On loops, cones, and stars: striving for the optimal shape

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Motto: Ubi materia, ibi geometria (Johannes Keppler)

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What brought us here?

Or a different question: do we have a colleague ready to examine critically every principle, even if comes from highest authorities? The answer is *yes!*

Round is nice

A trademark question in spectral geometry is about the *shape* which makes a given property *optimal*.

Quite often the optimal shape has a *rotational symmetry*; each of us can quote examples starting from the *Faber-Krahn inequality*

Note that such a symmetry always raised some fascination; recall how an old scifi described a *computer-generated ideal beauty:*

The canvas was covered by a complicated pattern of infinite number arabesques each of which decomposed into arabescues more minutious [...] and in the middle of the page was an empty, **ideally round white disc**.

Stanislaw Łem: The Magellanic cloud

An important example

RESEARCH ANNOUNCEMENTS

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PROOF OF THE PAYNE-PÓLYA-WEINBERGER CONJECTURE

MARK S. ASHBAUGH AND RAFAEL D. BENGURIA

In 1955 and 1956 Payne, Pólya, and Weinberger considered the problem of bounding ratios of eigenvalues for homogeneous membranes of arbitrary shape [PPW1, PPW2]. Among other things, they showed that the ratio λ_2/λ_1 of the first two eigenvalues was less than or equal to 3 and went on to conjecture that the optimal upper bound for λ_2/λ_1 was its value for the disk, approximately 2.539. It is this conjecture which we establish below.

Since 1956 various authors have attempted to prove the conjecture of Payne, Pólya, and Weinberger and some have been able to improve upon the constant 3. Specifically, Brands [Br] in 1964 obtained the value 2.686, de Vries [dV] in 1967 obtained 2.658, and Chiti [Ch2] in 1983 obtained 2.586. In addition, Thompson [Th] gave the natural extension of the PPW argument to dimension n, obtaining

$$\lambda_2/\lambda_1 \leq 1 + 4/n \tag{1}$$

as the bound for the analogous problem (eigenvalues of the Dirichlet Laplacian on a bounded domain in $\mathbf{R}^n$) and made the natural

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0273-0979/91 $1.00 + $.25 per page

Rafael certainly contributed to this fascination, through one his most famous results, obtained together with Mark: the proof, or rather proofs, of the *Payne-Pólya-Weinberger inequality*

It is useful to go through the history of attempts to demonstrate the PPW conjecture to realize what a *tour de force* this result is

But things are not always that simple

In this talk I will stay on what one can call the *Faber-Krahn level* asking shapes optimizing the *ground-state energy*

We will see that the rotational symmetry keeps playing an important role but the results may depend substantially on both the *boundary conditions* and *topology*

As the first example, consider the optimization problem for a *Robin Laplacian* associated with the quadratic form

$$\psi \mapsto \int_\Omega |\nabla\psi(x)|^2 \mathrm{d}x + \alpha \int_{\partial\Omega} |\psi(s)|^2 \mathrm{d}s$$

on $H^1(\Omega)$. As long as $\alpha > 0$ the result is similar to Faber-Krahn: the principal eigenvalue $\lambda_1^\alpha(\Omega)$ is *uniquely minimized* among the sets of *the same volume* by $\lambda_1^\alpha(\mathcal{B})$ where $\mathcal{B}$ is the *ball*

Attractive Robin boundary

The situation changes if $\alpha < 0$. In this case Bareket[1] conjectured that $\lambda_1^\alpha(\mathcal{B})$ is now *maximal* among ground states for sets of the same volume

This is true for local deformations of a ball, but *fails globally*: Freitas & Krejčiřík[2] showed that $\lambda_1^\alpha(\Omega) > \lambda_1^\alpha(\mathcal{B})$ may hold if Ω is a *spherical shell*.

