Isoperimetric problems for singular interactions in the plane

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Talk overview

- *Motivation:* some classical and less classical isoperimetric results

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- *Existence of a maximizer:* a local "continuous" result
- *Open questions*

Motivation

Isoperimetric problems are traditional in mathematical physics. Recall, e.g., the *Faber-Krahn inequality* for the Dirichlet Laplacian $-\Delta_D^M$ in a compact $M \subset \mathbb{R}^2$: among all regions with a fixed area the ground state is *uniquely minimized by the circle*,

$$\inf \sigma(-\Delta_D^M) \geq \pi\, j_{0,1}^2\, |M|^{-1}$$

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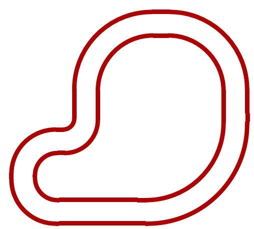
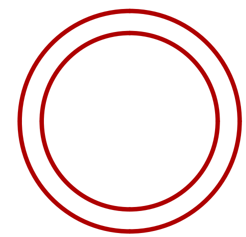
Another classical example is the *PPW conjecture* proved by *Ashbaugh* and *Benguria*: in the same situation we have

$$\frac{\lambda_2(M)}{\lambda_1(M)} \leq \left(\frac{j_{1,1}}{j_{0,1}} \right)^2$$

However, topology is important

If M is not simply connected, rotational symmetry may again lead to an extremum but its nature can be different. Recall a *a strip of fixed length and width* [E.-Harrell-Loss'99]

ground state of  $<$ ground state of 

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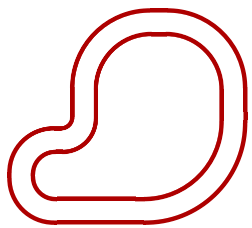
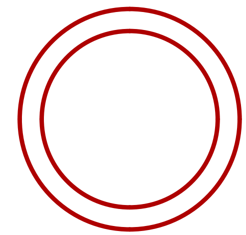
ground state of  $<$ ground state of 

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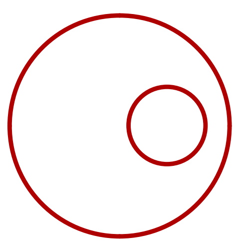
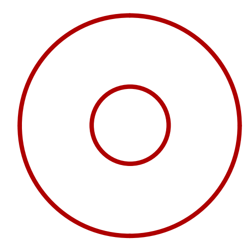
Another example is a *circular obstacle in circular cavity* [Harrell-Kröger-Kurata'01]

ground state of  $<$ ground state of 

whenever the obstacle is off center

Potential confinement

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First we take the simplest possible example where the confinement is due to a *closed array of δ potentials*, so the Hamiltonian can be written formally as

$$-\Delta + \tilde{\alpha} \sum_{j=1}^{N} \delta(x - y_j) \quad \text{in } L^2(\mathbb{R}^2) \,,$$

where the y_j's are vertices of an *equilateral polygon* $\mathcal{P}_N$

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where the y_j's are vertices of an *equilateral polygon $\mathcal{P}_N$*

Next we will consider an attractive δ potential supported by a *closed loop Γ of fixed length*, so formally we have

$$-\Delta - \alpha\delta(x - \Gamma) \quad \text{in} \;\; L^2(\mathbb{R}^2)$$

Remarks

- The two examples are related yet different in the character of the coupling, due the *codimension* of the interaction support. Roughly speaking, the 2D point interactions are a lot “more singular”

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- It is not apriori clear whether the same is true for *any coupling* (in our models the ground state always exists)
- There are extensions to *higher dimension*, which will mentioned later at appropriate places

A preliminary: 2D point interactions

Fixing the site y and "coupling constant" α we define them by b.c. which change *locally* the domain of $-\Delta$: we require

$$\psi(x) = -\frac{1}{2\pi} \log|x-y|\, L_0(\psi, y) + L_1(\psi, y) + \mathcal{O}(|x-y|)\,,$$

where the generalized b.v. $L_0(\psi, y)$ and $L_1(\psi, y)$ satisfy

$$L_1(\psi, y) + 2\pi\alpha L_0(\psi, y) = 0\,, \quad \alpha \in \mathbb{R}$$

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In this way we define our Hamiltonian $-\Delta_{\alpha,\mathcal{P}_N}$ in $L^2(\mathbb{R}^2)$ with N point interactions. We have $\sigma_{\rm disc}\left(-\Delta_{\alpha,\mathcal{P}_N}\right) \neq \emptyset$, i.e.

$$\epsilon_1 \equiv \epsilon_1(\alpha,\mathcal{P}_N) := \inf\sigma\left(-\Delta_{\alpha,\mathcal{P}_N}\right) < 0\,,$$

which is always true in two dimensions – cf. [AGHH’88, 05]

The point-interaction result

Theorem [E.'05]: Under the stated conditions, $\epsilon_1(\alpha, \mathcal{P}_N)$ is for fixed α and ℓ *locally sharply maximized* by a regular polygon, $\mathcal{P}_N = \tilde{\mathcal{P}}_N$.

