Geometrically induced spectrum in soft waveguides and potential well arrays

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My topic: geometric effects in the spectrum

They are numerous. What I am going to discuss here concerns their manifestations in what one can call *guided quantum dynamics*.

The simplest example of such an effect is provided by the Dirichlet Laplacian in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a *tubular region*.

It is well known that the spectrum depends on the *geometry of Ω*, for instance, if such a hard-wall tube is *bent, but asymptotically straight*, we have $\sigma_{\mathrm{disc}}(-\Delta_{\mathrm{D}}^{\Omega}) \neq \emptyset$; there is a number of related results including other dimensions, boundaries, and different geometric perturbations:

P.E., H. Kovařík: *Quantum Waveguides*, Springer, Cham 2015

Apart from a mathematical interest such geometrically induced bound states are of practical importance as models of various *waveguide effects*, mainly in quantum theory, but also in *electromagnetism* or *acoustics*

J.T. Londergan, J.P. Carini, D.P. Murdock: *Binding and Scattering in Two-Dimensional Systems. Applications to Quantum Wires, Waveguides and Photonic Crystals*, Springer LNP m60, Berlin 1999

E.B. Davies, L. Parnovski: Trapped modes in acoustic waveguides, *Quart. J. Mech. Appl. Math.* **51** (1998), 477–492.

A similar effect: singular Schrödinger operators

Considered as a model of guiding in real material structures, Dirichlet boundary is naturally an idealization; in the language of QM it means, in particular, that *tunneling* between different parts of the guide is forbidden

This motivated investigation *'leaky'* structures where the Hamiltonian is a *singular Schrödinger operator* formally written as $-\Delta - \alpha\delta(x-\Gamma)$ with $\alpha > 0$, Γ being is a curve, a graph, a surface, etc.

The geometry influences the spectrum again: as an example consider a *non-straight, piecewise C^1-smooth curve* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ such that
(a) $|s-s'| \geq |\Gamma(s)-\Gamma(s')| \geq c|s-s'|$ holds for some $c \in (0,1)$, and
(b) we have $1 - \frac{|\Gamma(s)-\Gamma(s')|}{|s-s'|} \leq d\left[1+|s+s'|^{2\mu}\right]^{-1/2}$ for some $d>0$, $\mu > \frac{1}{2}$ and $\omega \in (0,1)$ provided that $\omega < \frac{s}{s'} < \omega^{-1}$. Then we have

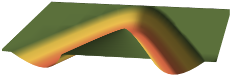
Theorem

Under these assumptions, $\sigma_{\mathrm{ess}}(-\Delta_{\Gamma,\alpha}) = [-\frac{1}{4}\alpha^2, \infty)$ and $-\Delta_{\Gamma,\alpha}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$.

P.E., T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys. A: Math. Gen.* **34** (2001), 1439–1450.

Soft quantum waveguides in two dimensions

Of course, the *zero width* is also an *idealization*; in a more realistic model the guiding would be due to a *regular potential 'ditch'*



Let thus $\Gamma:\ \mathbb{R}\to\mathbb{R}^2$ be an infinite and smooth planar curve without self-intersections, parametrized by its arc length s. We introduce the signed curvature $\gamma:\ \gamma(s)=(\dot\Gamma_2\ddot\Gamma_1-\dot\Gamma_1\ddot\Gamma_2)(s)$ and assume that

- (a) Γ is C^2-smooth so, in particular, $\gamma(s)$ makes sense,
- (b) γ is either of *compact support*, $\operatorname{supp}\gamma\subset[-s_0,s_0]$ for some $s_0>0$, or Γ is *C^4-smooth* and $\gamma(s)$, $\dot\gamma(s)$, $\ddot\gamma(s)$ tend to zero as $|s|\to\infty$,
- (c) $|\Gamma(s)-\Gamma(s')|\to\infty$ holds as $|s-s'|\to\infty$.

Recall that one can reconstruct the curve from γ, up to Euclidean transformations: putting $\beta(s_2,s_1):=\int_{s_1}^{s_2}\gamma(s)\,\mathrm{d}s$, we have

$$\Gamma(s)=\Big(x_1+\int_{s_0}^{s}\cos\beta(s_1,s_0)\,\mathrm{d}s_1,\,x_2-\int_{s_0}^{s}\sin\beta(s_1,s_0)\,\mathrm{d}s_1\Big),\quad s_0\in\mathbb{R},\ x=(x_1,x_2)\in\mathbb{R}^2.$$

Soft quantum waveguides: the Hamiltonian

We define the strip $\Omega^a := \{x \in \mathbb{R}^2 :\ \mathrm{dist}(x, \Gamma) < a\}$, in particular, $\Omega_0^a := \mathbb{R} \times (-a, a)$ corresponds to a straight line Γ_0, and assume that

- (d) Ω^a *does not intersect itself*, in particular, we have $a\|\gamma\|_\infty < 1$;

then points of Ω^a can be uniquely given in parallel (Fermi) coordinates,

$$x(s, u) = \big(\Gamma_1(s) - u\dot\Gamma_2(s), \Gamma_2(s) + u\dot\Gamma_1(s)\big).$$

This allows us to built a potential 'ditch' in Ω^a considering

- (e) a nonnegative $V \in L^\infty(\mathbb{R})$ with $\mathrm{supp}\, V \subset [-a, a]$

(we assume $V \geq 0$ and $\|V\|_\infty < \infty$ here for convenience only) and putting

$$\tilde V :\ \Omega^a \to \mathbb{R}_+,\ \ \tilde V(x(s, u)) = V(u),\ \text{ and }\ H_{\Gamma,V} = -\Delta - \tilde V(x);$$

in view of assumption (e) the operator domain is $D(-\Delta) = H^2(\mathbb{R}^2)$.

We also introduce the operator $h_V = -\partial_x^2 - V(x)$ on $L^2(\mathbb{R})$ which has in accordance with (e) a nonempty and finite discrete spectrum such that

$$\epsilon_0 := \inf \sigma_{\mathrm{disc}}(h_V) = \inf \sigma(h_V) \in \big(-\|V\|_\infty, 0\big),$$

where ϵ_0 is simple and associated with a positive $\phi_0 \in H^2(\mathbb{R})$.

The essential spectrum and asymptotic results

Obviously, $\sigma(H_{\Gamma_0,V}) = \sigma_{\mathrm{ess}}(H_{\Gamma_0,V}) = [\epsilon_0, \infty)$. If Ω_a is asymptotically straight in the sense of (b), the essential spectrum is preserved:

Proposition

Under assumptions (a)–(e) *we have* $\sigma_{\mathrm{ess}}(H_{\Gamma,V}) = [\epsilon_0, \infty)$

Recall further that $-\Delta - \alpha\delta(x - \Gamma)$ can be obtained as a *norm-resolvent limit* of Schrödinger operators with *scaled regular potentials*, namely $V_\varepsilon : \; V_\varepsilon(u) = \frac{1}{\varepsilon} V\left(\frac{u}{\varepsilon}\right)$; this follows from a general result obtained in

J. Behrndt, P. Exner, M. Holzmann, V. Lotoreichik: Approximation of Schrödinger operators with δ-interactions supported on hypersurfaces, *Math. Nachr.* **290** (2017), 1215–1248.

Proposition

Consider a non-straight C^2*-smooth curve* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ *such that* $|\Gamma(s) - \Gamma(s')| \geq c|s - s'|$ *holds for some* $c \in (0, 1)$. *If the support of its signed curvature* γ *is noncompact, assume, in addition to* (b), *that* $\gamma(s) = \mathcal{O}(|s|^{-\beta})$ *with some* $\beta > \frac{5}{4}$ *as* $|s| \to \infty$. *Then* $\sigma_{\mathrm{disc}}(H_{\Gamma,V_\varepsilon}) \neq \emptyset$ *holds for all* ε *small enough.*

Asymptotic results, continued

As another example, consider next a *flat-bottom* waveguide with

$$V_{J,0}(u) = V_0 \chi_J(u), \quad V_0 > 0,$$

where χ_J is the indicator function of an interval $J = [-a_1, a_2] \subset [-a, a]$. Using the fact that Dirichlet condition is the limit of a *high potential wall*,

J. Brasche, M. Demuth: Dynkin's formula and large coupling convergence, *J. Funct. Anal.* **129** (2005), 34–69.

we can easily prove the following result:

Proposition

Suppose that Γ *is not straight and assumptions* (a)–(d) *are satisfied, then the operator* $H_{\Gamma, V_{J,0}}$ *referring to the flat-bottom potential has nonempty discrete spectrum for all* V_0 *large enough.*

However, we would like to know whether the curvature can induce the existence of discrete spectrum also *beyond the asymptotic regime*.

Birman-Schwinger analysis

There are two main ways how to do that: (a) to use *Birman-Schwinger principle*, or (b) variationally, by constructing suitable *trial functions*.

Let us begin with the first option. For $z \in \mathbb{C} \setminus \mathbb{R}_+$ we put

$$K_{\Gamma,V}(z) := \tilde{V}^{1/2}(-\Delta - z)^{-1}\tilde{V}^{1/2}$$

with $\tilde{V}$ introduced above; we are particularly interested in the negative values of the spectral parameter, $z = -\kappa^2$ with $\kappa > 0$. In view of (e) it is a bounded operator, $L^2(\Omega^a) \to L^2(\Omega^a)$, positive for $z = -\kappa^2$. The discrete spectrum of $H_{\Gamma,V}$ can be found using the following claim:

Proposition

$z \in \sigma_{\rm disc}(H_{\Gamma,V})$ holds if and only if $1 \in \sigma_{\rm disc}(K_{\Gamma,V}(z))$. The function $\kappa \mapsto K_{\Gamma,V}(-\kappa^2)$ is continuous and decreasing in $(0,\infty)$, tending to zero in the norm topology, that is, $\|K_{\Gamma,V}(-\kappa^2)\| \to 0$ holds as $\kappa \to \infty$

Birman-Schwinger analysis, continued

Note that if g is an eigenfunction of $K_{\Gamma,V}(-\kappa^2)$ with eigenvalue one, the corresponding eigenfunction of $H_{\Gamma,V}$ is given by

$$\phi(x) = \int_{\mathrm{supp}\,\tilde{V}} G_\kappa(x,x')\,\tilde{V}(x')^{1/2} g(x')\,\mathrm{d}x',$$

where G_κ is the integral kernel of $(-\Delta+\kappa^2)^{-1}$.