On the other hand, note that by Antunes et al.[3] the analogous inequality *does hold* if we compare sets of *the same perimeter*

Furthermore, in *two dimensions* Krejčiřík & Lotoreichik[4] showed that $\lambda_1^\alpha(\Omega) \geq \lambda_1^\alpha(\mathcal{B})$ holds if Ω is the *exterior of a convex set of the same area/perimeter* as $\mathcal{B}$; under additional geometrical constraints the result extends to non-convex domains and higher dimensions[5]

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[1] M. Bareket: On an isoperimetric inequality for the first eigenvalue of a b.v.p., *SIAM J. Math. Anal.* **8** (1977), 280-287

[2] P. Freitas, D. Krejčiřík: The first Robin eigenvalue with negative boundary parameter, *Adv. Math.* **280** (2015), 322-339

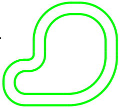
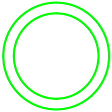
[3] P. Antunes, P. Freitas, D. Krejčiřík: Bounds and extremal domains for Robin ..., *Adv. Calc. Var.* **10** (2017), 357-380

[4] D. Krejčiřík, V. Lotoreichik: Optimisation of the lowest Robin eigenvalue ..., *J. Convex Anal.* **25** (2018), 319-337

[5] D. Krejčiřík, V. Lotoreichik: Optimisation ..., II: non-convex domains and higher dim..., `arXiv:1707.02269 [math.SP]`

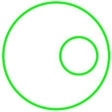
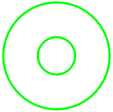
Back to the Dirichlet case

Even when the boundary is Dirichlet, the situation is not simple and *topology may play role*. Let us mention pictorially two examples in maximum symmetry may lead to *maximum* of the principal eigenvalue If we seek extremum among *strips of fixed length and width* we have[6]

ground state of  $<$ ground state of 

whenever the strip is not a circular annulus.

Similarly, for a *circular obstacle in circular cavity* we have[7]

ground state of  $<$ ground state of 

whenever the obstacle is off center; the minimum is reached when it is touching the boundary

[6] P.E., E.M. Harrell, M. Loss: Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature, in *Proceedings of QMath7*, Birkhäuser, Basel 1999; pp. 47-53

[7] E.M. Harrell, P. Kröger, K. Kurata: On the placement of an obstacle ..., *SIAM J. Math. Anal.* **33** (2001), 240-259

Singular potentials

The main aim of this talk is to show that similar *isoperimetric-type problems* arise in $\mathbb{R}^d$ if the particle is subject to suitable classes of *singular potentials*, specifically for Schrödinger operators of the type

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \quad \alpha > 0\,,$$

in $L^2(\mathbb{R}^d)$, where Γ is a manifold or a more general subset of $\mathbb{R}^d$ with some (not very strong, Lipschitz is enough) regularity properties

Motivation: (a) Interesting mathematical objects, in particular, since their spectral properties reflect the geometry of Γ
(b) an alternative model of *quantum graphs* and *generalized graphs* with the advantage that tunneling between edges is not neglected

Remarks: (a) In this talk we will consider primarily situations where Γ is a manifold or complex of *codimension one*
(b) However, we will also say something about *more singular* interactions, either of *δ' type*, or with support of *codimension two*

The δ-interaction supported by a manifold

A natural tool to define the corresponding singular Schrödinger operator is to employ the appropriate quadratic form, namely

$$q_{\delta,\alpha}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \alpha\|f|_\Gamma\|^2_{L^2(\Gamma)}$$

with the domain $H^1(\mathbb{R}^d)$ and to use the first representation theorem.

If Γ is a *smooth manifold* with $\operatorname{codim}\Gamma = 1$ one can easily check that the form defines a unique self-adjoint operator $H_{\alpha,\Gamma}$, which can alternatively characterized by boundary conditions: it acts as $-\Delta$ on functions from $H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\Big|_+ - \frac{\partial\psi}{\partial n}(x)\Big|_- = -\alpha(x)\psi(x)$$

This explains the formal expression as describing the attractive δ-interaction of strength $\alpha(x)$ perpendicular to Γ at the point x.

Alternatively, one sometimes uses the symbol $-\Delta_{\delta,\alpha}$ for this operator.

Planar loops

Let Γ be a *loop* in $\mathbb{R}^d$, $d \geq 2$, parametrized by its arc length, i.e. a piecewise differentiable function $\Gamma : [0, L] \to \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot{\Gamma}(s)| = 1$ for all but finitely many $s \in [0, L]$. We have[8]

Theorem ([E-Harrell-Loss'06])

Let $d = 2$. For any $\alpha > 0$ and $L > 0$ we have $\lambda_1(\alpha, \Gamma) \leq \lambda_1(\alpha, \mathcal{C})$, where $\mathcal{C}$ is a circle of perimeter L, the inequality being sharp unless Γ is congruent with $\mathcal{C}$.