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Proof will be reduced to the following *geometric problem:* Let $\mathcal{P}_N$ be an equilateral polygon. Given a fixed integer $m = 2, \ldots, [\frac{1}{2}N]$ we denote by $\mathcal{D}_m$ the *sum of lengths of all* m*-diagonals*, i.e. we put $\mathcal{D}_m := \sum_{i=1}^{N} |y_i - y_{i+m}|$

$D^1_{N,\ell}(m)$ The quantity $\mathcal{D}_m$ is, in the set of equilateral polygons $\mathcal{P}_N \subset \mathbb{R}^2$ with a fixed edge length $\ell > 0$, *uniquely maximized* by $\tilde{\mathcal{D}}_m$ referring to the (family of) regular polygon(s) $\tilde{\mathcal{P}}_N$.

Geometric reformulation

By Krein's formula, the spectral condition is reduced to an algebraic problem. Using $k = i\kappa$ with $\kappa > 0$, we find the ev's $-\kappa^2$ of our operator from

$$\det \Gamma_k = 0 \quad \text{with} \quad (\Gamma_k)_{ij} := (\alpha - \xi^k)\delta_{ij} - (1 - \delta_{ij}) g_{ij}^k \,,$$

where the off-diagonal elements are $g_{ij}^k := G_k(y_i - y_j)$, or equivalently

$$g_{ij}^k = \frac{1}{2\pi} K_0(\kappa |y_i - y_j|)$$

and the regularized Green's function at the interaction site is

$$\xi^k = -\frac{1}{2\pi} \left(\ln \frac{\kappa}{2} + \gamma_{\mathrm{E}} \right)$$

Geometric reformulation, continued

The ground state refers to the point where the *lowest* ev of $\Gamma_{i\kappa}$ vanishes. Using smoothness and monotonicity of the κ-dependence we have to check that

$$\min \sigma(\Gamma_{i\tilde{\kappa}_1}) < \min \sigma(\tilde{\Gamma}_{i\tilde{\kappa}_1})$$

holds locally for $\mathcal{P}_N \neq \tilde{\mathcal{P}}_N$, where $-\tilde{\kappa}_1^2 := \epsilon_1(\alpha, \tilde{\mathcal{P}}_N)$

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There is a *one-to-one relation* between an ef $c = (c_1, \dots, c_N)$ of $\Gamma_{i\kappa}$ at that point and the corresponding ef of $-\Delta_{\alpha,\mathcal{P}_N}$ given by $c \leftrightarrow \sum_{j=1}^N c_j G_{i\kappa}(\cdot - y_j)$, up to normalization. In particular, the lowest ev of $\tilde{\Gamma}_{i\tilde{\kappa}_1}$ corresponds to the eigenvector $\tilde{\phi}_1 = N^{-1/2}(1, \dots, 1)$. Hence

$$\min \sigma(\tilde{\Gamma}_{i\tilde{\kappa}_1}) = (\tilde{\phi}_1, \tilde{\Gamma}_{i\tilde{\kappa}_1}\tilde{\phi}_1) = \alpha - \xi^{i\tilde{\kappa}_1} - \frac{2}{N}\sum_{i<j} \tilde{g}_{ij}^{i\tilde{\kappa}_1}$$

Geometric reformulation, continued

On the other hand, we have $\min \sigma(\Gamma_{i\tilde{\kappa}_1}) \leq (\tilde{\phi}_1, \Gamma_{i\tilde{\kappa}_1}\tilde{\phi}_1)$, and therefore it is sufficient to check that

$$\sum_{i<j} G_{i\kappa}(y_i - y_j) > \sum_{i<j} G_{i\kappa}(\tilde{y}_i - \tilde{y}_j)$$

holds *for all* $\kappa > 0$ and $\mathcal{P}_N \neq \tilde{\mathcal{P}}_N$.

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holds *for all* $\kappa > 0$ and $\mathcal{P}_N \neq \tilde{\mathcal{P}}_N$. Call $\ell_{ij} := |y_i - y_j|$ and $\tilde{\ell}_{ij} := |\tilde{y}_i - \tilde{y}_j|$ and define $F : (\mathbb{R}_+)^{N(N-3)/2} \to \mathbb{R}$ by

$$F(\{\ell_{ij}\}) := \sum_{m=2}^{[N/2]} \sum_{|i-j|=m} \left[G_{i\kappa}(\ell_{ij}) - G_{i\kappa}(\tilde{\ell}_{ij}) \right] ;$$

