm{d}\lambda \to 1$, the other two integrals diverge giving

$$\frac{c}{3-\beta} N^{3-\beta} - \left( \frac{c}{2-\beta} \right)^2 N^{4-2\beta}$$

as the asymptotic behavior of the left-hand side, where the first term is dominating; this gives $\dot{P}(0+) = -\infty$ so AZ effect occurs.

Note also that for $\beta > 2$ we have Zeno effect, so that *the gap between the Zeno and anti-Zeno extremes is rather narrow!*

# Anti-Zeno effect: a sufficient condition

Moreover, even for $\beta = 2$ the appropriate limit *need not exist*: for instance, choosing $\mathrm{d}\omega(\lambda) = \frac{2}{\pi}(1+\lambda^2)^{-1}\Theta(\lambda)\mathrm{d}\lambda$ we can compute explicitly

$$v(t) = \mathrm{e}^{-t} - \frac{i}{\pi}\left(\mathrm{e}^{-t}\mathrm{Ei}(t) - \mathrm{e}^{t}\mathrm{Ei}(-t)\right) = \mathrm{e}^{-t}\left[1 - \frac{2i}{\pi}\left(t\ln t + \mathcal{O}(t)\right)\right].$$

This means, in particular, that $\arg v\left(\frac{t}{n}\right)^n$ is for $n \to \infty$ dominated by the *fast oscillating term* $\frac{2t}{\pi}\ln\frac{n}{t}$ and the limit does not exist.

It is not difficult to extend the argument to the case $\dim P > 1$ using an orthonormal basis in $P\mathcal{H}$. To write the result concisely, we denote $I_N := E_H(\Delta_N)$ with $\Delta_N := (-N, N)$ and $H_N^\beta := H^\beta I_N$.

**Theorem**

*In the above notation, suppose that*

$$\left(\langle H_N^2 P I_N\rangle_\psi - \|PH_N\psi\|^2\right)^{-1} = o(N^{-1})$$

*holds as $N \to \infty$, uniformly with respect to $\psi \in P\mathcal{H}$, then the permanent observation causes the anti-Zeno effect.*

P.E.: Sufficient conditions for the anti-Zeno effect, *J. Phys. A: Math. Gen.* **38** (2005), L449–L454.

# Zeno dynamics

Let us return to the Zeno effect in a system the Hamiltonian of which is *bounded from below*. In the non-trivial situation, $\dim \mathcal{H} > 1$, there is an important question that remains open: does the limit

$$(P\mathrm{e}^{-iHt/n}P)^n \longrightarrow \mathrm{e}^{-iH_P t}$$

hold as $n \to \infty$, *in which sense*, and what is then the *Zeno dynamics generator*, that is, the operator $H_P$?

Consider the quadratic form $u \mapsto \|H^{1/2}Pu\|^2$ with the form domain $D(H^{1/2}P)$ which is closed. By [Chernoff'74, loc.cit.] the associated self-adjoint operator, $(H^{1/2}P)^*(H^{1/2}P)$, is a natural candidate for the role of $H_P$ *(while, in general, PHP is not!)*

Without loss of generality, we may suppose that $H$ is *positive*. In addition to the semiboundedness, we have to assume that $H$ is *densely defined*. We have encountered counterexamples illustrating this claim, for others see

M. Matolcsi, R. Shvidkoy: Trotter's product formula for projections, *Arch. der Math.* **81** (2003), 309–317.

# Zeno dynamics, $\dim P < \infty$

The simplest situation occurs when the subspace to which permanent measurement localizes the state is *finite-dimensional*:

**Proposition**

*Let $H$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$, bounded from below, and assume that $P$ is a finite-dimensional orthogonal projection on $\mathcal{H}$. If $P\mathcal{H} \subset \mathcal{Q}(H)$, then for any $\psi \in \mathcal{H}$ and $t \geq 0$ we have*

$$\lim_{n\to\infty} (P\mathrm{e}^{-iHt/n}P)^n\psi = \mathrm{e}^{-iH_P t}\psi,$$

*uniformly on any compact interval of the variable $t$.*

*Proof* (following unpublished notes of G.-M. Graf and his student A. Guekos): first we have to check that

$$\lim_{t\to 0} t^{-1}\left\|P\mathrm{e}^{-itH}P - P\mathrm{e}^{-itH_P}P\right\| = 0,$$

because it implies $\left\|(P\mathrm{e}^{-itH/n}P)^n - \mathrm{e}^{-itH_P}\right\| = n\, o(t/n)$ as $n \to \infty$ by means of a natural telescopic estimate.