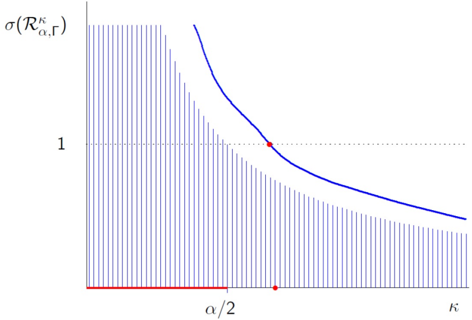
Using the knowledge of the Laplacian resolvent we can write the action of $K_{\Gamma,V}(-\kappa^2)$ explicitly: it is an integral operator with the kernel

$$K_{\Gamma,V}(x,x';-\kappa^2) = \frac{1}{2\pi}\,\tilde{V}^{1/2}(x) K_0(\kappa|x-x'|)\,\tilde{V}^{1/2}(x'),$$

where K_0 is the Macdonald function, mapping $L^2(\Omega^a)$ to itself.

In analogy with [E-Ichinose'01, loc.cit.], where instead of the sandwiched resolvent in $K_{\Gamma,V}(x,x';-\kappa^2)$ we have its *trace* at the points of Γ, we want to treat the geometry of Ω^a as a perturbation of the straight case.

Recall the BS proof scheme in the singular case



- in the straight case $\sigma(\mathcal{R}^{\kappa}_{\alpha,\Gamma_0}) = [0, \frac{\alpha}{2\kappa}]$ is checked directly
- since $\kappa \mapsto \frac{1}{2\pi} K_0(\kappa|x-x'|)$ is *decreasing*, the perturbation is *sign-definite*; it is not difficult to check that $\sup \sigma(\mathcal{R}^{\kappa}_{\alpha,\Gamma}) > \frac{\alpha}{2\kappa}$
- from the asymptotic straightness, the perturbation is *compact* so that the 'added' spectrum consists of eigenvalues at most
- the spectrum depends *continuously* on κ and *shrinks to zero* as $\kappa \to \infty$, hence there is a crossing *to the right* of $\frac{1}{2}\alpha$

Straightening the strip

Now the *compact support* of V is vital: we cannot write the operator in terms of the arc length of Γ, but we can still use the *Fermi coordinates*; passing to them and getting rid of the Jacobian, we get the unitary map

$$L^2(\Omega^a) \to L^2(\Omega_0^a), \quad (U\psi)(s,u) = (1+u\gamma(s))^{1/2}\psi(x(s,u))$$

We can thus replace $K_{\Gamma,V}(-\kappa^2)$ with the unitarily equivalent operator, $\mathcal{R}^\kappa_{\Gamma,V} := UK_{\Gamma,V}(-\kappa^2)U^{-1}$, an integral one on $L^2(\Omega_0^a)$ with the kernel

$$\mathcal{R}^\kappa_{\Gamma,V}(s,u;s',u') = \frac{1}{2\pi}\, W(s,u)^{1/2} K_0(\kappa|x-x'|)\, W(s',u')^{1/2},$$

where $x = x(s,u)$, $x' = x(s',u')$, and $W(s,u) := (1+u\gamma(s))\, V(u)$.

The biggest difference from the singular case is that we cannot guarantee that the bend would *decrease* the Euclidean distance of two points of Ω^a,

$$|x(s,u)-x(s',u')|^2 = |\Gamma(s)-\Gamma(s')|^2 + u^2 + u'^2 - 2uu'\cos\beta(s,s') + 2(u\cos\beta(s,s')-u')\int_{s'}^{s}\sin\beta(\xi,s')\,\mathrm{d}\xi\,;$$

in fact, it may increase on the outer side of the strip.

Existence of bound states

This forces us to make the sign-definiteness an *assumption* which then yields the following sufficient condition:

Theorem

Let assumptions (a)–(e) be valid and set

$$\begin{aligned}\mathcal{C}^{\kappa}_{\Gamma,V}(s,u;s',u') &= \frac{1}{2\pi}\,\phi_0(u)V(u)\,\big[(1+u\gamma(s))\,K_0(\kappa|x(s,u)-x(s',u')|)\,(1+u'\gamma(s')) \\ &\qquad -K_0(\kappa|x_0(s,u)-x_0(s',u')|)\,\big]\,V(u')\phi_0(u')\end{aligned}$$

for all (s,u), $(s',u')\in\Omega_0^a$*, then we have* $\sigma_{\mathrm{disc}}(H_{\Gamma,V})\neq\emptyset$ *provided*

$$\int_{\mathbb{R}^2}\mathrm{d}s\mathrm{d}s'\int_{-a}^{a}\int_{-a}^{a}\mathrm{d}u\mathrm{d}u'\,\mathcal{C}^{\kappa_0}_{\Gamma,V}(s,u;s',u')>0$$

holds for $\kappa_0=\sqrt{-\epsilon_0}$.

P.E.: Spectral properties of soft quantum waveguides, *J. Phys. A: Math. Theor.* **53** (2020), 355302.

In contrast to the asymptotic results above, this one has a quantitative character

Sketch of the proof

Since $\sigma_{\mathrm{ess}}(\mathcal{R}^{\kappa_0}_{\Gamma,V})$ is preserved, it is enough to find $\psi \in L^2(\Omega^a_0)$ such that

$$(\psi, \mathcal{R}^{\kappa_0}_{\Gamma,V}\psi) - \|\psi\|^2 > 0.$$

The trial function will be chosen to combine the generalized eigenfunction, g_0, associated with the edge of $\sigma_{\mathrm{ess}}(\mathcal{R}^{\kappa_0}_{\Gamma,V})$ and a suitable mollifier, that is, $\psi_\eta(s,u) = h_\eta(s)g_0(u)$. In the straight case, $\Gamma = \Gamma_0$, the operator $\mathcal{R}^{\kappa_0}_{\Gamma_0,V}$ is of a convolution character in s which allows to prove the following claim:

Lemma

Choosing $h_\eta(s) = h(\eta s)$ such that $h \in C_0^\infty(\mathbb{R})$ and $h(s) = 1$ holds in the vicinity of $s = 0$, we have

$$(\psi_\eta, \mathcal{R}^{\kappa_0}_{\Gamma_0,V}\psi_\eta) - \|\psi_\eta\|^2 = \mathcal{O}(\eta) \quad \text{as } \eta \to 0.$$

Then we consider the difference of the Birman-Schwinger operators,

$$\mathcal{D}^{\kappa}_{\Gamma,V} := \mathcal{R}^{\kappa}_{\Gamma,V} - \mathcal{R}^{\kappa}_{\Gamma_0,V}$$

Proof sketch, continued

This perturbation is an integral operator in $L^2(\Omega_0^a)$ with the kernel

$$\mathcal{D}^{\kappa}_{\Gamma,V}(s,u;s',u') = \frac{1}{2\pi}\Big(W(s,u)^{1/2}K_0(\kappa|x(s,u)-x(s',u')|)\,W(s',u')^{1/2} \\ -V(u)^{1/2}K_0(\kappa|x_0(s,u)-x_0(s',u')|)\,V(u')^{1/2}\Big)$$

To prove the theorem, we need to check that $\sup\sigma(\mathcal{R}^{\kappa_0}_{\Gamma_0,V}) > 1$, and this happens if

$$\lim_{\eta\to 0}(\psi_\eta, \mathcal{D}^{\kappa_0}_{\Gamma,V}\psi_\eta) > 0.$$

Given our choice of the mollifier h_η, this is equivalent to the requirement

$$\int_{\mathbb{R}^2}\mathrm{d}s\mathrm{d}s' \int_{-a}^{a}\int_{-a}^{a}\mathrm{d}u\mathrm{d}u'\, g_0(u)\,\mathcal{D}^{\kappa_0}_{\Gamma,V}(s,u;s',u')\,g_0(u') > 0,$$

which is nothing else than the condition stated in the theorem. □

Remarks

• The sufficient condition for the discrete spectrum existence can be extended to soft waveguides in *three dimensions* under the assumption that profile potential V is *rotationally symmetric* w.r.t. the tube axis.

• If it is not the case, the result still holds if the channel profile is fixed in a particular frame which is, modulo technicalities, the one which *rotates w.r.t. the Frenet frame* of the generating curve Γ and the *angular velocity* of this rotation coincides with the *torsion* of Γ.

P.E.: Soft quantum waveguides in three dimensions, *J. Math. Phys.* **63** (2022), 042103

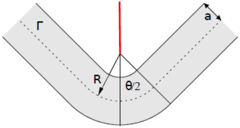
• If this condition is not satisfied, the problems is open; recall that for *Dirichlet tubes* twisting gives rise to an effective *repulsive* interaction.

• The results we have obtained by Birman-Schwinger method are quantitative, true, but it is not easy the evaluate the corresponding integrals. In general we can infer from them the bound state existence in the situations when the 'ditch' is *deep and narrow enough* but this is akin to the asymptotic results we had before.

Variational approach

An alternative to Birman-Schwinger is to apply variational estimates to the original operator $H_{\Gamma,V}$. The trouble is to find a suitable trial function which – in contrast to Dirichlet tubes where this approach works well – is that such a function is now supported in the whole plane/space.