Using the *convexity* of $G_{i\kappa}(\cdot)$ for a fixed $\kappa > 0$ we get

$$F(\{\ell_{ij}\}) \geq \sum_{m=2}^{[N/2]} \nu_m \left[G_{i\kappa} \left(\frac{1}{\nu_m} \sum_{|i-j|=m} \ell_{ij} \right) - G_{i\kappa}(\tilde{\ell}_{1,1+m}) \right] ,$$

where ν_n is the number of the appropriate diagonals

Geometric reformulation, continued

Since $G_{i\kappa}(\cdot)$ is also *monotonously decreasing* in $(0, \infty)$, we need

$$\tilde{\ell}_{1,m+1} \geq \frac{1}{\nu_n} \sum_{|i-j|=m} \ell_{ij}$$

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The claim we made is then implied by the following result:

Proposition: The property $D^1_{N,\ell}(m)$ holds *locally* for any $m = 2, \dots, [\frac{1}{2}N]$

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Remark: The argument carries through *for point interactions in* $\mathbb{R}^3$ because the Green's function is again convex and monotonous

Local validity of $D^1_{N,\ell}(m)$

We are looking for constrained local maxima of the function

$$f_m : \; f_m(y_1, \ldots, y_N) = \frac{1}{N} \sum_{i=1}^{N} |y_i - y_{i+m}|$$

with $g_i(y_1, \ldots, y_n) := \ell - |y_i - y_{i+1}| = 0, \; i = 1, \ldots, N$. There are in fact $(N-2)(d-1)-1$ independent variables because $2d-1$ parameters are related to Euclidean transformations

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It is straightforward to check that $\nabla_j K_m(y_1,\ldots,y_N)$ vanish for a regular polygon, $K_m := f_m + \sum_{r=1}^{N} \lambda_r g_r$, with all the Lagrange multipliers taking the same value

$$\lambda = \frac{\sigma_m}{N\Upsilon_m} \quad \text{with} \quad \sigma_m := \frac{\sin^2\frac{\pi m}{N}}{\sin^2\frac{\pi}{N}},\; \Upsilon_m := \ell^{-1}|\tilde{y}_j - \tilde{y}_{j\pm m}|$$

Local validity of $D^1_{N,\ell}(m)$, continued

Negative definiteness of the Hessian needs more computation. A simple estimate then shows that it is sufficient to establish negative definiteness of the form

$$\xi \mapsto S_m[\xi] := \sum_j \left\{ |\xi_j - \xi_{j+m}|^2 - \sigma_m |\xi_j - \xi_{j+1}|^2 \right\}$$

on $\mathbb{R}^{2N}$ (the case $m = 2$ needs an additional argument)

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The two parts can be simultaneously diagonalized; using their ev's one rewrites the condition as the inequality

$$U_{m-1}\left(\cos\frac{\pi}{N}\right) > \left|U_{m-1}\left(\cos\frac{\pi r}{N}\right)\right|, \; r = 2, \dots, m-1\,,$$

for Chebyshev polynomials of the second kind which can be checked directly $\square$

Attractive δ loops

To formulate the continuous analogue we have first to give meaning the formal operator

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)\,, \quad \alpha > 0\,,$$

in $L^2(\mathbb{R}^2)$, where Γ is a loop in the plane; we suppose that it has no *zero-angle* self-intersections

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$H_{\alpha,\Gamma}$ can be naturally associated with the quadratic form,

$$\psi \mapsto \|\nabla\psi\|^2_{L^2(\mathbb{R}^2)} - \alpha \int_\Gamma |\psi(x)|^2 \mathrm{d}x\,,$$

which is closed and below bounded in $W^{1,2}(\mathbb{R}^2)$; the second term makes sense in view of Sobolev embedding. This definition also works for various “wilder” sets Γ

Definition by boundary conditions

If Γ is *piecewise smooth* with *no cusps* we can use an *alternative definition* by boundary conditions: $H_{\alpha,\Gamma}$ acts as $-\Delta$ on functions from $W^{2,1}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\bigg|_{+} - \frac{\partial\psi}{\partial n}(x)\bigg|_{-} = -\alpha\psi(x)$$

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Remarks:

- this definition has an illustrative meaning which corresponds to a δ potential in the cross cut of Γ
- using the quadratic form associated with $H_{\alpha,\Gamma}$ one can check directly that the discrete spectrum *is not void* for any $\alpha > 0$; one has, of course, $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [0,\infty)$

The loop result

Let $\Gamma : [0, L] \to \mathbb{R}^2$ be *a closed curve*, $\Gamma(0) = \Gamma(L)$, parametrized by its arc length, which is C^1*-smooth, piecewise* C^2, and has *no cusps*. We will always consider classes of Euclidean transforms of Γ; it is clear that the *circle* class, $\mathcal{C} := \{\, ((L/2\pi)\cos s, (L/2\pi)\sin s) : \; s \in [0, L] \,\}$, belongs to this family

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Theorem [E.'05]: Within the specified class of curves,

$$\epsilon_1 \equiv \epsilon_1(\alpha, \Gamma) := \inf \sigma \left(H_{\alpha,\Gamma} \right)$$

is for any fixed $\alpha > 0$ and $L > 0$ *locally sharply maximized* by a circle, $\Gamma = \mathcal{C}$.