# Zeno dynamics, $\dim P < \infty$

Without loss of generality one may assume $H \geq cI$ for some $c > 0$. To begin with, we check that

$$t^{-1}\left[(f, P\mathrm{e}^{-itH}Pg) - (f,g) - it(\sqrt{H}Pf, \sqrt{H}Pg)\right] \longrightarrow 0$$

holds as $t \to 0$ for all $f, g$ from $D(\sqrt{H}P) = P\mathcal{H}$. Indeed, this expression equals $\left(\sqrt{H}Pf, \left[\frac{\mathrm{e}^{-itH}-I}{tH} - i\right]\sqrt{H}Pg\right)$ and by functional calculus, the square bracket tends to zero strongly. In the same way we find that

$$t^{-1}\left[(f, P\mathrm{e}^{-itH_P}Pg) - (f,g) - it(\sqrt{H_P}f, \sqrt{H_P}g)\right] \longrightarrow 0$$

holds as $t \to 0$ for any $f, g \in P\mathcal{H}$.

Next we note that $(\sqrt{H_P}f, \sqrt{H_P}g) = (\sqrt{H}Pf, \sqrt{H}Pg)$ holds by definition, which means that $t^{-1}(P\mathrm{e}^{-itH}P - P\mathrm{e}^{-itH_P}P) \to 0$ weakly as $t \to 0$, however, the weak and strong topologies are equivalent if $\dim P < \infty$. $\square$

# Zeno dynamics, general case

Without the dimensional restriction, the situation becomes much more complicated. For instance, one can prove the product formula, but with an additional restriction and the convergence *in a weaker topology*:

**Theorem**

*Let $\mathcal{H}$ be separable, then we have for any $\psi \in \mathcal{H}$ and any $T > 0$ the relation*

$$\lim_{n\to\infty} \int_0^T \|(P\mathrm{e}^{-iHt/n}P)^n\psi - \mathrm{e}^{-iH_Pt}\psi\|^2 \,\mathrm{d}t = 0,$$

*and the same with $(0, T)$ replaced by an arbitrary open interval.*

P.E., T. Ichinose: A product formula related to quantum Zeno dynamics, *Ann. Henri Poincaré* **6**(2) (2005), 195–215.

One might argue that such a result can be regarded as *sufficient from the viewpoint of physics* due to the fact that every measurement, in particular, that of time is burdened with errors, and any actual experiment typically involves averaging over a large number of system copies.

It is desirable, though, to answer the question *without such a underpinning* by demonstrating the convergence in the strong operator topology.

# Zeno dynamics, general case

This proved to be a challenge. One can derive a modified formula:

**Theorem**

*Under same assumptions, except that $\mathcal{H}$ need not be separable,*

$$\lim_{n\to\infty} (PE_H([0,\pi n/t))\, \mathrm{e}^{-iHt/n}P)^n\psi = \mathrm{e}^{-iH_P t}\psi,$$

*uniformly on any compact interval of the variable t.*

P.E., T. Ichinose, H. Neidhardt, and V.A. Zagrebnov: Zeno product formula revisited, *Integral Eq. Oper. Theory* **57**(1) (2007), 67–81.

**Corollary**

*Strong convergence holds if H is bounded.*

Moreover, the analogous result holds for $(P\phi(tH/n)P)^n$ with a function satisfying $|\phi(x)| \le 1$ and $\phi(0) = i\phi'(0) = 1$ provided $\mathrm{Im}\,\phi(x) \le 0$.