The only existing result in the literature concerns a simple example of the so-called *bookcover-shaped* potential ditch localized in the following Ω^a:



Source: the cited paper

S. Kondej, D. Krejčiřík, J. Kříž: Soft quantum waveguides with a explicit cut locus, *J. Phys. A: Math. Theor.* **54** (2021), 30LT01

The potential here is not assumed to be nonnegative and may be arbitrarily shallow. Note also that the generating curve here is not C^2.

Before proceeding with the existence question, let us mention application to the variational method to another related problem.

Finite soft guides: an optimization

The question we have in mind concerns the *spectral optimization* in analogy with what is known in Dirichlet and δ potential cases

P.E., E.M. Harrell, M. Loss: Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature, in *Mathematical Results in Quantum Mechanics*, Birkhäuser, Basel 1999; pp. 47–53.

P.E., E.M. Harrell, M. Loss: Inequalities for means of chords, with application to isoperimetric problems, *Lett. Math. Phys.* **75** (2006), 225–233; addendum **77** (2006), 219.

Let Γ be a C^2-smooth *loop* without self-intersections of a *fixed length L*. For small enough positive $d_\pm$ the map $[0,L)\times\mathcal{J}\ni(s,u)\mapsto\Gamma(s)+u\nu(s)$, where $\mathcal{J}=[-d_-,d_+]$ and $\nu=(-\dot\Gamma_2,\dot\Gamma_1)$ is the normal to Γ, is *bijective*.

We consider operators $H_{\gamma,\mu}$ corresponding the measure-type interaction

$$\mu(M):=\int_0^L\int_{-d_-}^{d_+}\chi_M\big(\Gamma(s)+u\nu(s)\big)\,(1+u\gamma(s))\,\mathrm{d}\mu_\perp(t)\mathrm{d}s,$$

where the positive transverse measure $\mu_\perp$ can describe both a regular attractive potential channel we are discussing here, $\mu_\perp(u)=V(u)\mathrm{d}u$, as well as a δ potential, and more.

Ground state optimization

We define $H_{\Gamma,\mu}$ as the self-adjoint operator associated with the form

$$h_{\Gamma,\mu}[\psi] := \|\nabla\psi\|^2 - \int_{\mathbb{R}^2} |\psi|^2 \mathrm{d}\mu, \quad \mathrm{dom}\, h_{\Gamma,\mu} = H^1(\mathbb{R}^2).$$

It is not difficult to check that the essential spectrum of $H_{\Gamma,\mu}$ is $[0,\infty)$ and $\sigma_{\mathrm{disc}}(H_{\Gamma,\mu}) \neq \emptyset$. Let $\mathcal{C}$ be a *circle of radius* $\frac{L}{2\pi}$. By $\mu_\circ$ we denote the corresponding measure generated by $\mu_\perp$ and giving rise to operator $H_{\Gamma,\mu_\circ}$.

Theorem

The lowest eigenvalues $\lambda_1(\mu)$ and $\lambda_1(\mu_\circ)$, respectively, of $H_{\Gamma,\mu}$ and of $H_{\Gamma,\mu_\circ}$ satisfy the inequality

$$\lambda_1(\mu) \leq \lambda_1(\mu_\circ).$$

We *conjecture* that the inequality is strict unless Γ and $\mathcal{C}$ are congruent.

Note also that this provides an *alternative proof* of the leaky loop result.

Ground state optimization

The claim follows by a simple *variational argument*: the appropriate trial function is obtained using the lowest eigenfunction of $H_{\Gamma,\mu_\circ}$ and *'transplanting'* it to the parallel coordinates.

More specifically, we take trial functions ψ the values which, inside and outside the loop, are of the form $u(\mathrm{dist}(x,\Gamma))$ where u is a C_0^∞ function. Using appropriate changes of the variables, we check that the inequality $h_{\Gamma,\mu}[\psi] \le h_{\mathcal{C},\mu}[\psi]$ holds for any such u; comparing then the Rayleigh quotients we arrive at the result. It has a slight generalization:

Theorem

Let χ, respectively $\chi_\circ$, be the indicator function of the open set inside the loop strip. The lowest spectral points $\lambda_1^\beta(\mu)$ and $\lambda_1^\beta(\mu_\circ)$ of $H_{\Gamma,\mu}+\beta\chi$ and $H_{\Gamma,\mu_\circ}+\beta\chi_\circ$, respectively, satisfy then the inequality

$$\lambda_1^\beta(\mu) \le \lambda_1^\beta(\mu_\circ).$$

In particular, $\sigma_{\mathrm{disc}}(H_{\Gamma,\mu_\circ}+\beta\chi_\circ) \ne \emptyset$ implies $\sigma_{\mathrm{disc}}(H_{\Gamma,\mu}+\beta\chi) \ne \emptyset$.

P.E., V. Lotoreichik: Optimization of the lowest eigenvalue of a soft quantum ring, *Lett. Math. Phys.* **111** (2021), 28

Another optimization result

One can also optimize with respect to the *channel profile*:

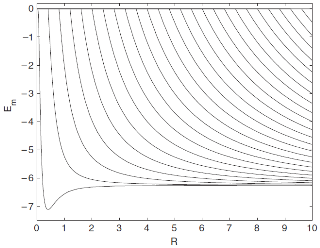
Theorem

Put $\alpha := \mu_\perp(\mathcal{J})$ *and consider Schrödinger operators* $H_{\Gamma_t,\alpha}$, *where* Γ_t *is 'parallel' to* Γ *at the distance* t, *then the lowest eigenvalues* $\lambda_1(\mu)$ *and* $\lambda_1(\alpha\Gamma_t)$ *of* $H_{\Gamma,\mu}$ *and of* $H_{\Gamma_t,\alpha}$, *respectively, satisfy the inequality*

$$\lambda_1(\mu) \geq \min_{u\in\mathcal{J}} \lambda_1(\alpha\Gamma_t).$$

This is again easy to prove variationally; one has to check that the function $\mathcal{J} \ni t \mapsto \|\psi|_{\Gamma_t}\|^2$ is continuous so that it attains its maximum value at some $t_\star = t_\star(\mu) \in \mathcal{J}$.

Depending on α, the position of $t_\star$ in $\mathcal{J}$ may be at different; recall how the eigenvalues $H_{\mathcal{C},\alpha}$, here with $\alpha = 5$, depend on the circle radius

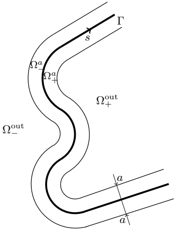


The existence: more complicated guides

Let us return to the variational approach of the $\sigma_{\rm disc}(H_{\Gamma,V})$ and see whether can go beyond the bookcover example of [KKK'21]

We adopt the following assumptions:

- s1 Γ is *C^3-smooth*, *non-straight* but *straight outside a compact*; its curved part consists of a finite number of segments such that on each of them the monotonicity character of the signed curvature $\kappa(\cdot)$ of Γ and its sign are preserved,
- s2 $|\Gamma(s_+) - \Gamma(s_-)| \to \infty$ as $s_\pm \to \pm\infty$,
- s3 the strip $\Omega^a := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < a\}$ does not intersect itself.



The potential, including a possible bias

We consider the channel profile operator of the form

$$h := -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + v(t) + V_0\chi_{[a,\infty)}(t), \quad V_0 \geq 0$$

and use (some of) the following assumptions:

- p1 $v \in L^2(\mathbb{R})$ and $\operatorname{supp} v \subset [-a, a]$,
- p2 sometimes we use *mirror symmetry*, $v(t) = v(-t)$ for $t \in [-a, a]$,
- p3 $\inf \sigma(h)$ is a negative (ground state) eigenvalue μ associated with a real-valued eigenfunction ϕ_0 normalized by $\phi_0(-a) = 1$, or
- p4 operator h has a *zero-energy resonance*, meaning that $h \geq 0$ and $-(1-\varepsilon)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + v(t) + V_0\chi_{[a,\infty)}(t)$ has a negative eigenvalue for any $\varepsilon > 0$. In that case, equation $h\phi = 0$ has a real-valued solution $\phi_0 \in H^2_{\mathrm{loc}}(\mathbb{R})$ not increasing at infinity; we set again $\phi_0(-a) = 1$.

The soft waveguide Hamiltonian

As before, the object of our interest is the Schrödinger operator

$$H_{\Gamma,V} = -\Delta + V(x)$$

on $L^2(\mathbb{R}^2)$ with the potential defined using the locally orthogonal coordinates (s,t) in the strip as

$$V(x) = \begin{cases} v(t) & \text{if } x \in \Omega^a \\ V_0 & \text{if } x \in \Omega_+ \setminus \Omega^a \\ 0 & \text{otherwise} \end{cases}$$

We drop the subscript of $H_{\Gamma,V}$ if it is clear from the context. We have:

Proposition

Under assumptions (s1)–(s3), (p1) and (p3), the operator is self-adjoint with $D(H_{\Gamma,V}) = H^2(\mathbb{R}^2)$, and $\sigma_{\mathrm{ess}}(H_{\Gamma,V}) = [\mu, \infty)$. If $h \geq 0$, the same is true with $\mu = 0$.

The unbiased case

The zero-energy resonance situation is easier:

Theorem

Assume (s1)–(s3), (p1) and (p4). If $V_0 = 0$ and

$$[\phi_0(a)^2 - \phi_0(-a)^2] \int_{\mathbb{R}} \kappa(s)\,\mathrm{d}s \leq 0$$

holds, then $H_{\Gamma,V}$ has at least one negative eigenvalue.