Birman-Schwinger reformulation

We employ the generalized Birman-Schwinger principle [BEKŠ'94]. One starts from the free resolvent R_0^k which is an integral operator in $L^2(\mathbb{R}^2)$ with the kernel

$$G_k(x-y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|)$$

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Then we introduce embedding operators associated with R_0^k for measures μ, ν which are the Dirac measure m supported by Γ and the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^2$; by $R_{\nu,\mu}^k$ we denote the integral operator from $L^2(\mu)$ to $L^2(\nu)$ with the kernel G_k, i.e. we suppose that

$$R_{\nu,\mu}^k \phi = G_k * \phi\mu$$

holds ν-a.e. for all $\phi \in D(R_{\nu,\mu}^k) \subset L^2(\mu)$

BS reformulation, continued

Proposition [BEKŠ'94, Posilicano'04]: (i) There is $\kappa_0 > 0$ s.t. $I - \alpha R^{i\kappa}_{m,m}$ on $L^2(m)$ has a bounded inverse for $\kappa \geq \kappa_0$

(ii) Let $\mathrm{Im}\, k > 0$ and $I - \alpha R^k_{m,m}$ be invertible with

$$R^k := R^k_0 + \alpha R^k_{\mathrm{d}x,m}[I - \alpha R^k_{m,m}]^{-1} R^k_{m,\mathrm{d}x}$$

from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ everywhere defined. Then k^2 belongs to $\rho(H_{\alpha,\Gamma})$ and $(H_{\alpha,\Gamma} - k^2)^{-1} = R^k$

(iii) $\dim\ker(H_{\alpha,\Gamma} - k^2) = \dim\ker(I - \alpha R^k_{m,m})$ for $\mathrm{Im}\, k > 0$

(iv) an ef of $H_{\alpha,\Gamma}$ associated with k^2 can be written as

$$\psi(x) = \int_0^L R^k_{\mathrm{d}x,m}(x,s)\phi(s)\,\mathrm{d}s\,,$$

where ϕ is the corresponding ef of $\alpha R^k_{m,m}$ with the ev one

BS reformulation, continued

Putting $k = i\kappa$ with $\kappa > 0$ we look thus for solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s)-\Gamma(s')|)\,,$$

on $L^2([0,L])$. The function $\kappa \mapsto \mathcal{R}^{\kappa}_{\alpha,\Gamma}$ is strictly decreasing in $(0,\infty)$ and $\|\mathcal{R}^{\kappa}_{\alpha,\Gamma}\| \to 0$ as $\kappa \to \infty$, hence we seek the point where the *largest* ev of $\mathcal{R}^{\kappa}_{\alpha,\Gamma}$ crosses one

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Putting $k = i\kappa$ with $\kappa > 0$ we look thus for solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s)-\Gamma(s')|)\,,$$

on $L^2([0,L])$. The function $\kappa \mapsto \mathcal{R}^{\kappa}_{\alpha,\Gamma}$ is strictly decreasing in $(0,\infty)$ and $\|\mathcal{R}^{\kappa}_{\alpha,\Gamma}\| \to 0$ as $\kappa \to \infty$, hence we seek the point where the *largest* ev of $\mathcal{R}^{\kappa}_{\alpha,\Gamma}$ crosses one

We observe that this ev is *simple,* since $\mathcal{R}^{\kappa}_{\alpha,\Gamma}$ is positivity improving and ergodic. The ground state of $H_{\alpha,\Gamma}$ is, of course, also simple. Using its rotational symmetry and the claim (iv) of the Proposition we find that the respective eigenfunction of $\mathcal{R}^{\tilde\kappa_1}_{\alpha,\mathcal{C}}$ corresponding to the unit eigenvalue is constant; we can choose it as $\tilde\phi_1(s) = L^{-1/2}$.