An example of such a function is $(1+ix)^{-1}$, but unfortunately *this class fails to include* $\mathrm{e}^{-ix}$ corresponding to our unitary group $\mathrm{e}^{-itH}$.

# A new result

## Theorem

*Let $H$ be a nonnegative self-adjoint operator on a separable Hilbert space $\mathcal{H}$, and $P$ an orthogonal projection onto a closed subspace of $\mathcal{H}$. Suppose that $H^{1/2}P$ is densely defined, so that $H_P := (H^{1/2}P)^*(H^{1/2}P)$ is a self-adjoint operator. Let $P(\cdot)$ be a strongly continuous projection-valued function satisfying $P(0) = P$ and*

$$\lim_{\tau\to 0+} [\tau^{-1}(I - \mathrm{e}^{-it\tau H})]^{1/2} P(\tau)v = \mathrm{e}^{\pi i/4}(tH)^{1/2}Pv,$$

*for every $v \in D[H^{1/2}P]$. Then for any $f \in \mathcal{H}$ and $\varepsilon = \pm 1$ we have*

$$\lim_{n\to\infty} (P(1/n)\,\mathrm{e}^{-\varepsilon itH/n}P(1/n))^n f = \mathrm{e}^{-\varepsilon itH_P}Pf,$$

$$\lim_{n\to\infty} (\mathrm{e}^{-\varepsilon itH/n}P(1/n))^n f = \mathrm{e}^{-\varepsilon itH_P}Pf,$$

$$\lim_{n\to\infty} (P(1/n)\,\mathrm{e}^{-\varepsilon itH/n})^n f = \mathrm{e}^{-\varepsilon itH_P}Pf,$$

*in the norm of $\mathcal{H}$, uniformly on every bounded $t$-interval in $\mathbb{R}$.*

P. Exner, T. Ichinose: Note on a product formula related to quantum Zeno dynamics, *Ann. H. Poincaré* **22** (2021), to appear; `arXiv:2012.15061`

# Proof sketch

The idea is to use Chernoff's theorem as Kato did proving a modified Trotter formula. This would work, were the exponentials *real*. For complex ones, however, we get in this way an oscillatory term – recall the condition $\mathrm{Im}\,\phi(x) \le 0$ in [EINZ'07] mentioned above – which requires additional, and rather involved considerations.

Given $H \ge 0$ with the spectral representation $H = \int_{0-}^{\infty} \lambda E(\mathrm{d}\lambda)$ we put

$$K(\kappa) := \frac{1}{\kappa}\big[I - \mathrm{e}^{-i\kappa H}\big] = G(\kappa) + iH(\kappa)$$

for $\kappa > 0$, where $G(\kappa) := \frac{I - \cos\kappa H}{\kappa} \ge 0$ and $H(\kappa) := \frac{\sin\kappa H}{\kappa}$ are bounded self-adjoint operators, and furthermore

$$F(\zeta;\tau) := P(\tau)\,\mathrm{e}^{-\zeta\tau H}P(\tau), \quad S(\zeta;\tau) := \tau^{-1}[I - F(\zeta;\tau)]$$

for $\zeta = it$. The former is obviously a contraction and we have

$$\mathrm{Re}\,(f, S(it;\tau)f) \ge 0$$

for all $f \in \mathcal{H}$, so that $S(it;\tau)$ is an *m*-accretive operator. Then $I + S(it;\tau)$ has a bounded inverse and $(I + S(it;\tau))^{-1}$ is also a *contraction*.

# Proof sketch

To prove the result are going to use Chernoff's result and verify that

$$(I + S(it;\tau))^{-1} \xrightarrow{s} (I + itH_P)^{-1}P \quad \text{as} \;\; \tau \to 0+$$

holds pointwise for any fixed $t \in \mathbb{R}$. To be precise, function $F(it;\tau)$ differs slightly from $F(t)$ appearing in condition (a) of Chernoff's theorem, but one can adapt his proof of the implication (a) ⇒ (b) to our situation.