Recall that $\kappa \neq 0$. The condition is naturally satisfied if $\phi_0(a) = \phi_0(-a)$, in particular, under assumption (p2). The integral equals $\pi - \theta$ where θ is the asymptote angle, hence if $\phi_0(a) \neq \phi_0(-a)$, at least one bound state exists if $\theta = \pi$ or $\theta \in (0, \pi)$ and ϕ_0 is larger at the 'outer' side of Ω^a.

Theorem

Assume (s1)–(s3) and (p1)–(p3). Let further $V_0 = 0$, then $H_{\Gamma,V}$ has at least one eigenvalue below the continuum threshold μ.

P.E., S. Vugalter: Bound states in bent soft waveguides, arXiv:2304.14776

A rough sketch of the proof

We seek a trial function $\psi \in H^1(\mathbb{R}^2)$ such that $Q[\psi] < \mu\|\psi\|^2$, where

$$Q[\psi] = \|\nabla\psi\|^2 + \int_{\Omega^a} v(t)|\psi(x(s,t))|^2 \mathrm{d}s\mathrm{d}t$$

Let us fix the geometry. We choose the origin O of the coordinates so that the asymptotes are symmetric w.r.t. x axis at angles $\pm\theta_0$, and s_0 so that $\Gamma(\pm s)$ there have the *same Euclidean distance from O*.

We begin with *trial function inside the strip* choosing s_0, such that the points $\Gamma(\pm s_0)$ lay outside the curved part of Γ, and $s^* > s_0$, defining

$$\chi_{\mathrm{in}}(s) := \begin{cases} 1 & \text{if } |s| < s_0 \\ \ln\frac{s^*}{|s|}\left(\ln\frac{s^*}{s_0}\right)^{-1} & \text{if } s_0 \le |s| \le s^* \\ 0 & \text{if } s_0 \le |s| \le s^* \end{cases}$$

Recalling that ϕ_0 satisfies $h\phi_0 = \mu\phi_0$, we put

$$\psi(s,t) = \phi_0(t)\chi_{\mathrm{in}}(s) + \nu g(s,t), \quad |t| \le a,$$

where ν and a compactly supported function g will be chosen later.

Sketch of the proof, continued

We denote by $Q_{\rm int}[\psi]$ the contribution from the strip to the shifted form, $Q[\psi]-\mu\|\psi\|^2$; using the parallel coordinates we can write it as

$$Q_{\rm int}[\psi] = \int_{|t|\le a} \Big\{ \Big(\frac{\partial\psi}{\partial s}\Big)^2 (1-\kappa(s)t)^{-1} + \Big(\frac{\partial\psi}{\partial t}\Big)^2 (1-\kappa(s)t)$$

$$+(v(t)-\mu)|\psi|^2(1-\kappa(s)t)\Big\}\,\mathrm{d}s\mathrm{d}t.$$

It is quadratic in ν, we can choose g so that *linear term is nonzero*. Denoting for brevity (keeping the bias V_0 for further purposes)

$$\phi_\pm = \phi_0(\pm a), \quad \xi_+ = -\sqrt{|\mu|+V_0}, \quad \xi_- = \sqrt{|\mu|},$$

we can estimate the internal contribution as follows,

$$Q_{\rm int}[\psi] \le -\frac{1}{2}\delta\nu + \big[\xi_+\phi_+^2 - \xi_-\big]\|\chi_{\rm in}\|^2 - \big[\xi_+\phi_+^2 + \xi_-\big]\, a\int_{\mathbb{R}} \kappa(s)\,\mathrm{d}s$$

$$+\frac{1}{2}(\phi_+^2-1)\int_{\mathbb{R}}\kappa(s)\,\mathrm{d}s + \tau_0^{-1}\|\phi_0\restriction_{[-a,a]}\|^2\|\chi_{\rm in}'\|^2.$$

where $\tau_: = 1 - a\|\kappa\|_\infty$ and the last term on the right-hand side can be made arbitrarily small choosing $s^*\gg s_0$.

The zero-energy resonance case

If $V_0 = \mu = 0$ we have $\xi_\pm = 0$ and since $(\phi_+^2 - 1)\int_{\mathbb{R}} \kappa(s)\,\mathrm{d}s \le 0$ holds by assumption, the above estimate simply becomes $Q_{\rm int}[\psi] \le -\frac14 \delta\nu$.

To conclude the proof we have thus to choose the outer part of trial function so that its contribution can be made smaller than any fixed positive number.

If $V_0 = \mu = 0$, we have $\phi_0(t) = \mathrm{const}$ for $|t| \ge a$, and to get an H^1 trial function, we have to multiply this constant (possibly different in $\Omega_\pm$) by a suitable mollifier $\chi_{\rm out}$ of which we require

- in $\mathbb{R}^2 \setminus \Omega^a$ the function depends on $\rho = \mathrm{dist}(x, O)$ only,
- continuity at the boundary of Ω^a: at points $x(s, \pm a)$ the relation $\chi_{\rm out}(x) = \chi_{\rm in}(s)$ holds.

With a bit of computing one can check that the goal is achieved for $s^* \gg s_0$; this concludes the proof of the first theorem.

The case $\mu < 0$

The proof of the second theorem is much more complicated. In view of the symmetry and absence of the bias, we have $\phi_+ = 1$ and $\xi_+ = -\xi_-$. Keeping thus $\psi(s,t) = \phi_0(t)\chi_{\rm in}(s) + \nu g(s,t)$ for the interior part, we have

$$Q_{\rm int}[\psi] \le -\frac{1}{4}\delta\nu - 2|\mu|^{1/2}\|\chi_{\rm in}\|^2.$$

To construct the outer part, we adopt first an additional assumption,

- s4 the curved part of Γ is piecewise C^∞-smooth consisting of a *finite array of circular arcs*; at its endpoints it is C^1-smoothly connected to the halflines,

in other words, the signed curvature $\kappa(\cdot)$ of such a Γ is a step function.

In $\Omega_{\rm out}$ we now define a function with the appropriate exponential decay,

$$\phi(x) := \exp\{-\xi(\mathrm{dist}(x,\Gamma) - a)\}, \quad x \in \mathbb{R}^2 \setminus \Omega^a,$$

where $\xi := \xi_- = -\xi_+ = |\mu|^{1/2}$; the sought trial function will be then of the form $\psi_{\rm out} = \phi\chi_{\rm out}$ with the mollifier $\chi_{\rm out}$ to be specified below.

The external mollifier

To construct it, we consider several regions in the plane:

- the disc $B_{\frac{1}{2}r_0}(O)$ containing the curved part of Γ
- the doubled disc $B_{r_0}(O)$ such that $\chi_{\rm out}(x) = 1$ on $B_{r_0}(O) \setminus \Omega^a$
- disjoint *conical sectors* $K_\pm$ of angle $2\theta_0$ in $\mathbb{R}^2 \setminus B_{r_0}(O)$ centered around the asymptotes of Γ; within them one can use the parallel coordinates and define $\chi_{\rm out}(s,t) = \chi_{\rm in}(s)$

In the remaining part of the plane we choose $\chi_{\rm out}$ as a *function of the distance ρ from the origin O only*, and such that $\chi_{\rm out}$ is *continuous* in $\Omega_{\rm out}$; it is clear that the radial decay of such an external mollifier is determined by the function $\chi_{\rm in}(s)$

As usual one has first to check that the mollifier effect in the kinetic part of the quadratic form can be made small. With the above choice we have

$$\int_{\Omega_{\rm out}} |\nabla\psi_{\rm out}(x)|^2 {\rm d}x \le \int_{\Omega_{\rm out}} |\nabla\phi(x)|^2 \chi_{\rm out}^2(x)\, {\rm d}x + \mathcal{O}(r_0^{-1}) \;\text{ as }\; r_0 \to \infty;$$

choosing r_0 large enough, the error term can be made $\frac{1}{8}|\mu|\delta\nu$ with the $\delta\nu$ we used in estimating the interior part.

Proof sketch, continued

Next we note that $|\nabla\phi|^2 - \mu|\phi|^2 = 2|\nabla\phi|^2$ holds almost everywhere in $\Omega_{\rm out}$, hence we can estimate the whole exterior contribution to the form $Q[\psi] - \mu\|\psi\|^2$ by doubling the kinetic term and neglecting the one containing the eigenvalue μ

Combining with the interior part estimate, we see that to prove the theorem it is *sufficient to check the inequality*

$$2\int_{\Omega_{\rm out}} |\nabla\phi(x)|^2\chi_{\rm out}^2(x)\,\mathrm{d}x \le 2|\mu|^{1/2}\|\chi_{\rm in}\|^2 + \frac{|\mu|}{8}\delta\nu,$$

which is in view of $|\nabla\phi|^2 = -\mu|\phi|^2$ further *equivalent to*

$$\int_{\Omega_{\rm out}} |\phi(x)\chi_{\rm out}(x)|^2\,\mathrm{d}x \le |\mu|^{-1/2}\|\chi_{\rm in}\|^2 + \frac{1}{16}\delta\nu$$

Recall that for $x = (\rho,\theta)$ in the conical sectors $K_\pm$ with $\rho \ge r_0$ we can use the (s,t) coordinates simultaneously with the polar ones. We choose an $\hat{s} \ge r_0$ so that the parts of Γ with $|s| \ge \hat{s}$ lay outside $B_{r_0}(O)$, and at the same time we choose s_0 of $\chi_{\rm in}$ is such a way that $s_0 > \hat{s}$.

Proof sketch, continued

With this choice it is easy to check that

$$\int_{\Omega_{\mathrm{out}}\cap\{K_+\cup K_-\}} |\phi(x)\chi_{\mathrm{out}}(x)|^2\,\mathrm{d}x \le |\mu|^{-1/2}\|\chi_{\mathrm{in}}\|^2_{L^2((-\infty,-\hat{s}]\cup[\hat{s},\infty))}$$

and it remains to estimate the integral over $\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}$ which can only increase if we remove χ_{out}, hence we have to check that

$$\int_{\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}} |\phi(x)|^2\,\mathrm{d}x \le 2\hat{s}\,|\mu|^{-1/2} + \frac{1}{16}\delta\nu,$$

where we have used the fact that $\|\chi_{\mathrm{in}}\|^2_{L^2((-\hat{s},\hat{s}))} = 2\hat{s}$.