BS reformulation, continued

Then we have

$$\max \sigma(\mathcal{R}^{\tilde{\kappa}_1}_{\alpha,\mathcal{C}}) = (\tilde{\phi}_1, \mathcal{R}^{\tilde{\kappa}_1}_{\alpha,\mathcal{C}} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}^{\tilde{\kappa}_1}_{\alpha,\mathcal{C}}(s,s')\, \mathrm{d}s \mathrm{d}s' \,,$$

and on the other hand, for the same quantity referring to a general Γ a simple variational estimate gives

$$\max \sigma(\mathcal{R}^{\tilde{\kappa}_1}_{\alpha,\Gamma}) \geq (\tilde{\phi}_1, \mathcal{R}^{\tilde{\kappa}_1}_{\alpha,\Gamma} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}^{\tilde{\kappa}_1}_{\alpha,\Gamma}(s,s')\, \mathrm{d}s \mathrm{d}s' \,.$$

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Hence it is sufficient to show that

$$\int_0^L\int_0^L K_0(\kappa|\Gamma(s)-\Gamma(s')|)\,\mathrm{d}s\mathrm{d}s' \ge \int_0^L\int_0^L K_0(\kappa|\mathcal{C}(s)-\mathcal{C}(s')|)\,\mathrm{d}s\mathrm{d}s'$$

holds *for all* $\kappa > 0$ and Γ in the vicinity of $\mathcal{C}$

Convexity argument

By a simple change of variables the claim is equivalent to positivity of the functional

$$F_\kappa(\Gamma) := \int_0^{L/2} \mathrm{d}u \int_0^L \mathrm{d}s \Big[K_0\big(\kappa|\Gamma(s+u)-\Gamma(s)|\big) - K_0\big(\kappa|\mathcal{C}(s+u)-\mathcal{C}(s)|\big) \Big]\,;$$

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The (strict) convexity of K_0 yields by means of Jensen inequality the estimate

$$\frac{1}{L} F_\kappa(\Gamma) \geq \int_0^{L/2} \left[K_0\left(\frac{\kappa}{L} \int_0^L |\Gamma(s+u)-\Gamma(s)|\mathrm{d}s \right) - K_0\left(\frac{\kappa L}{\pi} \sin\frac{\pi u}{L} \right) \right] \mathrm{d}u \,,$$

where the inequality is sharp unless $\int_0^L |\Gamma(s+u)-\Gamma(s)|\mathrm{d}s$ is independent of s

Monotonicity argument

Finally, we observe that K_0 is decreasing in $(0, \infty)$, hence it is sufficient to check the inequality

$$\int_0^L |\Gamma(s+u) - \Gamma(s)| \, \mathrm{d}s \le \frac{L^2}{\pi} \sin \frac{\pi u}{L}$$

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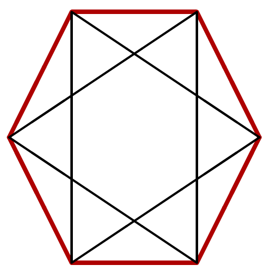
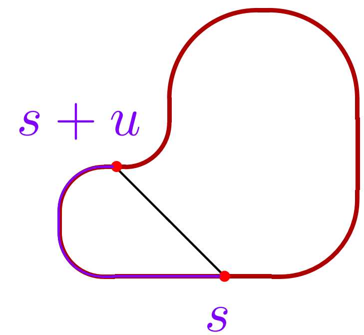
Remark: There was nothing *local* so far, hence proving the above inequality for all Γ would give the global result. Likewise, we have not used the C^2 smoothness

Common feature: summing chord lengths





Both geometric reformulations have a common feature: for polygons *we sum diagonal lengths* between vertices whose indices differ by a fixed m, for a loop *we integrate chord lengths* between points separated by a fixed arc length u

Mean-chord inequalities

Consider a wider family of inequalities – without knowing whether they are valid. Let $\Gamma:\ [0,L]\to\mathbb{R}^2$ be again a loop in the plane, with unspecified regularity properties. Take all the arcs of Γ having length $u\in(0,\frac{1}{2}L]$ and write

$$C_L^p(u):\quad \int_0^L |\Gamma(s+u)-\Gamma(s)|^p\,\mathrm{d}s \le \frac{L^{1+p}}{\pi^p}\sin^p\frac{\pi u}{L}\,,\quad p>0\,,$$

$$C_L^{-p}(u):\quad \int_0^L |\Gamma(s+u)-\Gamma(s)|^{-p}\,\mathrm{d}s \ge \frac{\pi^p L^{1-p}}{\sin^p\frac{\pi u}{L}}\,,\quad p>0\,.$$

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A discrete counterpart for an equilateral polygon $\mathcal{P}_N$ of N vertices $\{y_n\}$, side length $\ell > 0$, and $m = 1, \dots, [\frac{1}{2}N]$ reads

$$D_{N,\ell}^p(m): \quad \sum_{n=1}^N |y_{n+m}-y_n|^p \le \frac{N \ell^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}}, \quad p>0,$$

$$D_{N,\ell}^{-p}(m): \quad \sum_{n=1}^N |y_{n+m}-y_n|^{-p} \ge \frac{N \sin^p \frac{\pi}{N}}{\ell^p \sin^p \frac{\pi m}{N}}, \quad p>0.$$

Observations

- The right-hand sides correspond to the cases with *maximum symmetry,* i.e. to the circle and regular polygon $\tilde{\mathcal{P}}_N$, respectively

