

Inequalities for means of chords, with applications to isoperimetric problems

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Talk overview

The spectra of the systems considered may be complex but I will speak about a single eigenvalue only

- *Motivation:* some classical and less classical isoperimetric problems for Schrödinger operators



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- *Applications* to the described situations
- *A discrete analogue* of our problem



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- *Applications* to the described situations
- *A discrete analogue* of our problem
- *Summary and outlook*



Motivation

Isoperimetric problems are traditional in mathematical physics. Recall, e.g., the *Faber-Krahn inequality* for the Dirichlet Laplacian $-\Delta_D^M$ in a compact $M \subset \mathbb{R}^2$: among all regions with a fixed area the ground state is *uniquely minimized by the circle*,

$$\inf \sigma(-\Delta_D^M) \geq \pi j_{0,1}^2 |M|^{-1};$$

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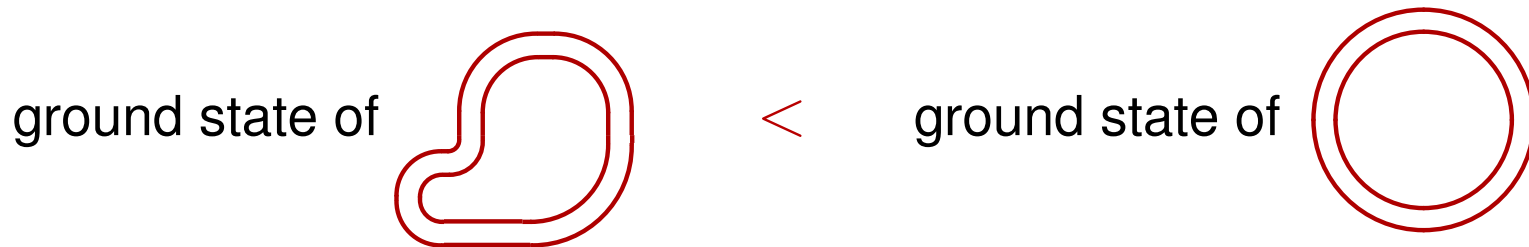
Another classical example is the *PPW conjecture* proved by *Ashbaugh* and *Benguria*: in the 2D situation we have

$$\frac{\lambda_2(M)}{\lambda_1(M)} \leq \left(\frac{j_{1,1}}{j_{0,1}} \right)^2$$



Notice that topology is important

If M is not simply connected, rotational symmetry may again lead to an extremum but its nature can be different. Recall a *a strip of fixed length and width* [E.-Harrell-Loss'99]

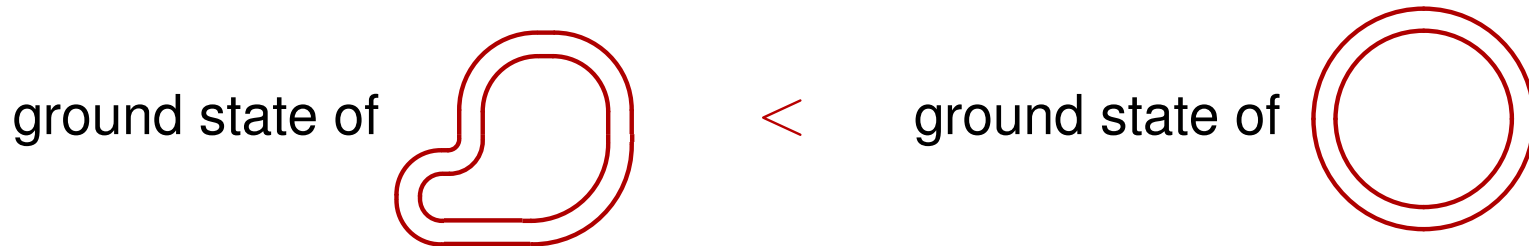


whenever the strip is not a circular annulus



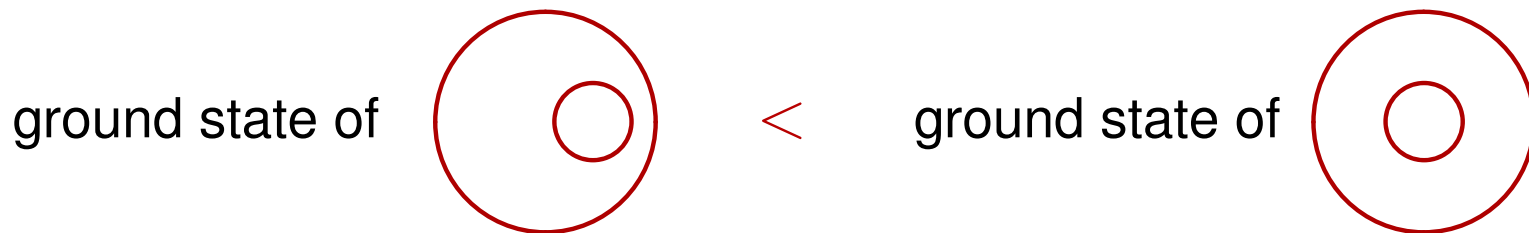
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whenever the strip is not a circular annulus