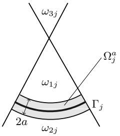
Now we employ the additional assumption (s4). The function $d_x : \mathbb{R}\to\mathbb{R}_+$ defined by $d_x(s) := \mathrm{dist}(x,\Gamma(s))$ is C^1 smooth for any $x\in\mathbb{R}^2$ and *piecewise monotonous* because on each arc it can have at most one extremum. Since $d_x(s)\to\infty$ holds as $s\to\pm\infty$, the function has a *global minimum*, and it may also have a finite number of *local extrema* which come in *pairs*, a minimum adjacent to a maximum.

Proof sketch, continued

Let s_x^0 be the coordinate of the global minimum and s_x^i refer to all the extrema; the index sets $M_x^\uparrow$ and $M_x^\downarrow$ refer to maxima and minima, respectively. Then for all $x \in \Omega_{\text{out}}$ we obviously have

$$\exp\{-2\xi(d_x(s_x^0) - a)\} \leq -\sum_{s_x^i \in M_x^\uparrow} \exp\{-2\xi(d_x(s_x^i) - a)\} + \sum_{s_x^i \in M_x^\downarrow} \exp\{-2\xi(d_x(s_x^i) - a)\}$$

We will combine this inequality with some simple geometrical facts:



Proposition

Let Γ_j be one the arcs of Γ and $\omega_{1j}, \omega_{2j}, \omega_{3j}$ and Ω_j^a as in the figure

- (i) *If $x \in \omega_{1j} \cup \omega_{2j}$, then $d_x(\cdot)$ has a minimum in the interior of Γ_j.*
- (ii) *If $x \in \omega_{3j}$, then $d_x(\cdot)$ has a maximum in the interior of Γ_j.*
- (iii) *$x \notin \bar\omega_{1j} \cup \bar\omega_{2j} \cup \bar\omega_{3j} \cup \bar\Omega_j^a$, then $d_x(\cdot)$ has no extremum on Γ_j.*
- (iv) *$d_x(\cdot)$ has no more than one critical point in the interior of Γ_j.*
- (v) *If $x \in \omega_{kj}$ for any of $k = 1, 2, 3$, then the one-sided derivative $d_x'(s) \neq 0$ at the endpoints of Γ_j.*

Proof sketch, continued

Within the regions introduced the minimal and maximal distances are easily expressed,

$$d_x(s_x^i) = \mathrm{dist}(x, \Gamma_j) \quad \text{if } s_x^i \in \Gamma_j \cap M_x^{\downarrow},$$
$$d_x(s_x^i) = |\kappa_j|^{-1} + \mathrm{dist}(x, O_j) \quad \text{if } s_x^i \in \Gamma_j \cap M_x^{\uparrow}.$$

Thus allows us to replace the right-hand side terms in the above estimate almost everywhere by

$$-\sum_j \exp\{-2\xi(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\}\iota_j^3(x) \quad \text{and} \quad \sum_j \exp\{-2\xi(\mathrm{dist}(x, \Gamma_j) - a)\}\iota_j^{1,2}(x),$$

respectively, where $\iota_j^{1,2}$ and ι_j^3 are the appropriate characteristic functions, hence $\int_{\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}} \exp\{-2\xi(d_x(s_x^0) - a)\}\,\mathrm{d}x$ is bound from above by

$$\sum_j \int_{(\omega_{1j}\cup\omega_{2j})\cap\{\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}\}} \exp\{-2\xi(\mathrm{dist}(x, \Gamma_j) - a)\}\,\mathrm{d}x$$
$$-\sum_j \int_{\omega_{3j}\cap\{\Omega_{\mathrm{out}}\setminus\{K_+\cup K_-\}\}} \exp\{-2\xi(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\}\,\mathrm{d}x,$$

where the sums include the straight segments with $|s| > \hat{s}$. There is a *double counting* here as x may belong to different ω_{kj}; this does not matter as long as we consider the contributions referring of a given Γ_j together.

Proof sketch, continued

To simplify the estimate, we note that the last bound can only increase if we replace the integration domains by $(\omega_{1j} \cup \omega_{2j}) \setminus \{K_+ \cup K_-\}$ and $\omega_{3j} \setminus \{K_+ \cup K_-\}$, respectively. This follows from the fact that any fixed j the three region are in Ω_{out}, i.e. $\omega_{kj_0} \cap \Omega^a_{j_0} = \emptyset$ holds for $k = 1, 2, 3$.

As mentioned, the summation includes the straight parts of Γ; without going into details, one can check that their contribution is estimated by a multiple of $\mathrm{e}^{-\xi \sin 2\Delta_0 \theta \cdot \rho(\hat{s})}$ becoming thus negligible for large r_0.

To get rid of the conical sectors, we note that that the positive part of the estimate cannot decrease if we enlarge the integration domain replacing $(\omega_{1j} \cup \omega_{2j}) \setminus \{K_+ \cup K_-\}$ by $\omega_{1j} \cup \omega_{2j}$.

We can also replace $\omega_{3j} \setminus \{K_+ \cup K_-\}$ by ω_{3j}. This enlarges the negative part, however, regions ω_{3j} exist only for the curved segments of Γ and those are by assumption inside $B_{\frac{1}{2}r_0}(O)$, while the regions $K_\pm$ are by construction outside $B_{r_0}(O)$, which implies that such an error is $\mathcal{O}(\mathrm{e}^{-3\xi r_0/2})$ and can be again neglected.

Conclusion of the proof

The estimate now contains only integrals over sectors ω_{kj} which are easy to evaluate explicitly; this proves the theorem under assumption (p3).

To complete the proof we use the following *approximation result*:

Theorem (Sabitov-Slovesnov (2010))

Let Γ be a C^3-smooth curve consisting of a finite number of segments such that on each of them the monotonicity character of the signed curvature $\kappa(\cdot)$ of Γ and its sign are preserved. Then Γ can be approximated by a C^1-smooth function $\hat{\Gamma}$ of the same length, the curvature of which is piecewise constant having jumps at the points $s_1 < s_2 < \cdots < s_N$, in the sense that the estimates

$$\|\Gamma^{(m)} - \hat{\Gamma}^{(m)}\|_\infty \le C \max_{1\le k\le N-1} (s_{k+1} - s_k)^{3-m}, \quad m = 0, 1, 2,$$

hold with some $C > 0$ for the function Γ and its two first derivatives.

It is straightforward to check that all the used estimates persist when we approximate our curve by a family of arc arrays, $\Gamma_n \to \Gamma$.

Convexity and potential bias

Theorem

Assume $V_0 \geq 0$ together with (s1)–(s3) and (p1). If one of the regions $\Omega_\pm$ is convex and (p3) holds, then $H_{\Gamma,V}$ has at least one discrete eigenvalue. If $V_0 > 0$ and Ω_+ is convex, the operator $H_{\Gamma,V}$ has at least one discrete eigenvalue provided that (p4) holds.

Note that these claims *do not need mirror symmetry* of the potential v.

The construction of the trial function proceed as in the previous case but we have to distinguish the two sides, $\Omega_\pm$, with different $\xi_\pm$; this requires the indicated stronger geometric restrictions.

In the *zero-energy resonance* situation the quadratic form is estimated by

$$Q_{V_0}[\psi] = -\frac{1}{8}\delta\nu - \int_{\mathbb{R}} \kappa(s)\,\mathrm{d}s + o(\psi),$$

where the error term can be made arbitrarily small by choosing large enough parameters r_0 and s^*; it obviously works in the *convex* case only when the integral is positive.

Many questions remain open

- Another weak-coupling problem concerns the effect of a *slight bend* for a soft guide of a constant profile. One conjectures that in analogy to the Dirichlet tubes and leaky curves the leading term would be proportional to the *fourth power of the bending angle*.
- Back to non-asymptotic problems, one may ask what happens in 3D waveguides when the profile does not have rotational symmetry and *Tang condition is violated*. It is known that in Dirichlet waveguides with *torsion* gives rise to an effective with *repulsive interaction* but for leaky and soft guides the problem may be more complicated.

  T. Ekholm, H. Kovařík, D. Krejčiřík: A Hardy inequality in twisted waveguides, *Arch. Rat. Mech. Anal.* **188** (2008), 245–264.

- Potential channels of a more complicated geometry, in first place *branched ones* built over a metric graph. Of course, to have the problem well defined one must specify the potential in the vicinity of the graph vertices because the spectrum would depend on it.

More problems

- Another question concerns *scattering* in a bent or locally perturbed potential channel including possible *resonance effects* in narrow and sufficiently deep channels.
- Another extension to three dimensions concerns *potential layers*, that is potentials of a fixed transverse profile built over an infinite surface Σ in $\mathbb{R}^3$. One can again establish the discrete spectrum existence for potential layers with the profile deep enough, while in the regime different from the asymptotic one, the question is open.
- For layers the spectrum may depend on the *global* geometry of the interaction support. An example of a *conical* potential layer was found, and recently the conclusion have been extended to layers with *asymptotically cylindrical ends*.

S. Egger, J. Kerner, K. Pankrashkin: Discrete spectrum of Schrödinger operators with potentials concentrated near conical surfaces, *Lett. Math. Phys.* **110** (2020), 945–968.

D. Krejčiřík, J. Kříž: Bound states in soft quantum layers, arXiv:2205.04919

- and the list may continue, *ad libitum*

Another model: quantum dot arrays

Given a $\rho > 0$ and a *nonzero* real-valued function $V \in L^2(0,\rho)$ we define *radial potential* supported in $B_\rho(y)$ centered at $y \in \mathbb{R}^\nu$, $\nu = 2,3$.