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- If $p = 0$ the inequalities turn into trivial identities
- By *scaling* one can put, for instance, $L = 1$ and $\ell = 1$ without loss if generality
- In the polygon case it is clear that the claim *may not be true for* $p > 2$ as the example of a rhomboid shows: $D^p_{4,\ell}(2)$ is equivalent to $\sin^p \phi + \cos^p \phi \le 2^{1-(p/2)}$ for $0 < \phi < \pi$

Properties and conjecture

Using convexity of $x \mapsto x^\alpha$ in $(0,\infty)$ for $\alpha > 1$ we get

Proposition: $C_L^p(u) \Rightarrow C_L^{p'}(u)$ and $D_{N,\ell}^p(m) \Rightarrow D_{N,\ell}^{p'}(m)$ if $p > p' > 0$

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Conjecture: We expect the above inequalities to be valid for any $p \leq 2$, without substantial regularity restrictions in the continuous case

What is known for $D^p_{N,\ell}(m)$?

We have shown that $D^1_{N,\ell}(m)$ holds *locally* for any $m = 2, \ldots, [\frac{1}{2}N]$, i.e. in the vicinity of the regular polygon, and consequently, $D^{\pm p}_{N,\ell}(m)$ holds locally for any $p \in (0, 1]$

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Proof: Call β_i the "bending angle" at i-th vertex, then the mean length of the 2-diagonals is $M_2 = \frac{2\ell}{N}\sum_{i=1}^N \cos\frac{\beta_i}{2}$. Using *strict convexity* of the function $u \mapsto -\cos\frac{u}{2}$ in $(-\pi, \pi)$ together with $\sum_{i=1}^N \beta_i = 2\pi w,\ w \in \mathbb{Z}$, we find

$$-\sum_{i=1}^N \cos\frac{\beta_i}{2} \geq -N\cos\left(\sum_{i=1}^N \frac{\beta_i}{2}\right) = -N\cos\frac{\pi}{N}\ ;$$

the inequality is sharp unless all the β_i's are the same $\square$

$C_L^p(u)$ in terms of curvature

Under our regularity assumption we can characterize Γ by its (signed) *curvature* $\gamma := \dot{\Gamma}_2\ddot{\Gamma}_1 - \dot{\Gamma}_1\ddot{\Gamma}_2$ which is piecewise continuous in $[0, L]$. Up to Euclidean transf's we have

$$\Gamma(s) = \left(\int_0^s \cos\beta(s')\,\mathrm{d}s', \int_0^s \sin\beta(s')\,\mathrm{d}s'\right) ,$$

where $\beta(s) := \int_0^s \gamma(s')\,\mathrm{d}s'$ is bending angle relative to $s = 0$

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To ensure that the curve is closed, we have to require

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The left-hand side of $C_L^p(u)$ can be now rewritten as

$$c_\Gamma^p(u) = \int_0^L \mathrm{d}s \left[\int_s^{s+u} \mathrm{d}s' \int_s^{s+u} \mathrm{d}s'' \cos(\beta(s') - \beta(s''))\right]^{p/2}$$

Proof of $C_L^2(u)$

It is sufficient to check that $c_\Gamma^2(u)$ is maximized by the circle, i.e. by $\beta(s) = \frac{2\pi s}{L}$. Rearranging the integrals we get

$$c_\Gamma^2(u) = \int_0^L \mathrm{d}s' \int_{s'-u}^{s'+u} \mathrm{d}s'' \, \left[u - |s' - s''|\right] \cos(\beta(s') - \beta(s''))$$

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Next we change the integration variables to $x := s' - s''$ and $z := \frac{1}{2}(s' + s'')$, and use the even parity of the functions involved to obtain

$$c_\Gamma^2(u) = 2\int_0^u \mathrm{d}x\,(u-x) \int_0^L \mathrm{d}z\, \cos\left(\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} \gamma(s)\,\mathrm{d}s\right)$$

A partial global result

In an analogy with $D^1_{N,\ell}(2)$ we can get a global result for u small enough:

Proposition: Let Γ have no self-intersections and the inequality $\beta(z+\frac{1}{2}u) - \beta(z-\frac{1}{2}u) \le \frac{1}{2}\pi$ is valid for all $z \in [0, L]$, then $C^2_L(u)$ holds

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Proof: We employ concavity of cosine in $(0,\frac{1}{2}\pi)$ obtaining

$$\begin{aligned} c^2_\Gamma(u) &\le 2L\int_0^u \mathrm{d}x\,(u-x)\cos\left(\frac{1}{L}\int_0^L \mathrm{d}z\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x}\gamma(s)\,\mathrm{d}s\right) \\ &= 2L\int_0^u \mathrm{d}x\,(u-x)\cos\frac{2\pi x}{L}=\frac{L^3}{\pi^2}\sin^2\frac{\pi u}{L}\,, \end{aligned}$$

since $\int_0^L\gamma(s)\,\mathrm{d}s=\pm 2\pi$. The function $z\mapsto\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x}\gamma(s)\,\mathrm{d}s$ is constant for $x\in(0,u)$ *iff* $\gamma(\cdot)$ is constant, hence the circle gives a sharp maximum. $\square$