The expression the convergence of which we study can be rewritten as

$$(I + S(it;\tau))^{-1} = (1+\tau^{-1})^{-1}(I - P(\tau)) \oplus \left[P(\tau)(I + tG(t\tau) + itH(t\tau))P(\tau)\right]^{-1}.$$

and by spectral theorem we have $G(\kappa)^{1/2}u \longrightarrow 0$, $H^+(\kappa)^{1/2}u \longrightarrow H^{1/2}u$, and $H^-(\kappa)^{1/2}u \longrightarrow 0$ for any $u \in D[H^{1/2}]$, where $H^\pm(\kappa)$ is the positive and negative part of $H(\kappa)$, respectively.

For an arbitrary but fixed $f \in \mathcal{H}$, $\tau > 0$, and $t \in \mathbb{R}$ we put

$$u_\tau(t) := (I + S(it;\tau))^{-1}f.$$

Obviously, $u_\tau(\cdot)$ is *uniformly bounded* and *strongly continuous*; our aim is to show that for each fixed $t \in \mathbb{R}$, the family $\{u_\tau(t)\}$ converges strongly to some $u(t) \in \mathcal{H}$ as $\tau \to 0+$ and that $u(t) = (I + itH_P)^{-1}Pf$.

# Proof sketch

For any $\tau > 0$ we have the identities

$$\langle (I - P(\tau))u_\tau(t), f\rangle = (1+\tau^{-1})\|(I-P(\tau))u_\tau(t)\|^2$$

$$\mathrm{Re}\,\langle P(\tau)u_\tau(t), f\rangle = \|P(\tau)u_\tau(t)\|^2 + \|(|t|G(t\tau))^{1/2}P(\tau)u_\tau(t)\|^2$$

which implies that the families $\{P(\tau)u_\tau(t)\}$ and $\{(I-P(\tau))u_\tau(t)\}$, as well as $\{\tau^{-1}(I-P(\tau))u_\tau(t)\}$, are uniformly bounded by $\|f\|$, and the same is true for $\{(|t|G(\tau))^{1/2}P(\tau)u_\tau(t)\}$.

It follows that for each $t \in \mathbb{R}$, there is a (sub)sequence $\{\tau'\}_{0<\tau'\le 1}$ along which the sequences $\{u_{\tau'}(t)\}$, $\{(\tau')^{-1/2}(I-P(\tau'))u_{\tau'}(t)\}$ and $\{t^{1/2}G(|t|\tau')^{1/2}u_{\tau'}(t)\}$ converge *weakly* to vectors $u(t)$, $u_0(t)$ and $g(t)$, respectively. This also means that $\{P(\tau')u_{\tau'}(t)\}$ converges weakly to $Pu(t)$ where, in general, the limit may depend on the chosen subsequence.

Note in passing that, unfortunately, we cannot use the same argument for

$$\mathrm{Im}\,\langle P(\tau)u_\tau(t), f\rangle = \|(|t|H^+(t\tau))^{1/2}P(\tau)u_\tau(t)\|^2 - \|(|t|H^-(t\tau))^{1/2}P(\tau)u_\tau(t)\|^2;$$

the trouble is that we do know whether each of the two terms on the right-hand side is *separately* uniformly bounded.

# Proof sketch

The limiting vectors can be specified, so we have, as $\tau' \to 0+$

$$u_{\tau'}(t) \xrightarrow{w} u(t) = Pu(t), \;\; (\tau')^{-1/2}(I - P(\tau'))u_{\tau'}(t) \xrightarrow{w} 0,$$
$$P(\tau')u_{\tau'}(t) \xrightarrow{w} Pu(t), \;\; (|t|G(t\tau'))^{1/2}P(\tau')u_{\tau'}(t) \xrightarrow{w} 0.$$

On the other hand, given $\{u_{\tau'}(t)\}$, the most we have been able to prove about the families $\{(|t|H^{\pm}(t\tau'))^{1/2}P(\tau')u_{\tau'}(t)\}$ is that they are Cauchy sequences, and as such they are weakly bounded, only in terms of the *$\sigma(\mathcal{H}, D[H^{1/2}])$-weak topology*, and the 'negative' one converges to zero in it. This does not allow us to conclude that $Pu(t)$ belongs to $D[H^{1/2}]$ and forces us to use a different strategy.