We consider a family of points, $Y = \{y_i\} \subset \mathbb{R}^\nu$, such that the balls $B_\rho(y_i)$ do not overlap, $\mathrm{dist}(y_i, y_j) \geq 2\rho$ if $i \neq j$, and denote $V_i : x \mapsto V(x - y_i)$. The object of our interest is the Schrödinger operator

$$H_{\lambda V,Y} = -\Delta - \lambda \sum_i V_i(x)$$

To visualise better the geometry of the system we suppose that the points of Y are distributed over a curve $\Gamma \subset \mathbb{R}^\nu$

If Y consists of a single point, we use the abbreviated symbol $H_{\lambda V}$. It is straightforward to check that $\sigma_{\mathrm{ess}}(H_{\lambda V}) = [0,\infty)$ and the discrete spectrum, written as an ascending sequence $\{\epsilon_n\}$, is at most finite.

In two dimensions it is nonempty provided $\int_0^\rho V(r)\, r\mathrm{d}r > 0$, for $\nu = 3$ the existence of bound states requires a critical interaction strength.

A straight array

Consider first the geometrically trivial case where the set $Y = Y_0$ is invariant w.r.t. discrete translations, i.e. the $\Gamma = \Gamma_0$ is a straight line:

Proposition

$\sigma(H_{V,Y_0}) \supset [0,\infty)$. If $\int_0^\rho V(r)\, r^\nu \mathrm{d}r > 0$, we have $\inf \sigma(H_{V,Y_0}) < 0$, and the spectrum may or may not have gaps. Their number is finite and does not exceed $\#\sigma_{\mathrm{disc}}(H_V)$. This bound is saturated for the spacing a large enough if $\nu = 2$, in the case $\nu = 3$ there may be one gap less which happens if the potential is weak, i.e. for $H_{\lambda V,Y_0}$ with λ sufficiently small.

- For positive energies it is easy to construct a Weyl sequence
- In the negative part by Floquet decomposition we consider a *single potential well in a slab S^a* of width a using two-sided estimates by the symmetric/antisymmetric solutions
- Negative spectrum existence is proved using a a suitable trial function
- Note that $\inf \sigma(H_{V,Y_0}) < 0$ even if a single well in 3D is *subcritical*

The essential spectrum

Suppose now that $\Gamma : \mathbb{R} \to \mathbb{R}^\nu$ is a unit-speed curve, $|\dot\Gamma| = 1$, i.e., the curve is parametrized by its arc length, and the points of the array Y_Γ are *distributed equidistantly* with respect to this variable with a spacing satisfying again $a \ge 2\rho$, as required by the potential wells disjointness.

In addition, the potential components of the operator $H_{V,Y}$ must not overlap: we assume that $|\Gamma(s+a) - \Gamma(s)| \ge 2\rho$ holds for any $s \in \mathbb{R}$.

Using a (double-sided) Weyl-sequence argument, it is not difficult to prove the following claim:

Proposition

Let Γ be straight outside a compact set and let $|\Gamma(s) - \Gamma(-s)| \to \infty$ hold as $|s| \to \infty$, then $\sigma_{\mathrm{ess}}(H_{V,Y})$ is the same as in the case of a straight array of the same spacing.

Birman-Schwinger principle again

Suppose now that the array potentials are *purely attractive*, $V \geq 0$. The symmetry of the potentials V allows us to use *Birman-Schwinger principle* more effectively inspectig the spectrum of the operator

$$K_{V,Y}(z) := V_Y^{1/2}(-\Delta - z)^{-1} V_Y^{1/2}, \quad V_Y := \sum_i V_i.$$

Note that since the supports of the V_i's are disjoint, we can write $K_{V,Y}$ in the 'matrix' form with the 'entries' $K_{V,Y}^{(i,j)}(-\kappa^2) := V_i^{1/2}(-\Delta + \kappa^2)^{-1} V_j^{1/2}$.

The crucial part of the argument is the following equivalence:

Proposition

$z \in \sigma_{\rm disc}(H_{V,Y})$ *holds if and only if* $1 \in \sigma_{\rm disc}(K_{V,Y}(z))$ *and the dimensions of the corresponding eigenspaces coincide. The operator* $K_{V,Y}(-\kappa^2)$ *is bounded for any* $\kappa > 0$ *and the function* $\kappa \mapsto K_{V,Y}(-\kappa^2)$ *is continuously decreasing in* $(0, \infty)$ *with* $\lim_{\kappa\to\infty} \|K_{V,Y}(-\kappa^2)\| = 0$.

Curvature-induced bound states

Theorem

Suppose that $\Gamma \neq \Gamma_0$ *satisfy the stated assumptions and* $V \geq 0$*, then* $\inf \sigma(H_{V,Y}) < \epsilon_0 := \inf \sigma_{\mathrm{ess}}(H_{V,Y})$*, and consequently,* $\sigma_{\mathrm{disc}}(H_{V,Y}) \neq \emptyset$.

P.E.: Geometry effects in quantum dot families, arXiv:2305.12748

Sketch of the proof: We have to show that there is a $\kappa > \sqrt{-\epsilon_0}$ such that $K_{V,Y}(-\kappa^2)$ has eigenvalue one. Due to the mentioned monotonicity of the BS operator with respect to κ, it is sufficient to check that

$$\sup \sigma(K_{V,Y}(-\kappa^2)) > \epsilon_{\mathrm{ess}}(\kappa) := \sup \sigma_{\mathrm{ess}}(K_{V,Y}(-\kappa^2)$$

holds for any $\kappa > 0$. To this aim, we construct a trial function ψ such that

$$(\phi, K_{V,Y}(-\kappa_0^2)\phi) - \|\phi\|^2 > 0$$

where the first expression can be rewritten explicitly as

$$\sum_{i,j\in\mathbb{Z}} \int_{B_\rho(y_i)\times B_\rho(y_j)} \overline{\phi}(x) V_i^{1/2}(x)(-\Delta + \kappa_0^2)^{-1}(x,x') V_j^{1/2}(x')\phi(x')\,\mathrm{d}x\,\mathrm{d}x'.$$

Trial function

Denote by ϕ_0 the *generalized eigenfunction* of $K_{V,Y}(-\kappa_0^2)$ referring to $\inf \sigma(H_{V,Y_0})$; as the product of the corresponding *gef* of H_{V,Y_0} and $V_Y^{1/2}$, it is *periodic* and we regard it as *real-valued* and *positive*.

The restrictions $\phi_{0,i} = \phi_0 \upharpoonright B_\rho(y_i)$ are copies of the same function properly shifted, $\phi_{0,i}(\xi) = \phi_0(\xi + y_i)$ for $\xi \in B_\rho(0)$. The symmetries of ϕ_0 imply, in particular, that $\phi_{0,i}(-\xi) = \phi_{0,i}(\xi)$ holds for $\xi \in B_\rho(0)$.

For a given Y the functions ϕ_0^Y as an *'array of beads'*: its values in $B_\rho(y_i)$ would coincide with $\phi_{0,i}$ the axis of which is *aligned with the tangent*

to Γ at the point y_i. To make such a function an L^2 element, we need a suitable family of mollifiers; we choose it in the form

$$h_n(x) = \frac{1}{2n+1}\,\chi_{M_n}(x), \quad n \in \mathbb{N}.$$

where $M_n := \{x :\ \mathrm{dist}(x, \Gamma \upharpoonright [-(2n+1)a/2, (2n+1)a/2]) \le \rho\}$ is a 2ρ-wide closed tubular neighborhood of the $(2n+1)a$-long arc of Γ.

The inequality to be checked

The influence of such a cut-off can be made arbitrarily small:

Lemma

$(h_n\phi_0^Y, K_{V,Y}(-\kappa_0^2)h_n\phi_0^Y) - \|h_n\phi_0^Y\|^2 = \mathcal{O}(n^{-1})$ as $n \to \infty$.

Consequently, it is sufficient to check that

$$\lim_{n\to\infty}(h_n\phi_0^Y, K_{V,Y}(-\kappa_0^2)h_n\phi_0^Y) - (h_n\phi_0 K_{V,Y_0}(-\kappa_0^2)]h_n\phi_0) > 0,$$

or – with an abuse of notation neglecting the rotation of $\phi_{0,i}$ – that

$$(\phi_0, [K_{V,Y}^{(i,j)}(-\kappa^2) - K_{V,Y_0}^{(i,j)}(-\kappa^2)]\phi_0) \geq 0$$

holds any $\kappa > 0$ and all $i, j \in \mathbb{Z}$ being *positive* for some of them.

If $Y \neq Y_0$, however, there is a pair of indices for which this is not the case, $|y_i - y_j| < |i - j|a$, in fact, infinitely many such pairs. The *monotonicity* of the resolvent kernel is *not sufficient*, though, because bending of the chain may cause some distances between points of potential supports outside the ball centers to *increase*.

Convexity enters the game

Denoting the resolvent kernel by $G_{i\kappa}$, we can rewrite the expression as

$$\int_{B_\rho(0)}\int_{B_\rho(0)} \phi_0(\xi)V^{1/2}(\xi)\big[G_{i\kappa}(y_i-y_j+\xi-\xi')-G_{i\kappa}(y_i^{(0)}-y_j^{(0)}+\xi-\xi')\big]$$

$$\times V^{1/2}(\xi')\phi_0(\xi')\mathrm{d}\xi\,\mathrm{d}\xi'$$

$$=\frac{1}{2}\int_{B_\rho(0)}\int_{B_\rho(0)} \phi_0(\xi)V^{1/2}(\xi)\big[G_{i\kappa}(y_i-y_j+\xi-\xi')-G_{i\kappa}(y_i^{(0)}-y_j^{(0)}+\xi-\xi')$$

$$+G_{i\kappa}(y_i-y_j-\xi+\xi')-G_{i\kappa}(y_i^{(0)}-y_j^{(0)}-\xi+\xi')\big]V^{1/2}(\xi')\phi_0(\xi')\mathrm{d}\xi\,\mathrm{d}\xi,$$

where we used the symmetry, $\phi_0(\xi)V^{1/2}(\xi)=\phi_0(-\xi)V^{1/2}(-\xi)$.