Another example is a *circular obstacle in circular cavity* [Harrell-Kröger-Kurata'01]



whenever the obstacle is off center



Singular Schrödinger operators

Topology loses meaning when the confinement is due to a potential. For simplicity, we suppose is a singular one,

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma), \quad \alpha > 0,$$

in $L^2(\mathbb{R}^2)$, where Γ is a loop in the plane; we suppose that it has no *zero-angle* self-intersections



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$H_{\alpha,\Gamma}$ can be naturally associated with the quadratic form,

$$\psi \mapsto \|\nabla\psi\|_{L^2(\mathbb{R}^2)}^2 - \alpha \int_{\Gamma} |\psi(x)|^2 dx,$$

which is closed and below bounded in $W^{1,2}(\mathbb{R}^2)$; the second term makes sense in view of Sobolev embedding. This definition also works for various “wilder” sets Γ



Definition by boundary conditions

Since Γ is *piecewise smooth* with *no cusps* we can use an *alternative definition* by boundary conditions: $H_{\alpha,\Gamma}$ acts as $-\Delta$ on functions from $W_{\text{loc}}^{2,1}(\mathbb{R}^2 \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial \psi}{\partial n}(x) \Big|_+ - \frac{\partial \psi}{\partial n}(x) \Big|_- = -\alpha \psi(x)$$



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Remarks:

- this definition has an illustrative meaning which corresponds to a δ potential in the cross cut of Γ
- using the form associated with $H_{\alpha,\Gamma}$ one can check directly that $\sigma_{\text{disc}}(H_{\alpha,\Gamma})$ *is not void* for any $\alpha > 0$; one has, of course, $\sigma_{\text{ess}}(H_{\alpha,\Gamma}) = [0, \infty)$. We will ask about Γ of a fixed length which *maximizes the ground state*



Charged loops

Let us mention another problem which comes from *classical electrostatics* and at a glance it has a little in common with the quantum mechanical question posed above



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Let $\Gamma : [0, L] \rightarrow \mathbb{R}^3$ be a smooth loop and suppose that it is *homogeneously charged* and *non-conducting*. We ask about the shape which it will take in absence of external forces, i.e. about *minimum* of the potential energy of the Coulombic repulsion.

Remark: The latter has to be renormalized. The question makes sense because the divergent factor comes from the short-distance behavior being shape-independent



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We are going to show that both the mentioned problems reduce essentially to *the same geometric question*



Inequalities for L^p norms of chords

It is convenient to work in an arbitrary dimension $d \geq 2$.

Let Γ be a piecewise differentiable function $\Gamma : [0, L] \rightarrow \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot{\Gamma}(s)| = 1$ for any $s \in [0, L]$.

Consider *chords* corresponding to a *fixed arc length* $u \in (0, \frac{1}{2}L]$; we are interested in the inequalities

$$C_L^p(u) : \int_0^L |\Gamma(s+u) - \Gamma(s)|^p ds \leq \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L}, \quad p > 0,$$

$$C_L^{-p}(u) : \int_0^L |\Gamma(s+u) - \Gamma(s)|^{-p} ds \geq \frac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}}, \quad p > 0.$$



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The right sides correspond to the maximally symmetric case, the *planar circle*. It is clear that the inequalities are invariant under scaling, so without loss of generality we may fix the length, say, to $L = 2\pi$. Notice also that for $p = 0$ the inequalities $C_L^p(u)$ and $C_L^{-p}(u)$ turn into trivial identities.



Simple properties

Using convexity of $x \mapsto x^\alpha$ in $(0, \infty)$ for $\alpha > 1$ we get

Proposition: $C_L^p(u) \Rightarrow C_L^{p'}(u)$ if $p > p' > 0$



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The norm can be expressed through curvature of Γ . Using then a Fourier analysis, one can prove

Proposition [E'05b]: If Γ is C^2 , the inequality $C_L^2(u)$, and thus also $C_L^p(u)$ for $|p| \leq 2$, holds *locally*.