Proof idea: One employs a generalized *Birman-Schwinger principle* by which there is one-to-one correspondence between eigenvalues $-\kappa^2$ of $H_{\alpha,\Gamma}$ and solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \text{where } \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s) - \Gamma(s')|)$$

on $L^2([0, L])$, where K_0 is the Macdonald function

[8] P.E., E.M. Harrell, M. Loss: Inequalities for means of chords, with application ..., *Lett. Math. Phys.* **75** (2006), 242-233

Proof idea, continued

We employ *inequalities on mean values of chords* denoted as $C_L^p(u)$:

$$\int_0^L |\Gamma(s+u)-\Gamma(s)|^p \mathrm{d}s \le \frac{L^{1+p}}{\pi^p}\, sin^p \frac{\pi u}{L}\,, \quad p>0,\; u\in(0,\tfrac12 L]$$

This *may not be true for all $p>0$*, however, a simple Fourier analysis allows one to demonstrate the following result:

Proposition

$C_L^2(u)$ is valid for any $u\in(0,\frac12 L]$, and the inequality is strict unless Γ is a planar circle; by convexity the same is true for all $p<2$.

Using a variational argument together with the fact that $K_0(\cdot)$ appearing in the resolvent kernel is *strictly monotonous and convex* the optimization problem for $\mathcal{R}^\kappa_{\alpha,\Gamma}$ is reduced to the inequality $C_L^1(u)$ being thus proved $\square$

Remark: The (reverse) inequalities hold also for $p\in[-2,0)$ showing, e.g., that a *charged loop in the absence of gravity takes a circular form*

A discrete analogue: polymer loops

Consider the same loop as above with *point interactions* placed at the arc distances $\frac{jL}{N}$, $j = 0, \dots, N_1$, in other words, the formal Hamiltonian

$$H^N_{\alpha,\Gamma} = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta\left(x - \Gamma\left(\frac{jL}{N}\right)\right)$$

in $L^2(\mathbb{R}^d)$, $d = 2, 3$, where the last term has to be properly defined

We are interested in the shape of Γ which *maximizes* the ground state energy provided, of course, that the discrete spectrum of $H^N_{\alpha,\Gamma}$ is non-empty; this requirement is *nontrivial for $d = 3$*

Introduce the *generalized boundary values* as the coefficients in the expansion of H^*_Y where H_Y is the Laplacian restricted to functions vanishing at the vicinity of the points of Y

Point interactions 'necklaces'

A reminder: fixing the points $y_j \in Y$ the said expansion look as follows

$$\psi(x) = -\frac{1}{2\pi} \log|x - y_j|\, L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|), \quad d = 2,$$

$$\psi(x) = \frac{1}{4\pi|x - y_j|}\, L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|), \quad d = 3.$$

Local self-adjoint extension are then given by

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0\,, \quad \alpha \in \mathbb{R};$$

for details we refer to [AGHH'88, 05]. Then we have[9]

Theorem ([E'06])

The ground state of $H^N_{\alpha,\Gamma}$ is uniquely maximized by a N-regular polygon

[9] P.E.: Necklaces with interacting beads: isoperimetric ..., in *Proceedings UAB05*, AMS, Providence 2006; pp. 141-149

New effects in three dimensions

In three dimensions the discrete spectrum of $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)$ *may be empty* is α is small enough. As an example, for Γ being a *sphere of radius* R bound states are known[10] to exist *iff* $\alpha R > 1$

This raises the following question: given the *critical sphere*, $\alpha R = 1$, would its *deformation produce a discrete spectrum?* One answer is[11]

Theorem ([E-Fraas'09])

Let Γ_ϵ by a deformation of the sphere expressed in spherical coordinates as $r(\theta,\phi) = R(1+\epsilon\rho(\theta,\phi))$ where ρ is nonzero function of zero mean. If H_{α,Γ_0} is critical, $\sigma_{\rm disc}(H_{\alpha,\Gamma_\epsilon}) \neq \emptyset$ holds for all nonzero ϵ small enough.

Remarks: (a) The results *fails to hold globally*: if a *surface-preserving* deformation of a critical surface is *elongated enough*, the discrete spectrum is *empty*.