The integration over ξ can be split by orientation with respect to y_i-y_j, specifically, we have $\int_{B_\rho(0)}\mathrm{d}\xi=\int_{-\rho}^{\rho}\mathrm{d}\xi_\perp\int_{-\sqrt{\rho^2-s_\perp^2}}^{\sqrt{\rho^2-s_\perp^2}}\mathrm{d}\xi_{||}$.

Now not only the function $G_{i\kappa}(\cdot)$ is *convex*, but the same is true for $G_{i\kappa}(|y_i-y_j|+\cdot)-G_{i\kappa}(|y_i^{(0)}-y_j^{(0)}|+\cdot)$ as long as $|y_i-y_j|<|y_i^{(0)}-y_j^{(0)}|$, hence *Jensen's inequality* yields

$$G_{i\kappa}(|y_i-y_j|)-G_{i\kappa}(|y_i^{(0)}-y_j^{(0)}|)>0.$$

Proof conclusion and comments

In combination with the positivity of $\phi_0 V^{1/2}$ this proves that the right-hand side is *positive* whenever $|y_i - y_j| < |i - j|a$; this in turn concludes the proof.

Note the role of the symmetry of V. Without is, the deformation of Γ had to be strong enough to diminish *all* the distances between the points of the pairs of balls; this is true, e.g., if $|y_i - y_{i+1}| < a - 2\rho$ holds for neighboring balls, which is clearly far from optimal.

One the other hand, *shrinking* the potential wells using an appropriate *nonlinear scaling* one can *approximate point interactions*, which requires neither symmetry of V not it positivity.

S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: Solvable Models in Quantum Mechanics, 2nd edition, Amer. Math. Soc., Providence, R.I., 2005.

For the limiting operator the analogous result is known: an infinite 'locally equidistant' array of point interactions in dimension $\nu = 2, 3$ which not straight, but is asymptotically straight has a *nonempty discrete spectrum*.

P.E.: Bound states of infinite curved polymer chains, *Lett. Math. Phys.* **57** (2001), 87–96.

This suggests that our result is likely to hold under weaker assumptions.

Optimization for finite dot arrays

Consider the 2D situation and fix the curve Γ as a *circle* of radius R on which we place centers of the disks $B_\rho(y_i)$; without loss of generality we place the circle center to the origin of the coordinates. The support balls again do not overlap, $\rho \le R \sin \frac{\pi}{N}$, where $N := \# Y$.

It is again the maximum-symmetry configuration which maximizes the principal eigenvalue of $H_{V,Y}$:

Theorem

Up to rotations, $\epsilon_1(H_{V,Y}) := \inf \sigma(H_{V,Y})$ is uniquely maximized by the configurations in which all the neighboring points of Y have the same angular distance $\frac{2\pi}{N}$.

Proof sketch: The negative spectrum of $H_{V,Y}$ is now discrete and finite, and $\epsilon_1(H_{V,Y})$ is a simple eigenvalue. We denote by Y_{sym} the symmetric array. The real-valued eigenfunction ψ_{sym} associated with $\epsilon_1(H_{V,Y_{\mathrm{sym}}})$ has the appropriate symmetry: in polar coordinates we can express it as $\psi_{\mathrm{sym}}(r,\varphi) = \psi_{\mathrm{sym}}(r,\varphi + \frac{2\pi n}{N})$ for any $n \in \mathbb{Z}$.

Optimization for finite dot arrays

We use BS principle again and denote by $\phi_{\rm sym}$ the eigenfunction corresponding to the *largest eigenvalue* of $K_{V,Y_{\rm sym}}(\epsilon_{\rm sym})$, where $\epsilon_{\rm sym} = \inf \sigma(H_{V,Y_{\rm sym}})$.

It has the same symmetry and may be again regarded as real-valued and positive. In analogy with previous proof we we are looking for a trial function ϕ_Y such that

$$(\phi_Y, K_{V,Y}(-\kappa_0^2)\phi_Y) - \|\phi_Y\|^2 > 0, \quad \kappa_0 = \sqrt{-\epsilon_{\rm sym}}.$$

As before ϕ_Y will be an *'array of beads'*; we take $\phi_{\rm sym} \upharpoonright B_\rho(y_1)$ calling it $\phi_{{\rm sym},1}$ and use it to create $\phi_{{\rm sym},j}$, $j = 2, \dots, N$, by rotating this function on the angle $\sum_{i=1}^{j-1} \theta_i$ around the origin. For $Y = Y_0$ the left-hand side of the inequality vanishes by construction, hence it is sufficient to prove that

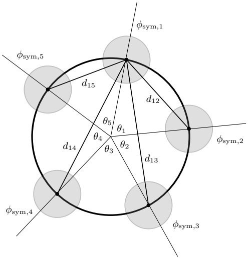
$$(\phi_Y, K_{V,Y}(-\kappa^2)\phi_Y) - (\phi_{\rm sym}, K_{V,Y_0}(-\kappa^2)\phi_{\rm sym}) > 0$$

holds for any $\kappa > 0$, in particular, for $\kappa = \kappa_0$, or explicitly

Optimization for finite dot arrays

$$\frac{1}{2\pi}\sum_{i,j=1}^{N}\Big\{\int_{B_\rho(0)}\int_{B_\rho(0)}\phi_{\rm sym}(\xi)V^{1/2}(\xi)\,K_0(\kappa|y_i+\xi-y_j-\xi'|)$$
$$\times V^{1/2}(\xi')\phi_{\rm sym}(\xi')\,\mathrm{d}\xi\,\mathrm{d}\xi'$$
$$-\int_{B_\rho(0)}\int_{B_\rho(0)}\phi_{\rm sym}(\xi)V^{1/2}(\xi)\,K_0(\kappa|y_i^{(0)}+\xi-y_j^{(0)}-\xi'|)$$
$$\times V^{1/2}(\xi')\phi_{\rm sym}(\xi')\,\mathrm{d}\xi\,\mathrm{d}\xi'\Big\}>0$$

We denote $d_{ij}:=|y_i-y_j|$ and $d_{ij}^{(0)}:=|y_i^{(0)}-y_j^{(0)}|$ as indicated in the figure,



and write the first part of the above expression as $\sum_{i,j=1}^{N}\tilde{G}_{i\kappa}(d_{ij})$.

Convexity again

Using this notation, the sought inequality takes the form

$$\sum_{i,j=1}^{N} \tilde{G}_{i\kappa}(d_{ij}) > \sum_{i,j=1}^{N} \tilde{G}_{i\kappa}(d_{ij}^{(0)}),$$

and rearranging the summation order, we have to check that

$$F(d_{ij}) := \sum_{m=1}^{[N/2]} \sum_{|i-j|=m} \left[\tilde{G}_{i\kappa}(d_{ij}) - \tilde{G}_{i\kappa}(d_{ij}^{(0)})\right] > 0$$

holds for every family $\{d_{ij}\}$ which is *not congruent* with $\{d_{ij}^{(0)}\}$.

The *composed map* $d_{ij} \mapsto K_0(\kappa|y_i + \xi - y_j - \xi'|)$ is easily seen to be convex for any $\xi, \xi' \in B_\rho(0)$, and the property persists at integration with a positive weight, hence by Jensen's inequality

$$F(d_{ij}) \geq \sum_{m=1}^{[N/2]} \nu_n \Big[\tilde{G}_{i\kappa}\Big(\frac{1}{\nu_n} \sum_{|i-j|=m} d_{ij}\Big) - \tilde{G}_{i\kappa}(d_{i,i+m}^{(0)})\Big],$$

where $\nu_n = N$ except for N even and $m = \frac{1}{2}N$, in which case $\nu_n = \frac{1}{2}N$.

Proof conclusion

It remains to use the *monotonicity* of the resolvent kernel, and thus of $\tilde{G}_{i\kappa}(\cdot)$; since $d_{ij} \mapsto |y_i + \xi - y_j - \xi'|$ is increasing, it is only necessary to check that

$$\frac{1}{\nu_n} \sum_{|i-j|=m} d_{ij} < d^{(0)}_{i,i+m}$$

for any fixed i. Denoting $\beta_{ij} = \sum_{k=i}^{j-1} \theta_k$, we have $d_{ij} = 2\sin\frac{1}{2}\beta_{ij}$ and $d^{(0)}_{i,i+m} = 2\sin\frac{\pi m}{N}$, and since the sine function is *strictly concave* in $(0,\pi)$, Jensen's inequality gives

$$\frac{1}{\nu_n} \sum_{|i-j|=m} 2\sin\frac{1}{2}\beta_{ij} < 2\sin\left(\frac{1}{\nu_n} \sum_{|i-j|=m} \frac{1}{2}\beta_{ij}\right) = 2\sin\frac{\pi m}{N} = d^{(0)}_{i,i+m}$$

for those families $\{d_{ij}\}$ of circle chords which are *not congruent* with $\{d^{(0)}_{ij}\}$; this concludes the proof.