The global result

Theorem [E-Harrell-Loss'05]: Let Γ be piecewise C^1 with no cusps. Then $C_L^2(u)$ is valid for any $u \in (0, \frac{1}{2}L]$, and the inequality is strict unless Γ is a planar circle.



The global result

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Proof: Without loss of generality we put $L = 2\pi$ and write

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n e^{ins}$$

with $c_n \in \mathbb{C}^d$. Since $\Gamma(s) \in \mathbb{R}^d$ the coefficients have to satisfy $c_{-n} = \bar{c}_n$; the absence of c_0 can be always achieved by a choice of the origin of the coordinate system.

In view of the Weierstrass theorem and continuity of the functional in question, we may suppose that Γ is C^2 , apart of the last part of the theorem.



Proof, continued

If Γ is C^2 its derivative is a sum of the uniformly convergent Fourier series

$$\dot{\Gamma}(s) = i \sum_{0 \neq n \in \mathbb{Z}} n c_n e^{ins}$$



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By assumption, $|\dot{\Gamma}(s)| = 1$, and hence from the relation

$$2\pi = \int_0^{2\pi} |\dot{\Gamma}(s)|^2 ds = \int_0^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} nm c_m^* \cdot c_n e^{i(n-m)s} ds,$$

where c_m^* denotes the row vector $(\bar{c}_{m,1}, \dots, \bar{c}_{m,d})$ and dot marks the inner product in \mathbb{C}^d , we infer that

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1$$



Proof, continued

In a similar way we can rewrite the right-hand side expression of $C_{2\pi}^2(u)$ using the Parseval relation as

$$\int_0^{2\pi} \left| \sum_{0 \neq n \in \mathbb{Z}} c_n (e^{inu} - 1) e^{ins} \right|^2 ds = 8\pi \sum_{0 \neq n \in \mathbb{Z}} |c_n|^2 \left(\sin \frac{nu}{2} \right)^2$$



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Thus the sought inequality is equivalent to

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 \left(\frac{\sin \frac{nu}{2}}{n \sin \frac{u}{2}} \right)^2 \leq 1$$

and it is sufficient to prove that $|\sin nx| \leq n \sin x$ holds for all positive integers n and all $x \in (0, \frac{1}{2}\pi]$.



Proof, continued

We use induction. The claim is valid for $n = 1$ and

$$(n+1) \sin x \mp \sin(n+1)x = n \sin x \mp \sin nx \cos x + \sin x(1 \mp \cos nx),$$

so if it holds for n , the sum of the first two terms at the *rhs* is non-negative, and the same is clearly true for the last one



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We also see that if $|\sin nx| < n \sin x$ the inequality is strict for $n + 1$ as well. Since this is true for $n = 2$, equality can occur only for $n = 1$. Hence $C_{2\pi}^2(u)$ is strict unless $c_n = 0$ for $|n| \geq 2$, being saturated only if the j th projection of Γ equals

$$\Gamma_j(s) = 2|c_{1,j}| \cos(s + \arg c_{1,j}).$$

Furthermore, $|\dot{\Gamma}(s)| = 1$ can be true only if there is a basis in \mathbb{R}^d where $c_{1,1} = ic_{1,2} = \frac{1}{2}$ and $c_{1,j} = 0$ for $j = 3, \dots, d$, in other words, if Γ is a planar circle



Proof, conclusion

It remains to check that the inequality cannot be saturated for a curve Γ that is not C^2 , so that the sum $\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2$ diverges



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This would require

$$\frac{\sum_{1 \leq n \leq N} n^2 |c_n|^2 \left(\frac{\sin \frac{nu}{2}}{n \sin \frac{u}{2}} \right)^2}{\sum_{1 \leq n \leq N} n^2 |c_n|^2} \rightarrow 1$$

as $N \rightarrow \infty$. This is impossible, however, because the sum in the numerator is bounded by $\sec^2 \frac{u}{2} \sum_{1 \leq n \leq N} |c_n|^2$ so it has a finite limit; this concludes the proof. \square



Application to charged loops

Let Γ be a closed C^2 curve in \mathbb{R}^3 , to be compared with a planar circle. The energy cost of such a deformation is $q^2 \delta(\Gamma)$, where

$$\delta(\Gamma) := 2 \int_0^{L/2} du \int_0^L ds \left[|\Gamma(s+u) - \Gamma(s)|^{-1} - \frac{\pi}{L} \operatorname{csc} \frac{\pi u}{L} \right]$$

and q is the charge density along the loop



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Corollary: $\delta(\Gamma)$ is finite and non-negative; it is zero if and only if $\Gamma = \mathcal{C}_L$, up to Euclidean equivalence.