(b) In contrast, deformation of a critical surface *always produces a nonvoid discrete spectrum* if it is *capacity preserving*

[10] J.-P. Antoine, F. Gesztesy, J. Shabani: Exactly solvable models of ..., *J. Phys. A: Math. Gen.* **20** (1987), 3687-3712.

[11] P.E., M. Fraas: On geometric perturbations of critical Schrödinger operators with ..., *J. Math. Phys.* **50** (2009), 112101

More singular interactions in 2D

So far I spoke of an old stuff. Let us now look at some fresh results. Consider again a planar loop a replace δ by *δ' interaction*. The latter can be defined by boundary condition or using the quadratic form,

$$q_{\delta',\beta}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{\beta}\|[f]_\Gamma\|^2_{L^2(\Gamma)}$$

defined on $H^1(\mathbb{R}^2 \setminus \Gamma)$, where $[f]_\Gamma := f_+|_\Gamma - f_+|_\Gamma$. Then we have[12]

Theorem ([Lotoreichik'18])

For any $\beta > 0$ we have $\max_{|\Gamma|=L} \lambda_1^\beta(\Gamma) = \lambda_1^\beta(\mathcal{C})$, where $\mathcal{C}$ is a circle of perimeter $L > 0$ and the maximum is taken over all C^2 smooth loops.

The Birman-Schwinger method does not work in this case, one has to use instead *locally orthogonal coordinates* in a way similar to those employed in [Krejčiřík-Lotoreichik'18] to treat exterior of a Robin obstacle

[12] V. Lotoreichik: Spectral isoperimetric inequality for the δ' interaction on a contour, arXiv:1810.05457

Cones

Let us now involve next some new geometries into the game.

Our now topic will be singular Schrödinger operators $H_{\alpha,\Gamma}$ having a *conical surface* as the interaction support Γ.

We start with some *definitions*: Let $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth loop on the 2D unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ of length $|\mathcal{T}|$ without self-intersections. We distinguish between *circular* and *non-circular loops*. A circle $\mathcal{C}$ on $\mathbb{S}^2$ has, of course, the length $|\mathcal{C}| \leq 2\pi$.

The C^2-smooth *cone* $\Sigma_R(\mathcal{T}) \subset \mathbb{R}^3$ of radius $R \in (0, \infty]$ with a C^2-smooth loop $\mathcal{T} \subset \mathbb{S}^2$ as its *cross-section* is

$$\Sigma_R(\mathcal{T}) := \left\{ r\mathcal{T} \in \mathbb{R}^3 \colon r \in [0, R) \right\};$$

it is called *finite* (or *truncated*) if $R < \infty$ and *infinite* otherwise

The cone $\Sigma_R(\mathcal{T})$ is called *circular* if its cross-section $\mathcal{T}$ is a circle and *non-circular* otherwise. An infinite circular cone with the cross-section length 2π is, in fact, a plane.

Results in the finite case

If $R < \infty$ it is not difficult to check that $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [0,\infty)$; we are interested in the principal eigenvalue $\lambda_1(H_{\alpha,\Gamma})$. We have[13]

Theorem ([E-Lotoreichik'17])

Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $L := |\mathcal{C}| = |\mathcal{T}| \in (0, 2\pi]$. Let $\Gamma_R := \Sigma_R(\mathcal{C})$ and $\Lambda_R := \Sigma_R(\mathcal{T})$ be finite cones of radius $R > 0$ with the cross-sections $\mathcal{C}$ and $\mathcal{T}$, respectively; then

- $\#\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma_R}) \geq 1$ *if and only if* $\alpha > \alpha_{\mathrm{crit}}$ *for a certain value* $\alpha_{\mathrm{crit}} = \alpha_{\mathrm{crit}}(L,R) > 0$.
- $\#\sigma_{\mathrm{disc}}(H_{\alpha,\Lambda_R}) \geq 1$ *for all* $\alpha \geq \alpha_{\mathrm{crit}}$ *(the borderline case $\alpha = \alpha_{\mathrm{crit}}$ is included) and the spectral isoperimetric inequality*

$$\lambda_1(H_{\alpha,\Lambda_R}) < \lambda_1(H_{\alpha,\Gamma_R})$$

is satisfied for all $\alpha > \alpha_{\mathrm{crit}}$.