Local validity of $C_L^2(u)$

Proposition: If Γ is C^1, piecewise C^2, the inequality $C_L^2(u)$ holds locally for any $L > 0$ and $u \in (0, \frac{1}{2}L]$, and consequently, $C_L^{\pm p}(u)$ holds locally for any $p \in (0, 2]$

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Proof: Gentle deformations of $\mathcal{C}$ can be characterized by

$$\gamma(s) = \frac{2\pi}{L} + g(s)\,,$$

where g is a piecewise continuous function, small in the sense that $\|g\|_\infty \ll L^{-1}$ and satisfying the condition $\int_0^L g(s)\,\mathrm{d}s = 0$. We employ the expansion

$$\cos\frac{2\pi x}{L} - \sin\frac{2\pi x}{L}\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s - \frac{1}{2}\cos\frac{2\pi x}{L}\left(\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s\right)^2 + \mathcal{O}(g^3)\,,$$

where the error term is a shorthand for $\mathcal{O}(\|Lg\|_\infty^3)$

Proof, continued

Substituting into the expression for $c_\Gamma^2(u)$ we find that the term linear in g vanishes, because

$$\int_0^L \mathrm{d}z \int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s = \int_0^L \mathrm{d}s\, g(s) \int_{s-\frac{1}{2}x}^{s+\frac{1}{2}x} \mathrm{d}z = 0\,,$$

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Hence the deformation shows in the 2nd order term only,

$$c_\Gamma^2(u) = \frac{L^3}{\pi^2}\sin^2\frac{\pi u}{L} - I_g(u) + \mathcal{O}(g^3)\,,$$

where

$$I_g(u) := \int_0^u \mathrm{d}x\,(u-x)\,\cos\frac{2\pi x}{L}\int_0^L \mathrm{d}z\left(\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s\right)^2$$

and we need to show that $I_g(u) > 0$ unless $g = 0$ identically. Notice that *for $u \le \frac{1}{4}L$ this property holds trivially*

Proof, continued

For $u \in (\frac{1}{4}L, \frac{1}{2}L]$ we notice that g is periodic and piecewise C^0, so we write it as Fourier series with zero term missing,

$$g(s) = \sum_{n=1}^{\infty} \left(a_n \sin \frac{2\pi n s}{L} + b_n \cos \frac{2\pi n s}{L} \right) ,$$

where $\sum_n (a_n^2 + b_n^2) < \infty$ (and small).

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$$\int_{z-\frac{1}{2}x}^{z+\frac{1}{2}x} g(s)\,\mathrm{d}s = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(a_n \sin \frac{2\pi n z}{L} + b_n \cos \frac{2\pi n z}{L} \right) \sin \frac{\pi n x}{L} ,$$

so using orthogonality of the Fourier basis one gets

$$I_g(u) = \int_0^u \mathrm{d}x \, (u - x) \, \cos \frac{2\pi x}{L} \sum_{n=1}^{\infty} \frac{L^3}{2\pi^2} \frac{a_n^2 + b_n^2}{n^2} \sin \frac{\pi n x}{L} .$$

Proof, continued

Summation and integration can be interchanged giving

$$I_g(u) = \frac{L^5}{2\pi^4} \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{n^2} F_n\left(\frac{\pi u}{L}\right) ,$$

where

$$F_n(v) := \int_0^v (v-y) \cos 2y \sin ny \, \mathrm{d}y \, .$$

Proof, continued

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These integrals are equal to

$$\begin{aligned} F_1(v) &= \frac{1}{18}(\,9\sin v - \sin 3v - 6v\,)\,, \\ F_2(v) &= \frac{1}{32}(\,4v - \sin 4v\,)\,, \\ F_n(v) &= \frac{nv}{n^2-4} - \frac{\sin(n-2)v}{2(n-2)^2} - \frac{\sin(n+2)v}{2(n+2)^2}\,, \quad n \geq 3\,. \end{aligned}$$

It is easy to see that $F_n(v) > 0$ for $v > 0$ and $n \geq 2$ and $F_1(v) > 0$ in the interval $(0, \frac{\pi}{2})$. Thus we have found that $I_g(u) > 0$ unless all the coefficients a_n, b_n are zero. $\square$

Remark

One may wonder what happened with the *closedness requirement*, $\int_0^L \cos\beta(s')\,\mathrm{d}s' = \int_0^L \sin\beta(s')\,\mathrm{d}s' = 0$. We proved the claim using the weaker property $\beta(0) = \beta(L)$. This is possible *for small deformations only!*