Proof: The integrand is ≥ 0 by $C_L^{-1}(u)$, strictly so if $\Gamma \neq \mathcal{C}_L$. Moreover, we have $|\Gamma(s+u) - \Gamma(s)|^{-1} = u^{-1} + \mathcal{O}(1)$ with the error dependent on curvature and torsion of Γ but uniform in s , hence the integral converges. \square



Application to leaky loops

Let us turn to the singular Schrödinger operators $H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$. We have mentioned that the discrete spectrum is nonempty and finite, in particular,

$$\epsilon_1 \equiv \epsilon_1(\alpha, \Gamma) := \inf \sigma (H_{\alpha,\Gamma}) < 0$$



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Theorem [E'05b]: Let $\Gamma : [0, L] \rightarrow \mathbb{R}^2$ have the indicated properties; then for any fixed $\alpha > 0$ and $L > 0$ the ground state $\epsilon_1(\alpha, \Gamma)$ is *globally uniquely maximized by the circle* of radius $L/2\pi$.

Proof is based on the generalized Birman-Schwinger principle, plus symmetry and convexity arguments



Birman-Schwinger reformulation

We employ the generalized Birman-Schwinger principle [BEKŠ'94]. One starts from the free resolvent R_0^k which is an integral operator in $L^2(\mathbb{R}^2)$ with the kernel

$$G_k(x-y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$$



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Then we introduce embedding operators associated with R_0^k for measures μ, ν which are the Dirac measure m supported by Γ and the Lebesgue measure dx on \mathbb{R}^2 ; by $R_{\nu, \mu}^k$ we denote the integral operator from $L^2(\mu)$ to $L^2(\nu)$ with the kernel G_k , i.e. we suppose that

$$R_{\nu, \mu}^k \phi = G_k * \phi \mu$$

holds ν -a.e. for all $\phi \in D(R_{\nu, \mu}^k) \subset L^2(\mu)$



BS reformulation, continued

Proposition [BEKŠ'94, Posilicano'04]: (i) There is $\kappa_0 > 0$ s.t. $I - \alpha R_{m,m}^{i\kappa}$ on $L^2(m)$ has a bounded inverse for $\kappa \geq \kappa_0$

(ii) Let $\text{Im } k > 0$ and $I - \alpha R_{m,m}^k$ be invertible with

$$R^k := R_0^k + \alpha R_{dx,m}^k [I - \alpha R_{m,m}^k]^{-1} R_{m,dx}^k$$

from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ everywhere defined. Then k^2 belongs to $\rho(H_{\alpha,\Gamma})$ and $(H_{\alpha,\Gamma} - k^2)^{-1} = R^k$

(iii) $\dim \ker(H_{\alpha,\Gamma} - k^2) = \dim \ker(I - \alpha R_{m,m}^k)$ for $\text{Im } k > 0$

(iv) an ef of $H_{\alpha,\Gamma}$ associated with k^2 can be written as

$$\psi(x) = \int_0^L R_{dx,m}^k(x, s) \phi(s) ds,$$

where ϕ is the corresponding ef of $\alpha R_{m,m}^k$ with the ev one



BS reformulation, continued

Putting $k = i\kappa$ with $\kappa > 0$ we look thus for solutions to the integral-operator equation

$$\mathcal{R}_{\alpha, \Gamma}^{\kappa} \phi = \phi, \quad \mathcal{R}_{\alpha, \Gamma}^{\kappa}(s, s') := \frac{\alpha}{2\pi} K_0(\kappa |\Gamma(s) - \Gamma(s')|),$$

on $L^2([0, L])$. The function $\kappa \mapsto \mathcal{R}_{\alpha, \Gamma}^{\kappa}$ is strictly decreasing in $(0, \infty)$ and $\|\mathcal{R}_{\alpha, \Gamma}^{\kappa}\| \rightarrow 0$ as $\kappa \rightarrow \infty$, hence we seek the point where the *largest* ev of $\mathcal{R}_{\alpha, \Gamma}^{\kappa}$ crosses one



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We observe that this ev is *simple*, since $\mathcal{R}_{\alpha,\Gamma}^{\kappa}$ is positivity improving and ergodic. The ground state of $H_{\alpha,\Gamma}$ is, of course, also simple. Using its rotational symmetry and the claim (iv) of the Proposition we find that the respective eigenfunction of $\mathcal{R}_{\alpha,\mathcal{C}}^{\tilde{\kappa}_1}$ corresponding to the unit eigenvalue is constant; we can choose it as $\tilde{\phi}_1(s) = L^{-1/2}$.