[13] P.E., V. Lotoreichik: A spectral isoperimetric inequality for cones, *Lett. Math. Phys.* **107** (2017), 717-732

Cones, continued

We see the effect we have encountered before with spheres:

Corollary

Any (fixed-radius, smooth, conical) deformation of a critical circular cone gives rise to a non-void discrete spectrum of the corresponding $H_{\alpha,\Gamma}$

On the other hand, the spectrum is different for *infinite cones*: we have $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [-\frac{1}{4}\alpha^2, \infty)$ and the discrete spectrum is *infinite* except in the trivial case of a plane

Moreover, we even know its *accumulation rate*: for circular cones we have[14]

$$\mathcal{N}_{-\frac{1}{4}\alpha^2 - E}(-\Delta_{\delta,\alpha}) \sim \frac{\cot\theta}{4\pi}\,|\ln E|\,, \quad E \to 0+\,,$$

and as a similar results also holds in the non-circular case[15]

[14] V. Lotoreichik, T. Ourmières-Bonafos: On the bound states of Schrodinger ..., *Comm. PDE* **41** (2016), 999-1028

[15] T. Ourmières-Bonafos, K. Pankrashkin: Discrete spectrum of interactions ..., *Appl. Anal.* **97** (2018), 1628-1649

Results in the infinite case

Theorem ([E-Lotoreichik'17])

Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $L := |\mathcal{C}| = |\mathcal{T}| \in (0, 2\pi)$. Let $\Gamma_\infty := \Sigma_\infty(\mathcal{C})$ and $\Lambda_\infty := \Sigma_\infty(\mathcal{T})$ be infinite cones with the cross-sections $\mathcal{C}$ and $\mathcal{T}$, respectively; then for any $\alpha > 0$ we have

- $\#\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma_\infty}) \cap (-\infty, -\frac{1}{4}\alpha^2) \geq 1$
- *the spectral isoperimetric inequality* $\lambda_1(H_{\alpha,\Lambda_\infty}) \leq \lambda_1(H_{\alpha,\Gamma_\infty})$ *is valid*

Once we prove the previous theorem, the present one can be demonstrated by analyzing the limit $R \to \infty$ of the finite-cone operators. It is not trivial but the argument is straightforward (with an extra work one can prove that the maximum is sharp)

Proof sketch in the finite case

The strategy is to employ the *generalized Birman-Schwinger principle* in combination with a minimization result about the energy of knots, cf. [E-Harrell-Loss'06] and an earlier paper by Abrams et al.[16]

The former we have used before; it can be written as

$$\dim\ker\left(H_{\alpha,\Sigma}+\kappa^2\right)=\dim\ker\left(I-\alpha S_\Sigma(\kappa)\right)$$

for any $\kappa>0$, where

$$\left(S_\Sigma(\kappa)\psi\right)(x):=\int_\Sigma G_\kappa(x-y)\psi(y)\,\mathrm{d}\sigma(y)$$

It implies, in particular, the following equivalences:

- $\#\sigma_{\rm disc}(H_{\alpha,\Sigma})\cap(-\infty,-\kappa^2)\geq 1$ *iff* $\mu_\Sigma(\kappa)>\alpha^{-1}$, where $\mu_\Sigma(\kappa)>0$ is the largest eigenvalue of $S_\Sigma(\kappa)$
- $\lambda_1(H_{\alpha,\Sigma})=-\kappa^2$ *iff* $\mu_\Sigma(\kappa)=\alpha^{-1}$

[16] A. Abrams et al.: Circles minimize most knot energies, *Topology* **42** (2003), 381-394

Proof sketch, continued

We also note the eigenvalue $\mu_\Sigma(\kappa)$ is *simple* and the corresponding eigenfunction can be chosen *positive*

To proceed, we need a suitable *parametrization of the cone*. We begin with arc-length parametrization of the cross section, $\tau\colon [0,L] \to \mathbb{S}^2$ with $|\dot\tau| \equiv 1$ and put

$$\sigma\colon [0,R) \times [0,L] \to \mathbb{R}^3, \quad \sigma(r,s) := r\tau(s)\,; \tag{1}$$

this defines natural co-ordinates (r,s) on Σ_R. We find easily

Proposition

Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\Gamma_R := \Sigma_R(\mathcal{C})$. Then the eigenfunction corresponding to the largest eigenvalue of the BS operator $S_{\Gamma_R}(\kappa)$ is rotationally invariant, i.e. it depends on the distance from the tip of the cone only.