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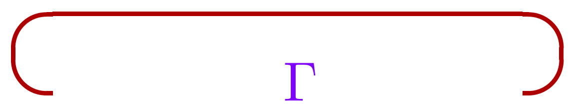


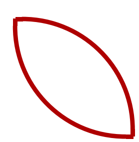
As an illustration, consider Γ in the form of an "overgrown paperclip" which satisfies the condition $\beta(0) = \beta(L)$ but not the *closedness requirement*. Making the U-turns small one can get $c_\Gamma^2(\frac{1}{2}L)$ *arbitrarily close to* $\frac{1}{3}L^3$ which is, of course, larger than L^3/π^2

Global validity of $C_L^2(u)$: an example

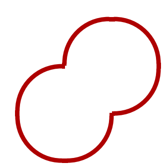
Let Γ be a curve consisting of two circular segments of radius $R > \frac{L}{4\pi}$, i.e. it is given by the equations

$$\left(x \pm R\cos\frac{L}{2R}\right)^2 + y^2 = R^2 \quad \text{for} \quad \pm x \geq 0$$

being "lens-shaped" for $R > \frac{L}{2\pi}$, "apple-shaped" for $\frac{L}{4\pi} < R < \frac{L}{2\pi}$ "apple-shaped" and a circle for $R = \frac{L}{2\pi}$



$R > \frac{L}{2\pi}$



$\frac{L}{4\pi} < R < \frac{L}{2\pi}$

Example, continued

It is straightforward exercise to compute

$$c_\Gamma^2(u) = 8R^3 \left\{ \frac{L}{2R} \sin^2 \frac{u}{2R} + 4 \left(\frac{u}{2R} \cos \frac{u}{2R} - \sin \frac{u}{2R} \right) \cos \frac{L}{4R} \cos \frac{L-2u}{4R} \right\}$$

Let us plot $c_\Gamma^2(u) \left(\frac{L^3}{\pi^2} \sin^2 \frac{\pi u}{L} \right)^{-1}$ for $L = 1$ w.r.t. R and u

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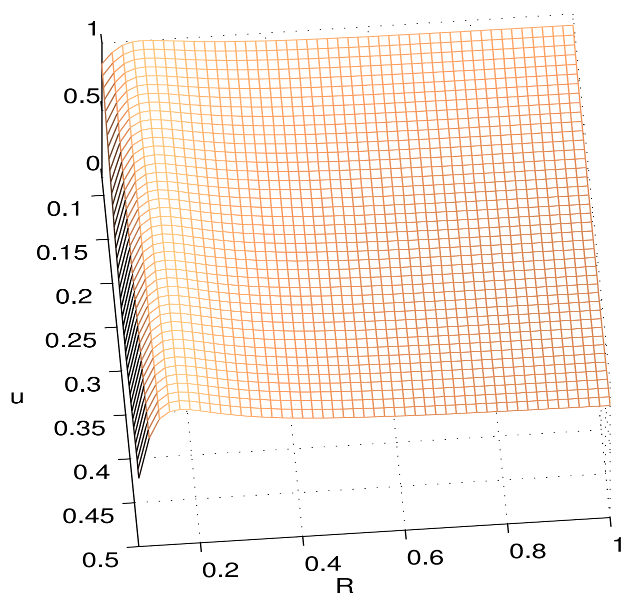

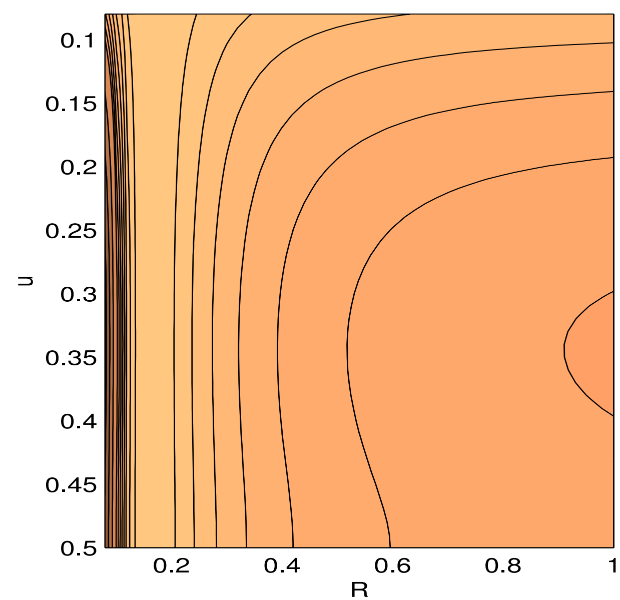

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- Find maximizers if the *interaction strength changes* along the curve (or surface), so the problem ceases to be purely geometric

The talk was based on

[E05a] P.E.: An isoperimetric problem for point interactions, *J. Phys. A: Math. Gen.* **A38** (2005), to appear; `math-ph/0406017`

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for more information see *http://www.ujf.cas.cz/~exner*