BS reformulation, continued

Then we have

$$\max \sigma(\mathcal{R}_{\alpha, \mathcal{C}}^{\tilde{\kappa}_1}) = (\tilde{\phi}_1, \mathcal{R}_{\alpha, \mathcal{C}}^{\tilde{\kappa}_1} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\alpha, \mathcal{C}}^{\tilde{\kappa}_1}(s, s') ds ds' ,$$

and on the other hand, for the same quantity referring to a general Γ a simple variational estimate gives

$$\max \sigma(\mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1}) \geq (\tilde{\phi}_1, \mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1}(s, s') ds ds' .$$



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Hence it is sufficient to show that

$$\int_0^L \int_0^L K_0(\kappa |\Gamma(s) - \Gamma(s')|) ds ds' \geq \int_0^L \int_0^L K_0(\kappa |\mathcal{C}(s) - \mathcal{C}(s')|) ds ds'$$

holds *for all* $\kappa > 0$ and Γ in the vicinity of \mathcal{C}



Convexity argument

By a simple change of variables the claim is equivalent to positivity of the functional

$$F_\kappa(\Gamma) := \int_0^{L/2} du \int_0^L ds \left[K_0(\kappa|\Gamma(s+u) - \Gamma(s)|) - K_0(\kappa|\mathcal{C}(s+u) - \mathcal{C}(s)|) \right];$$

the s -independent second term is equal to $K_0\left(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L}\right)$



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The (strict) convexity of K_0 yields by means of Jensen inequality the estimate

$$\frac{1}{L} F_\kappa(\Gamma) \geq \int_0^{L/2} \left[K_0 \left(\frac{\kappa}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)| ds \right) - K_0 \left(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L} \right) \right] du,$$

where the inequality is sharp unless $\int_0^L |\Gamma(s+u) - \Gamma(s)| ds$ is independent of s



Monotonicity argument

Finally, we observe that K_0 is decreasing in $(0, \infty)$, hence it is sufficient to check the inequality

$$\int_0^L |\Gamma(s+u) - \Gamma(s)| \, ds \leq \frac{L^2}{\pi} \sin \frac{\pi u}{L}$$

for all $u \in (0, \frac{1}{2}L]$ and furthermore, to show that it is strict unless Γ is a circle



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In this way our problem is reduced to the $C_L^1(u)$ inequality which follows from $C_L^2(u)$ proved above. \square



A discrete analogue: polymer loops

Consider a problem related to the above one; following [AGHH'88, 05] we can call it a *polymer loop*



A discrete analogue: polymer loops

Consider a problem related to the above one; following [AGHH'88, 05] we can call it a *polymer loop*

It is an extension of the “discrete” problem to a more general class of curves: we take a closed loop Γ and consider a class of singular Schrödinger operators in $L^2(\mathbb{R}^d)$, $d = 2, 3$, given formally by the expression

$$H_{\alpha, \Gamma}^N = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta \left(x - \Gamma \left(\frac{jL}{N} \right) \right)$$

We are interested in the shape of Γ which *maximizes* the ground state energy provided, of course, that the discrete spectrum of $H_{\alpha, \Gamma}^N$ is non-empty.



A reminder: 2D point interactions

Fixing the site y_j and “coupling constant” α we define them by b.c. which change *locally* the domain of $-\Delta$: we require

$$\psi(x) = -\frac{1}{2\pi} \log |x - y_j| L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|),$$

where the generalized b.v. $L_0(\psi, y_j)$ and $L_1(\psi, y_j)$ satisfy

$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0, \quad \alpha \in \mathbb{R}$$



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For $Y_\Gamma := \{y_j := \Gamma \left(\frac{jL}{N}\right) : j = 0, \dots, N-1\}$ we define in this way $-\Delta_{\alpha, Y_\Gamma}$ in $L^2(\mathbb{R}^2)$. It holds $\sigma_{\text{disc}}(-\Delta_{\alpha, Y_\Gamma}) \neq \emptyset$, i.e.

$$\epsilon_1 \equiv \epsilon_1(\alpha, Y_\Gamma) := \inf \sigma(-\Delta_{\alpha, Y_\Gamma}) < 0,$$

which is always true in two dimensions – cf. [AGHH'88, 05]



A reminder: 3D point interactions

Similarly, for y_j and “coupling” α we define them by b.c. which change locally the domain of $-\Delta$: we require

$$\psi(x) = \frac{1}{4\pi|x - y_j|} L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|),$$

where the b.v. $L_0(\psi, y_j)$ and $L_1(\psi, y_j)$ satisfy again

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$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0, \quad \alpha \in \mathbb{R},$$

giving $-\Delta_{\alpha, Y_\Gamma}$ in $L^2(\mathbb{R}^3)$. However, $\sigma_{\text{disc}}(-\Delta_{\alpha, Y_\Gamma}) \neq \emptyset$, i.e.