Proof sketch, continued

Now we employ an inequality related to $C_L^p(u)$ used earlier, known also from other sources:[17] for a C^2-smooth loop $\mathcal{T} \subset \mathbb{S}^2$ we put

$$\Phi_f[\mathcal{T}] := \int_0^L \int_0^L f(|\tau(s) - \tau(t)|^2)\mathrm{d}s\mathrm{d}t$$

Then we have

Proposition

Let $f \in C([0,\infty);\mathbb{R})$ be convex and decreasing. Let $\mathcal{C} \subset \mathbb{S}^2$ be a circle and $\mathcal{T} \subset \mathbb{S}^2$ be a C^2-smooth non-circular loop such that $|\mathcal{T}| = |\mathcal{C}| = L$ for some $L \in (0, 2\pi]$. Then the following isoperimetric inequality holds

$$\Phi_f[\mathcal{C}] < \Phi_f[\mathcal{T}].$$

[17] G. Lűko: On the mean length of the chords of a closed curve, *Israel J. Math.* **4** (1966), 23-32
J. O'Hara: *Energy of knots and conformal geometry*, World Scientific 2003

Proof sketch, concluded

In particular, the above proposition holds with the function

$$f(x) := \frac{e^{-a\sqrt{bx+c}}}{\sqrt{bx+c}}\,,$$

which is convex and decreasing for any positive a, b, c.

Comparing the BS operators for the circular and non-circular cones with the use of the indicated parametrization we employ such isoperimetric inequalities with

$$a(r,r') := \kappa, \quad b(r,r') := rr', \quad c(r,r') := (r-r')^2\,;$$

we have to exclude situations where $r=0$, $r'=0$ or $r=r'$, but this is a zero measure set. With a bit of work, this yields finally the result. $\square$

Another object of interest: stars

Let us return to planar leaky graphs and consider *star-shaped* ones:

We consider a *star graph* $\Sigma_N = \Sigma_N(L) \subset \mathbb{R}^2$, which has $N \geq 2$ edges of length $L \in (0, \infty]$ each, enumerated in the clockwise manner.

They are characterized by the angles $\phi = \phi(\Sigma_N) = \{\phi_1, \phi_2, \dots, \phi_N\}$ between the neighboring edges, $\phi_n \in (0, 2\pi)$ for all $n \in \{1, \dots, N\}$ and $\sum_{n=1}^{N} \phi_n = 2\pi$.

By Γ_N we denote the star graph with maximum symmetry, in other words, $\phi = \phi(\Gamma_N) = \left\{\frac{2\pi}{N}, \frac{2\pi}{N}, \dots, \frac{2\pi}{N}\right\}$.

Given $\alpha > 0$, we ask again about the *spectral threshold* of the operator H_{α,Σ_N} corresponding to the formal expression $-\Delta - \alpha\delta(x - \Sigma_n)$

An illustration

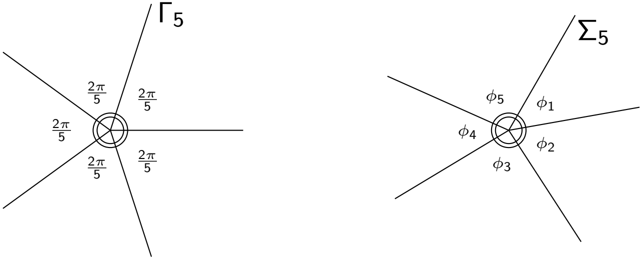


Figure: The star graphs Γ_5 and Σ_5 with $N = 5$ and $L < \infty$.

Star optimization

It is easy to see that $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [0,\infty)$ if $L < \infty$ and with the set $\sigma_{\mathrm{ess}}(H_{\alpha,\Sigma_N}) = [-\frac{1}{4}\alpha^2,\infty)$ if $L = \infty$.