Remarks

- By an easy modification with a *planar* circle, one can prove the analogous claim for a quantum-dot 'necklace' in *three dimensions*
- We *conjecture* that the claim extends to a wider class of functions: if points of Y are on a loop Γ of a fixed length in $\mathbb{R}^\nu$, $\nu = 2, 3$, *equidistantly in arc length*, and the balls $B_\rho(y_i)$ do not overlap, $\epsilon_1(H_{V,Y}) = \inf \sigma(H_{V,Y})$ is maximized, uniquely up to Euclidean transformations, by a *planar regular polygon* of $\#Y$ vertices.
- Optimizing a distribution on a *sphere* is much harder reminding the *Thomson problem*. We *conjecture* that if balls $B_\rho(y_i)$ centered at a sphere do not overlap, $\epsilon_1(H_{V,Y})$ is *maximized*, uniquely up to Euclidean transformations, by the following five configurations:
  - three *simplices*, with $N = 2$ (a pair antipodal points), $N = 3$ (equilateral triangle), and $N = 4$ (tetrahedron),
  - *octahedron* with $N = 6$,
  - *icosahedron* with $N = 12$.

  Note that both conjectures have proved point-interaction counterparts

  P.E.: An optimization problem for finite point interaction families, *J. Phys.: Math. Theor.* **52** (2019), 405302
- One can also consider the *minimization problem* in this context

If time allows: gentle deformations

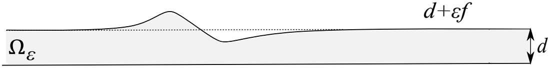
The weak-coupling behavior of the discrete spectrum in the situation when a straight Dirichlet strip suffers a *gentle deformation* was analyzed and the corresponding asymptotic expansions derived

W. Bulla, F. Gesztesy, W. Renger, B. Simon: Weakly coupled bound states in quantum waveguides, *Proc. Amer. Math. Soc.* **127** (1997), 1487–1495.

P.E., S.A. Vugalter: Bounds states in a locally deformed waveguide: the critical case, *Lett. Math. Phys.* **39** (1997), 59–68.

Let us pose the same question for a soft waveguide; as in the mentioned papers we assume for simplicity that the deformation is *one-sided*. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous compactly supported function and $\varepsilon\in[0,\varepsilon_0)$ with ε_0 sufficiently small. We put

$$\Omega_\varepsilon := \{(x_1,x_2)\in\mathbb{R}^2:\ 0<x_2<d+\varepsilon f(x_1)\}$$



Gentle deformations of a soft waveguide

A *flat-bottom* waveguide of the depth α referring to the deformed strip Ω_ε is described by the Hamiltonian

$$H_{\alpha,\varepsilon} = -\Delta - \alpha\chi_{\Omega_\varepsilon}, \quad \mathrm{dom}(H_{\alpha,\varepsilon}) = H^2(\mathbb{R}^2),$$

associated with the quadratic form

$$h_{\alpha,\varepsilon}[u] = \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\Omega_\varepsilon} |u|^2 \mathrm{d}x, \quad \mathrm{dom}(h_{\alpha,\varepsilon}) = H^1(\mathbb{R}^2).$$

The profile of the straight guide, $\varepsilon = 0$, is described by the operator

$$h_\alpha :\ h_\alpha\psi = -\psi'' - \alpha\chi_{[0,d]}\psi, \quad \mathrm{dom}(h_\alpha) = H^2(\mathbb{R}).$$

It is elementary to find its spectral properties. In particular, it has a finite and nonempty discrete spectrum $\{\mu_n\}_{n=1}^N$ with bounded eigenfunctions v_n. Equally easily, one can check that $\sigma(H_{\alpha,0}) = \sigma_{\mathrm{ess}}(H_{\alpha,0}) = [\mu_1, \infty)$ and

$$\sigma_{\mathrm{ess}}(H_{\alpha,\varepsilon}) = [\mu_1, \infty).$$

Asymptotics in the non-critical regime

We are interested in the spectral bottom, $\lambda_1^\alpha(\varepsilon) := \inf \sigma(H_{\alpha,\varepsilon})$, first in the *non-critical regime* when the area change is nonzero.

Theorem

(i) If $\int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1 > 0$, *we have* $N_\alpha(\varepsilon) = 1$ *for all sufficiently small* ε *and the corresponding eigenvalue admits the asymptotic expansion*

$$\lambda_1^\alpha(\varepsilon) = \mu_1 - \varepsilon^2 \Big(\frac{\alpha^2 v_1^4(d)}{4}\Big)\Big(\int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1\Big)^2 + \mathcal{O}(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0.$$

(ii) If $\int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1 < 0$, *we have* $N_\alpha(\varepsilon) = 0$ *for all sufficiently small* ε, *in other words, the discrete spectrum of* $H_{\alpha,\varepsilon}$ *is empty.*

P.E., S. Kondej, V. Lotoreichik: Bounds states of weakly deformed open waveguides, arXiv:2211.01989 [math.SP]

Remark: the claim (ii) holds *in the weak-coupling regime only*. Indeed, it is not difficult to check that if $f = f_+ + f_-$ with $\pm f_\pm > 0$ and the supports of f_+ and f_- are *far apart*, the discrete spectrum for a fixed ε is *nonempty* irrespective of the relation between $\int_{\mathbb{R}} f_+(x_1)\,\mathrm{d}x_1$ and $\int_{\mathbb{R}} f_-(x_1)\,\mathrm{d}x_1$.

Comparison with the Dirichlet guide

Since the limit $\alpha \to \infty$ of $H_{\alpha,\varepsilon}$ yields the Dirichlet Laplacian $-\Delta^{\mathrm{D}}_{\Omega_\varepsilon}$ (in the sense of generalized strong convergence [Weidmann'00]), it is appropriate to compare the expansion to that for $-\Delta^{\mathrm{D}}_{\Omega_\varepsilon}$, which reads

$$\lambda_1^{\mathrm{D}}(\varepsilon) = \left(\frac{\pi}{d}\right)^2 - \varepsilon^2 \left(\frac{\pi}{d}\right)^4 \left(\frac{1}{d}\int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1\right)^2 + \mathcal{O}(\varepsilon^3).$$

While it is not possible to derive this expression from the above theorem with an additional information about the error term, the two results are *consistent*: using

$$v_1(d) = \sqrt{\frac{2}{d}}\,\frac{\pi}{d\sqrt{\alpha}} + \mathcal{O}(\alpha^{-1}), \quad \mu_1 = -\alpha + \left(\frac{\pi}{d}\right)^2 + \mathcal{O}(\alpha^{-1/2}),$$

one can check that the first two terms in the asymptotic expansion of $\alpha + \lambda_1^\alpha(\varepsilon)$ converge to their Dirichlet counterparts as $\alpha \to \infty$.

The eigenfunction asymptotics

One can also find how the eigenfuction of $H_{\alpha,\varepsilon}$ looks in the leading order:

Theorem

If $\int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1 > 0$, *we have for all sufficiently small* ε *the expansion*

$$\psi_\varepsilon = u_\varepsilon + v_\varepsilon$$

for the unnormalized eigenfunction, in which the leading contribution is

$$u_\varepsilon(x) = \frac{v_1(x_2)}{\varepsilon} \int_{\mathbb{R}^2} e^{-\hat{\delta}(\varepsilon)|x_1 - x_1'|} v_1^2(x_2')\, V_{\alpha,\varepsilon}(x')\,\mathrm{d}x'$$

with $V_{\alpha,\varepsilon} = \sqrt{\alpha}(\chi_{\Omega_\varepsilon} - \chi_{\Omega_0})$ *and* $\hat{\delta}(\varepsilon) = \varepsilon\Big(\frac{\alpha v_1^2(d)}{2}\Big)^2 \int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1$. *The* L^2 *norms of the two constituents of* ψ_ε *are*

$$\|u_\varepsilon\| = \sqrt{2} v_1(d)\Big(\int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1\Big)^{1/2} + \mathcal{O}(\sqrt{\varepsilon}) \quad \text{and} \quad \|v_\varepsilon\| = \mathcal{O}(\sqrt{\varepsilon}).$$

Proof idea

One uses *Birman-Schwinger principle* again and applies the common trick to deal with the weak coupling in waveguides:

The BS kernel is split into two parts, one coming from the *projection on the lowest transverse mode*, v_1 in our case, and the rest. The latter is shown to be small as $\varepsilon \to 0$, from the former one can derive the leading term of the expansion. The idea is simple but implementation is laborious.

The argument does not work in the *critical case*. Instead one can use the variational method to find a sufficient condition for the existence of the discrete spectrum. Choosing the trial function

$$\psi_\varepsilon(x_1, x_2) = g(\varepsilon^3 x_1)[1 + \varepsilon\sqrt{-\mu_1} f(x_1)] v_1(x_2)$$

with ε small enough, where $g \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} g \subset [-2, 2]$ and such that $g(u) = 1$ for $u \in [-1, 1]$, we can prove the following claim:

Discrete spectrum in the critical regime

Theorem

Suppose that $f \in W^{1,\infty}$ and $\int_{\mathbb{R}} f(x_1)\,\mathrm{d}x_1 = 0$. If the condition

$$\frac{\int_{\mathbb{R}} |f'(x_1)|^2\,\mathrm{d}x_1}{\int_{\mathbb{R}} |f(x_1)|^2\,\mathrm{d}x_1} < \frac{\alpha |v_1(d)|^2}{\sqrt{-\mu_1}}$$

is satisfied, then $H_{\alpha,\varepsilon}$ has a unique simple eigenvalues below the bottom of the essential spectrum for all $\varepsilon > 0$ small enough.

Remarks: (a) The deformation must be *elongated enough*; taking $f_\gamma(x) = f(\gamma x)$, the above condition is satisfied for all γ small enough.

(b) A similar 'spread enough' condition is known in the Dirichlet case, however, a direct comparison is not possible; note that the right-hand side of the above inequality behaves as $\mathcal{O}(\alpha^{-1/2})$ as $\alpha \to \infty$.

(c) In contrast to the Dirichlet case, we lack a sufficient condition for the *nonexistence* of the discrete spectrum.

It is time to stop, and it remains naturally to say

Thank you for your attention!