$$\epsilon_1 \equiv \epsilon_1(\alpha, Y_\Gamma) := \inf \sigma(-\Delta_{\alpha, Y_\Gamma}) < 0,$$

is now a nontrivial requirement; it holds only for α below some critical value α_0 – cf. [AGHH'88, 05]



A geometric reformulation

By Krein's formula, the spectral condition is reduced to an algebraic problem. Using $k = i\kappa$ with $\kappa > 0$, we find the ev's $-\kappa^2$ of our operator from

$$\det \Gamma_k = 0 \quad \text{with} \quad (\Gamma_k)_{ij} := (\alpha - \xi^k) \delta_{ij} - (1 - \delta_{ij}) g_{ij}^k,$$

where the off-diagonal elements are $g_{ij}^k := G_k(y_i - y_j)$, or equivalently

$$g_{ij}^k = \frac{1}{2\pi} K_0(\kappa |y_i - y_j|)$$

and the regularized Green's function at the interaction site is

$$\xi^k = -\frac{1}{2\pi} \left(\ln \frac{\kappa}{2} + \gamma_E \right)$$



Geometric reformulation, continued

The ground state refers to the point where the *lowest* ev of $\Gamma_{i\kappa}$ vanishes. Using smoothness and monotonicity of the κ -dependence we have to check that

$$\min \sigma(\Gamma_{i\tilde{\kappa}_1}) < \min \sigma(\tilde{\Gamma}_{i\tilde{\kappa}_1})$$

holds locally for $\Gamma \neq \tilde{\mathcal{P}}_N$, where $-\tilde{\kappa}_1^2 := \epsilon_1(\alpha, \tilde{\mathcal{P}}_N)$



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There is a *one-to-one relation* between an ef $c = (c_1, \dots, c_N)$ of $\Gamma_{i\kappa}$ at that point and the corresponding ef of $-\Delta_{\alpha, \Gamma}$ given by $c \leftrightarrow \sum_{j=1}^N c_j G_{i\kappa}(\cdot - y_j)$, up to normalization. In particular, the lowest ev of $\tilde{\Gamma}_{i\tilde{\kappa}_1}$ corresponds to the eigenvector $\tilde{\phi}_1 = N^{-1/2}(1, \dots, 1)$; hence the spectral threshold is

$$\min \sigma(\tilde{\Gamma}_{i\tilde{\kappa}_1}) = (\tilde{\phi}_1, \tilde{\Gamma}_{i\tilde{\kappa}_1} \tilde{\phi}_1) = \alpha - \xi^{i\tilde{\kappa}_1} - \frac{2}{N} \sum_{i < j} \tilde{g}_{ij}^{i\tilde{\kappa}_1}$$



Geometric reformulation, continued

On the other hand, we have $\min \sigma(\Gamma_{i\tilde{\kappa}_1}) \leq (\tilde{\phi}_1, \Gamma_{i\tilde{\kappa}_1} \tilde{\phi}_1)$, and therefore it is sufficient to check that

$$\sum_{i < j} G_{i\kappa}(y_i - y_j) > \sum_{i < j} G_{i\kappa}(\tilde{y}_i - \tilde{y}_j)$$

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holds *for all* $\kappa > 0$ and $\Gamma \neq \tilde{\mathcal{P}}_N$. Call $\ell_{ij} := |y_i - y_j|$ and $\tilde{\ell}_{ij} := |\tilde{y}_i - \tilde{y}_j|$ and define $F : (\mathbb{R}_+)^{N(N-3)/2} \rightarrow \mathbb{R}$ by

$$F(\{\ell_{ij}\}) := \sum_{m=2}^{[N/2]} \sum_{|i-j|=m} \left[G_{i\kappa}(\ell_{ij}) - G_{i\kappa}(\tilde{\ell}_{ij}) \right] ;$$