We mentioned that $\sigma_{\mathrm{disc}}(H_{\alpha,\Sigma_N}) \neq \emptyset$ if $L < \infty$, and the same is true also for an infinite star[18] unless we have simultaneously $L = \infty$, $N = 2$, and $\phi = \{\pi,\pi\}$. For the lowest eigenvalue we have[19]

Theorem ([E-Lotoreichik'18])

For any $\alpha > 0$ we have the relation

$$\max_{\Sigma_N(L)} \lambda_1^\alpha\left(\Sigma_N(L)\right) = \lambda_1^\alpha\left(\Gamma_N(L)\right),$$

where the maximum is taken over all star graphs with $N \geq 2$ edges of a given length $L \in (0,\infty]$. If $L < \infty$ the equality is achieved iff Σ_N and Γ_N are congruent.

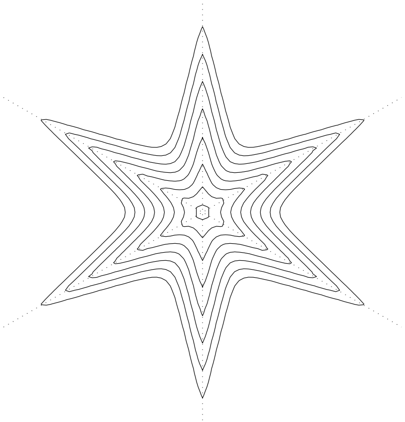
[18] P. Exner, T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys. A* **34** (2001), 1439-1450

[19] P. Exner, V. Lotoreichik: Optimization of the lowest eigenvalue for leaky ..., in *Proceedings of QMath13*, AMS 2018

Star optimization, continued

For *infinite stars* the condition $\Sigma_N(\infty) \cong \Gamma_N(\infty)$ is apparently also necessary and sufficient, just the method used in the proof of the theorem needs to be amended

Moreover, the ground state has then some *esthetic quality*:



Star optimization, concluded

Idea of the proof: We again employ Birman-Schwinger principle. Writing the BS operators for $L < \infty$, one can interchange integration over the variables parametrizing the edges and summation over the edges

In the internal part we then employ an inequality for *polygon chords* similar to the one used in the proof of the isoperimetric inequality for *point-interaction necklaces* mentioned above [E'06]

To establish the relation for $L = \infty$ one uses the *strong resolvent convergence* which gives, in particular,

$$\lim_{L\to\infty} \lambda_1^\alpha(\Sigma_N(L)) = \lambda_1^\alpha(\Sigma_N(\infty))$$

and the analogous relation for symmetric stars

Stars in three dimensions

Albeit technically nontrivial, the previous problem was simple in the sense that the result was easy to guess

This would not be the same if we consider an analogue of the star optimization problem *in three dimensions*, i.e. for Schrödinger operators with a singular interaction supported by a 'sea urchin' shape set Γ of N 'pins', finite or semi-infinite

The first question is how to define the operator. The interaction is more singular as we have now $\mathrm{codim}\,\Gamma = 2$ and one has no 'natural' quadratic form to be used

Without being too technical, one takes the Laplacian defined on functions that are H^2 outside Γ and imposed the *generalized boundary conditions*[20] defining 2D point interaction in the cross planes to the edges of Γ

[20] P.E., S. Kondej: Curvature-induced bound states for a δ interaction supported by ..., *Ann. H. Poincaré* **3** (2002), 967-981

Recall some related problems

Optimization problem for 3D stars is no doubt nontrivial. The first analogue coming to mind is the *Thomson problem*[21] about distribution of N point charges on the surface of a sphere

Recall that a rigorous solution is known for a few small N cases, for instance, a (computer-assisted) proof for $N = 5$ was presented only recently[22]. Note also that twenty years ago Stephen Smale included it into the list of eighteen 'new Hilbert problems' for the 21st century

Various generalizations of the problem triggered numerous mathematical investigations in algebraic combinatorics[23]

Unfortunately – and this is another point Rafael often stressed – *physics is forgotten at that!* They quote, for instance, *Tamme's problem* in botany but not Thomson. The *plum-pudding model* was wrong, of course, but still physics was the original inspiration here!