We have encountered this problem already in [EI'05]. There we did not restrict $\zeta$ to the purely imaginary values $\zeta = it$ and using *analyticity properties* of $S(\zeta)$ in combination with *Vitali theorem*, we established the above mentioned convergence in the topology of the Fréchet space $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{H}) = L^2_{\mathrm{loc}}(\mathbb{R}) \otimes \mathcal{H}$ with the topology induced by the family of semi-norms $v \mapsto \big(\int_a^b \|v(t)\|^2\,\mathrm{d}t\big)^{1/2}$ for any bounded interval $(a, b)$.

# Proof sketch

Repeated once more, the result of [EI'05] says that

$$\int_a^b \|u_\tau(t) - (I + itH_P)^{-1}Pf\|^2 \,\mathrm{d}t \longrightarrow 0 \quad \text{as } \tau \to 0+$$

and our intention is to use this claim as a departing point here.

The above relation implies that for every $f \in \mathcal{H}$, there is a set $M_f \subset \mathbb{R}$ of *Lebesgue measure zero*, possibly dependent on $f$, and a (sub)sequence $\{\tau'\}_{0<\tau'\leq 1}$ of $\{\tau\}_{0<\tau\leq 1}$ such that for all $s \in \mathbb{R} \setminus M_f$ we have

$$u_\tau(s) \longrightarrow (I + isH_P)^{-1}Pf \quad \text{in the norm of } \mathcal{H}.$$

Naturally, the set $\mathbb{R} \setminus M_f$ at which the convergence takes place is *dense* in $\mathbb{R}$. Furthermore, since $\mathcal{H}$ is *separable* by assumption, we can choose a countable dense subset $\mathcal{D} := \{f_l\}_{l=1}^\infty$ in $\mathcal{H}$. Putting $M = M_{\mathcal{D}} := \cup_{l=1}^\infty M_{f_l}$, which is also a set of Lebesgue measure zero, we get the validity of the above convergence *for all* $s \in \mathbb{R} \setminus M$ and for every $f \in \mathcal{D}$, and hence, in view of the density, also for every $f \in \mathcal{H}$.

# Proof sketch

To pass from the 'almost all $t$' to the 'all $t$' stage, one has to check that the exceptional set $M$ is in fact *empty*. This task may seem a small step, but in reality it proved to be a deep and highly nontrivial question.

We will do that in *four steps* starting from the following crucial claim:

**Lemma**

*Let $f \in \mathcal{H}$. Then the family $\{u_\tau(t)\}_{0<\tau\leq 1}$ is equicontinuous locally in $t$ with respect to the strong topology on $\mathcal{H}$. More explicitly, for every $\varepsilon > 0$ and for every $s \in \mathbb{R} \setminus \{0\}$ there exists an $s$-dependent constant $\delta = \delta(f; \varepsilon; s) > 0$ such that if $t, s > 0$ or $t, s < 0$ with $|t - s| < \delta$, then $\|u_\tau(t) - u_\tau(s)\| < \varepsilon$ holds for all $0 < \tau \leq 1$.*

The proof of this lemma is the toughest part of the task. It is based on the following factorization of the expression in question,

$$u_\tau(t) - u_\tau(s) = T_1(t; \tau)\, T_2(t, s; \tau)\, T_3(s; \tau)$$

with the help of $K(\kappa) = G(\kappa) + iH(\kappa)$ and $H_\tau := H(I + \tau H)^{-1}$, where