Using the *convexity* of $G_{i\kappa}(\cdot)$ for a fixed $\kappa > 0$ we get

$$F(\{\ell_{ij}\}) \geq \sum_{m=2}^{[N/2]} \nu_m \left[G_{i\kappa} \left(\frac{1}{\nu_m} \sum_{|i-j|=m} \ell_{ij} \right) - G_{i\kappa}(\tilde{\ell}_{1,1+m}) \right] ,$$

where ν_n is the number of the appropriate chords



Geometric reformulation, continued

It is easy to see that

$$\nu_m := \begin{cases} N & \dots & m = 1, \dots, [\frac{1}{2}(N - 1)] \\ \frac{1}{2}N & \dots & m = \frac{1}{2}N \quad \text{for } N \text{ even} \end{cases}$$

since for an even N one has to prevent double counting



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Since $G_{i\kappa}(\cdot)$ is also *monotonously decreasing* in $(0, \infty)$, we thus need only to demonstrate that

$$\tilde{\ell}_{1,m+1} \geq \frac{1}{\nu_n} \sum_{|i-j|=m} \ell_{ij}$$

with the sharp inequality for at least one m if $\mathcal{P}_N \neq \tilde{\mathcal{P}}_N$.

In this way the problem becomes again purely geometric



"Discrete" chord inequalities

Recall that for $\Gamma : [0, L] \rightarrow \mathbb{R}^2$ we have used the notation

$$y_j := \Gamma \left(\frac{jL}{N} \right), \quad j = 0, 1, \dots, N - 1;$$



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$$y_j := \Gamma \left(\frac{jL}{N} \right), \quad j = 0, 1, \dots, N-1;$$

For fixed $L > 0$, N and $m = 1, \dots, [\frac{1}{2}N]$ we consider the following inequalities for ℓ^p norms related to the chord lengths, that is, the quantities $\Gamma \left(\cdot + \frac{jL}{N} \right) - \Gamma(\cdot)$

$$D_{L,N}^p(m) : \sum_{n=1}^N |y_{n+m} - y_n|^p \leq \frac{N^{1-p} L^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}}, \quad p > 0,$$

$$D_{L,N}^{-p}(m) : \sum_{n=1}^N |y_{n+m} - y_n|^{-p} \geq \frac{N^{1+p} \sin^p \frac{\pi}{N}}{L^p \sin^p \frac{\pi m}{N}}, \quad p > 0.$$

The RHS's correspond to regular planar polygon $\tilde{\mathcal{P}}_N$



More on the "discrete" inequalities

In general, the inequalities *are not valid for* $p > 2$ as the example of a rhomboid shows: $D_{L,4}^p(2)$ is equivalent to $\sin^p \phi + \cos^p \phi \leq 2^{1-(p/2)}$ for $0 < \phi < \pi$ which obviously holds for $p \leq 2$ only



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Proposition: $D_{L,N}^p(m) \Rightarrow D_{L,N}^{p'}(m)$ if $p > p' > 0$ and $D_{L,N}^p(m) \Rightarrow D_{L,N}^{-p}(m)$ for any $p > 0$



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Theorem [E'05c]: The inequality $D_{L,N}^2(m)$ is valid

Remark: By $D_{L,N}^{-1}(m)$ this implies that the unique minimizers of the "discrete" electrostatic problem is the regular planar polygon $\tilde{\mathcal{P}}_N$



Global validity of $D_{L,N}^2(m)$

Let us to adapt the above proof the “discrete” case. We put $L = 2\pi$ and express Γ through its Fourier series,

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n e^{ins}$$

with $c_n \in \mathbb{C}^d$; since $\Gamma(s) \in \mathbb{R}^d$ one has to require $c_{-n} = \bar{c}_n$. Again, we can choose $c_0 = 0$ and the normalization condition $\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1$ follows from $|\dot{\Gamma}(s)| = 1$



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On the other hand, the left-hand side of $D_{2\pi,N}^2(m)$ equals

$$\sum_{n=1}^N \sum_{0 \neq j,k \in \mathbb{Z}} c_j^* \cdot c_k \left(e^{-2\pi imj/N} - 1 \right) \left(e^{2\pi imk/N} - 1 \right) e^{2\pi in(k-j)/N}$$



Global validity, continued

Next we change the order of summation and observe that $\sum_{n=1}^N e^{2\pi i n(k-j)/N} = N$ if $j = k \pmod{N}$ and zero otherwise; this allows us to write the last expression as

$$4N \sum_{l \in \mathbb{Z}} \sum_{\substack{0 \neq j, k \in \mathbb{Z} \\ j - k = lN}} |j| c_j^* \cdot |k| c_k \left| j^{-1} \sin \frac{\pi m j}{N} \right| \left| k^{-1} \sin \frac{\pi m k}{N} \right| .$$



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Hence the sought inequality $D_{2\pi, N}^2(m)$ is equivalent to

$$\left(d, (A^{(N, m)} \otimes I) d \right) \leq \left(\frac{\pi \sin \frac{\pi m}{N}}{N \sin \frac{\pi}{N}} \right)^2$$