[21] J.J.Thomson: On the structure of the atom: an investigation of the stability ..., *Phil. Mag.* **7** (1904), 237-265

[22] R.E. Schwartz: The five-electron case of Thomson's problem, *Experim. Math.* **22** (2013), 157-186

[23] H. Cohen, A. Kumar: Universally optimal distribution of point on the sphere, *J. AMS* **20** (2007), 99-148
E. Bannai, E. Bannai: A survey of spherical designs and algebraic combinatorics ..., *Eur. J. Combin.* **30** (2009), 1392-1425

Universal optimality by Cohen and Kumar

Consider N points $\{x_i\}_{i=1}^N$ living on the unit sphere S^2. They form an *M-spherical design* if for any polynomial $x \mapsto p(x)$ on $\mathbb{R}^3$ of total degree M the equivalence one has $\int_{S^2} p(x)\mathrm{d}x = \frac{1}{N}\sum_i^N p(x_i)$ holds

Let m be the number of *different* inner products between distinct $\{x_i\}_{i=1}^N$. They form a *sharp configuration if* it is $2m-1$ spherical design

By [Cohen-Kumar'07] sharp configurations are *universally optimal* meaning that they minimize *any* potential energy $f:[0,4]\to\mathbb{R}$ which is *completely monotonous*, i.e. it satisfies $(-1)^k f^{(k)} \ge 0$ for all $k \ge 0$
in three dimensions there are five sharp configurations:

- $N=2$, *antipodal points*
- $N=3$, *simplex* with inner product $-1/2$,
- $N=4$, *tetrahedron* – simplex with inner product $-1/3$,
- $N=6$, *octahedron* – cross polytope with inner products $-1, 0$,
- $N=12$, *icosahedron* with inner products $-1/, \pm 1/\sqrt{5}$.

Remark: The remaining Platonic solids, *cube* and *dodekahedron*, do not qualify for universality having m=3 and 4, respectively. Note that they *do not represent Thomson problem solutions!*

Application to star leaky graphs

One may wonder what has the mentioned *minimization* problem to do with the *maximization* of the ground state eigenvalues. The answer is that, as in the previous cases, that the problem is equivalent to minimization of the (norm of) the Birman-Schwinger operator. We have

Lemma

Consider an N-arm star with edges of length $L \in (0, \infty]$ determined by unit vectors $\{\bar{\gamma}_i\}_{i=1}^N$, and let $\{\bar{\sigma}_i\}_{i=1}^N$ corresponds to a sharp-configuration star. Then we have

$$\sum_{i,j\; i\neq j} T_{\kappa;s,t}(|\bar{\gamma}_i - \bar{\gamma}_j|^2) \geq \sum_{i,j\; i\neq j} T_{\kappa;s,t}(|\bar{\sigma}_i - \bar{\sigma}_j|^2)$$

for any $s, t \in [0, L]$ and the inequality is sharp unless the two stars are congruent. Here $T_{\kappa;s,t}(x) := \frac{\mathrm{e}^{-\kappa\sqrt{a+bx}}}{4\pi\sqrt{a+bx}}$ with $a = (s-t)^2$ and $b = st$

Application to star leaky graphs, continued

Next we use the fact that the largest eigenvalue of the Birman-Schwinger operator corresponding to a sharp-configuration star has the *maximum symmetry*, $\tilde{f}_\sigma = (f_\sigma, \dots, f_\sigma) \in \oplus_1^N L^2([0, L])$

Then $\sup Q_{\kappa,\gamma} \geq (Q_{\kappa,\gamma}\tilde{f}_\sigma, \tilde{f}_\sigma) \geq \sup Q_{\kappa,\sigma}$ holds according to the above lemma, which allows us to make the following conclusion[24]

Theorem

Assume that $N \in \{2, 3, 4, 6, 12\}$, then the ground state energy of the N-arm leaky star assumes the unique maximum for $\gamma = \sigma$, where σ is the corresponds to the appropriate sharp configuration listed above.

[24] P.E., S. Kondej, *in preparation*

Open questions

Ignoring various technical ones which appeared in the course of the presentation, there are deeper and more interesting questions, for instance

- what a *non-constant coupling strength* α would do with the optimization?
- what would an optimal shape for a 2D δ' *star*? The answer is easy to guess when N is even but *not at all for an odd number of edges*
- no need to stress how beautiful and difficult the *3D star optimization* problem described above is; one want to know what happens for the N's different from the five listed cases
- can one say something about *higher eigenvalues*, say, along the lines of Payne-Pólya-Weinberger-Ashbaugh-Benguria?
- etc., etc.

It remains to say

¡Muchos años más felices, Rafael!