# Proof sketch

$$T_1(t;\tau) = T^P(t;\tau) = [P(\tau)(I + tK(t\tau))P(\tau)]^{-1}$$
$$T_2(t,s;\tau) = (sK(s\tau) - tK(t\tau))(I + |s|H_\tau)^{-1},$$
$$T_3(s;\tau) = (I + |s|H_\tau)P(\tau)[P(\tau)(I + sK(s,\tau))P(\tau)]^{-1}.$$

together with properties of operators which may be thought of as the $\tau$-limits of the first and last of the above families,

$$T_1(t) := (I + itH_P)^{-1}P, \quad T_3(s) := (I + |s|H)[(I + isH_P)^{-1}P],$$

of which the first is a contraction and the second can be extend to a bounded operator with $\|T_3(s)\| \le \sqrt{2}$.

In view of the *density of M* it is sufficient to establish the claim for $s \in \mathbb{R} \setminus M$. For those $s$ we use the equivalence the strong resolvent and strong graph limits to show that the family $\{T_3(s;\tau)\}_{0<\tau\le 1}$ converges strongly to $T_3(s)$ as $\tau \to 0+$, next we use the uniform boundedness principle to establish the existence of a $C_{T_3}(s) \ge \sqrt{2}$ such that

$$\|T_3(s;\tau)\| \le C_{T_3}(s) \quad \text{for all } \tau \in (0,1].$$

# Proof sketch

Since we have clearly $\|T_1(t;\tau)\| \leq 1$, the claim of the lemma depends on the properties of the middle factor $T_2(t,s;\tau)$.

Using spectral theorem, we rewrite its norm as

$$\|T_2(t,s;\tau)g\|^2 = \int_{0-}^{\infty} \left|\frac{\frac{2}{\tau}\sin(\frac{t-s}{2}\tau\lambda)}{I+|s|\lambda(I+\tau\lambda)^{-1}}\right|^2 \|E(\mathrm{d}\lambda)g\|^2\,;$$

this allows us to check that the family is uniformly bounded, locally uniformly for $t,s \in \mathbb{R}\setminus\{0\}$, and moreover, that for all $f \in \mathcal{H}$ and $\varepsilon > 0$, there is an $s$-dependent number $\delta = \delta(f;\varepsilon;s) > 0$ such that

$$t \in \mathbb{R}\setminus\{0\} \;\text{ with }\; |t-s| < \delta \implies \|T_2(t,s;\tau)f\| < \varepsilon,$$

uniformly with respect to $\tau \in (0,1]$.

This is the core of the argument; together with the appropriate dose of 'abstract nonsense', this proves the lemma. □

# Concluding the proof

The *second step* is to show that the family $\{u_\tau(t)\}_{0<\tau\le 1}$ converges as $\tau \to 0+$ for each fixed $t \in \mathbb{R}$ to some $u(t) \in \mathcal{H}$ in the *weak topology* of $\mathcal{H}$, and that the convergence is even locally uniform with respect to $t \in \mathbb{R} \setminus \{0\}$; the limit function $u(t)$ turns out to be *continuous* in $t \in \mathbb{R}$, again in the weak topology of $\mathcal{H}$.

Since $\mathcal{H}$ is separable by assumption, the ball $B_{\|f\|}(0)$ is metrizable in the weak topology and the claim follows from the *Ascoli–Arzelà theorem*.

*Step III:* We have established convergence of the family $\{u_\tau(t)\}_{0<\tau\le 1}$ in two different topologies, the *weak topology* of $\mathcal{H}$, (locally) uniformly for $t \in \mathbb{R} \setminus \{0\}$, and the the convergence to $(I + itH_P)^{-1}Pf$ in $L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{H})$ in the strong, and therefore also weak sense. This allows us to conclude that the two limit coincide,

$$u(t) = (I + itH_P)^{-1}Pf \quad \text{for all } t.$$

# Concluding the proof

*Step IV:* The rest is easy, to check that the family $\{u_\tau(t)\}_{0<\tau\le 1}$ converges as $\tau \to 0+$ for any fixed $t \in \mathbb{R}$ to $u(t) \equiv (I + itH_P)^{-1}Pf$ also in the *strong topology* of $\mathcal{H}$, and furthermore, that the convergence is even locally uniform with respect to $t \in \mathbb{R} \setminus \{0\}$.