Global validity, continued

Here the vector $d \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^d$ has the components $d_j := |j|c_j$ and the operator $A^{(N,m)}$ on $\ell^2(\mathbb{Z})$ is defined as

$$A_{jk}^{(N,m)} := \begin{cases} |j^{-1} \sin \frac{\pi mj}{N}| |k^{-1} \sin \frac{\pi mk}{N}| & \text{if } 0 \neq j, k \in \mathbb{Z}, j - k = lN \\ 0 & \text{otherwise} \end{cases}$$

$A^{(N,m)}$ is obviously bounded because its Hilbert-Schmidt norm is finite; we have to estimate its norm



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Remark: The “continuous” case corresponds formally to $N = \infty$. Then $A^{(N,m)}$ is a multiple of I and it is only necessary to employ $|\sin jx| \leq j \sin x$ for any $j \in \mathbb{N}$ and $x \in (0, \frac{1}{2}\pi]$. Here due to *infinitely many side diagonals* such a simple estimate yields an unbounded Toeplitz-type operator, and one has to use the *matrix-element decay*



Global validity, continued

For a given $j \neq 0$ and $d \in \ell^2(\mathbb{Z})$ we have

$$\left(A^{(N,m)} d \right)_j = \left| j^{-1} \sin \frac{\pi m j}{N} \right| \sum_{\substack{0 \neq k \in \mathbb{Z} \\ k = j \pmod{N}}} \left| k^{-1} \sin \frac{\pi m k}{N} \right| d_k$$



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The norm $\|A^{(N,m)} d\|$ is then easily estimated by means of Schwarz inequality,

$$\begin{aligned} \|A^{(N,m)} d\|^2 &= \sum_{0 \neq j \in \mathbb{Z}} j^{-2} \sin^2 \frac{\pi m j}{N} \left| \sum_{\substack{0 \neq k \in \mathbb{Z} \\ k = j(\text{mod } N)}} \left| k^{-1} \sin \frac{\pi m k}{N} \right| d_k \right|^2 \\ &\leq \sum_{n=0}^{N-1} \sin^4 \frac{\pi m n}{N} S_n^2 \sum_{\substack{n + lN \neq 0 \\ l \in \mathbb{Z}}} |d_{n+lN}|^2 \end{aligned}$$



Global validity, concluded

Here we have introduced

$$S_n := \sum_{\substack{n + lN \neq 0 \\ l \in \mathbb{Z}}} \frac{1}{(n + lN)^2} = \sum_{l=1}^{\infty} \left\{ \frac{1}{(lN - n)^2} + \frac{1}{(lN - N + n)^2} \right\}$$

which is easily evaluated to be $S_n = \left(\frac{\pi}{N \sin \frac{\pi n}{N}} \right)^2$



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The sought claim, the validity of $D_{L,N}^2(m)$, then follows from

$$\sin \frac{\pi m}{N} \sin \frac{\pi r}{N} > \left| \sin \frac{\pi}{N} \sin \frac{\pi m r}{N} \right|, \quad 2 \leq r < m \leq \left[\frac{1}{2} N \right]$$

This can be also equivalently written as the inequalities $U_{m-1} \left(\cos \frac{\pi}{N} \right) > \left| U_{m-1} \left(\cos \frac{\pi r}{N} \right) \right|$ for Chebyshev polynomials of the second kind; they are verified directly \square



Summary and outlook

There is a host of open questions, for instance

- to find out whether $C_L^p(u)$ is *invalid for $p > 2$* .

A “stadium-perimeter” example shows it is the case for $p \gtrsim 3.15295$



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- to find extrema in situations *without a built-in symmetry*, i.e. with different couplings or source spacing. This problem is no longer purely geometric
- to analyze *regular-potential analogues* of our problem, i.e. systems with a loop-shaped potential ditch
- to find *higher-dimensional analogues* of the inequalities discussed here, etc.



The talk was based on

- [E05a] P.E.: An isoperimetric problem for point interactions, *J. Phys. A: Math. Gen.* **A38** (2005), 4795-4802
- [E05b] P.E.: An isoperimetric problem for leaky loops and related mean-chord inequalities, *J. Math. Phys.* **46** (2005), 062105
- [E05c] P.E.: Necklaces with interacting beads: isoperimetric problems, *Proceedings of the "International Conference on Differential Equations and Mathematical Physics" (Birmingham 2005)*, AMS "Contemporary Mathematics" Series; *to appear*
- [EHL05] P.E., E. Harrell, M. Loss: Global mean-chord inequalities with application to isoperimetric problems, *submitted*



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for more information see <http://www.ujf.cas.cz/~exner>