Since we have already established the *weak* convergence, we need only to show that the $\tau$-families of the *norms* of these vectors converge.

In analogy with an earlier argument, we observe how $\mathrm{Re}\,\langle u_\tau(t), P(\tau)f\rangle$ behaves as $\tau \to 0+$, to conclude that $P(\tau)u_\tau(t) \longrightarrow Pu(t)$. As the part of $u_\tau(t)$ on the orthogonal complement to $P(\tau)\mathcal{H}$ is trivial, we arrive finally at the result we seek. □

*Remark:* We also get $(|t|G(t\tau))^{1/2}P(\tau)u_\tau(t) \longrightarrow 0$, but predictably no result for $H^\pm$ associated with the imaginary part.

**Corollary**

*For a constant projection-valued function, $P(\tau) = P$ we get the validity of the Zeno product formula in the strong operator topology.*

# Example: position measurement

Consider a *perpetual position ascertaining* to an open domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary, thought of as the *detector volume*, and associate with it the orthogonal projection $P$ acting as multiplication operator by the indicator function $\chi_\Omega$.

Suppose that, apart from the measurement, the particle is *free*, that is, its dynamics is described by the Hamiltonian $H = -\Delta$, obviously positive.

The domain density assumption of $H^{1/2}P = (-\Delta)^{1/2}\chi_\Omega$ is satisfied, since it contains $C_0^\infty(\Omega) \cup C_0^\infty(\mathbb{R}^d \setminus \overline{\Omega})$, where $\overline{\Omega}$ is the closure of $\Omega$.

Consider further the *Dirichlet Laplacian* $-\Delta_\Omega$ in $L^2(\Omega)$ defined as the *Friedrichs extension* of the appropriate quadratic form. It is not difficult to check that

$$(-\Delta)_P := ((-\Delta)^{1/2}P)^*(-\Delta)^{1/2}P$$

is densely defined and its restriction to $L^2(\Omega)$ is nothing but $-\Delta_\Omega$ with the domain $D[-\Delta_\Omega] = W_0^1(\Omega) \cap W^2(\Omega)$.

# Example: position measurement

In this situation, our main result says that

$$\operatorname*{s-lim}_{n\to\infty} \left(P\, \mathrm{e}^{-it(-\Delta/n)} P\right)^n = \mathrm{e}^{-it(-\Delta_\Omega)} P$$

holds in the strong operator topology of $\mathcal{B}(L^2(\mathbb{R}^d))$, the Banach space of bounded linear operators on $L^2(\mathbb{R}^d)$.

In other words, the perpetual reduction of the wave function forces the particle to move within the region $\Omega$ as if its boundary was Dirichlet, i.e. *hard wall*. This is, of course, the expected conclusion; on the formal level the limit was calculated using the stationary phase method in

P. Facchi, S. Pascazio, A. Scardicchio, and L.S. Schulman: Zeno dynamics yields ordinary constraints, *Phys. Rev.* **A65** (2001), 012108.

Only the present result, however, justifies this claim rigorously.

# Where to go further

- The argument we presented is not exactly simple, and it would be good to have an *alternative proof*. It seems useful to think of methods *not* based on Chernoff's theorem.

- Various open problems of Zeno type can be found in the *open system* setting where one studies convergence of the expressions $(M\mathrm{e}^{t\mathcal{L}/n})^n$, where $\mathcal{L}$ is the *generator of a dynamical semigroup* on the space $\mathcal{T}(\mathcal{H})$ of *trace-class operators*, and $M = \{M_j\}$ is a *quantum operation*, which means a *completely positive*, *trace non-increasing* map on $\mathcal{T}(\mathcal{H})$. For some recent results see

S. Becker, N. Datta, R. Salzmann: Quantum Zeno effect for open quantum systems, arXiv:2010.04121

# It remains to say

Thank you for your attention!